

Finite-volume effects and meson scattering in the 2-flavour Schwinger model

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We investigate the 2-flavour Schwinger model in the canonical formulation with fixed fermion numbers. We use Wilson fermions and a formalism which describes the determinant of the Dirac operator in terms of dimensionally reduced canonical transfer matrices. These transfer matrices allow the direct examination of arbitrary multi-particle (meson) sectors and the determination of the corresponding ground-state energies. We discuss the finite-volume effects in the meson mass. From the 2-meson energies, we determine the scattering phase shifts and compare the 3-meson energies at finite volume to predictions based on 3-particle quantization conditions.

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1. Introduction

The Schwinger model [1] is of great interest since it shares many similarities with Quantum Chromodynamics (QCD), such as confinement, chiral symmetry breaking, charge shielding, and a topological θ -vacuum [2, 3]. Thanks to these similarities the model is often used as a toy model to test new computational strategies. In our case, we perform numerical computations in the 2-flavour Schwinger model using the canonical formulation. The corresponding canonical partition functions allow one to consider the physics of the system in sectors with a fixed number of particles, i.e., with fixed numbers of fermions, and to determine the corresponding ground-state energies. Using appropriate ratios of the canonical partition functions we directly access the energy spectrum of (multi-)meson states without resorting to the computation of correlation functions. These become more and more complicated with an increasing number of mesons [4]. In contrast, the complexity for the computation of the partition functions is independent of the number of mesons involved.

We use the (multi-)meson ground-state energies to perform some meson-scattering analysis. In the 2-flavour Schwinger model, the canonical sectors with fixed fermion numbers are characterized by their isospin content. Consequently, the corresponding states with the lowest energies are the n -meson states of maximal isospin. The lowest-lying energies in the isospin $I = 1$ sector, for example, describe single-meson energies, while the lowest-lying energies in the isospin $I = 2$ sector correspond to the energies of the 2-meson scattering states of maximal isospin, and so forth. For the investigation of the meson scattering, we determine the mass of the isospin $I = 1$ meson (corresponding to the pion in QCD) on a large range of spatial volumes in order to control the finite-volume effects. Then, we calculate the scattering phase shifts from the energies of the 2-meson states (corresponding to 2-pion scattering states in QCD). Finally, from the isospin $I = 3$ sector, we determine the 3-meson energies and compare them to predictions from quantization conditions for 3-particle energies based on the 2-meson scattering phase shift [5, 6].

The computation of the canonical partition functions is based on the dimensional reduction of the fermion determinant in terms of transfer matrices [7]. Using those, it is then straightforward to project onto the canonical determinants describing the dynamics of the fermions in the sectors with fixed fermion numbers [8]. In the context of QCD, the canonical formulation has been used with staggered and Wilson fermions, see Refs. [9–11] and [12], respectively, for some early applications. In some cases, the canonical formulation is also useful to solve fermion sign problems, see Refs. [13–15]. Here we consider the 2-flavour Schwinger model with Wilson fermions at fixed isospin density where the fermion sign problem is not present.

2. The 2-flavour Schwinger model in the canonical formulation

Using the doublet $\psi = (u, d)$ to describe the two flavours of mass-degenerate fermions with opposite isospin charges (in correspondence with the up and down quarks in QCD), the continuum Lagrangian \mathcal{L} of the 2-flavour variant of the Schwinger model is given by

$$\mathcal{L}[\bar{\psi}, \psi, A_\mu] = \bar{\psi}(x)[i\not{D} - m_0]\psi(x) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \quad (1)$$

where $D_\mu = \partial_\mu + igA_\mu(x)$ is the covariant derivative with the Abelian gauge field $A_\mu(x)$, m_0 the mass of the two fermions, g the gauge coupling and $F_{\mu\nu}(x) = A_\mu(x)A_\nu(x) - A_\nu(x)A_\mu(x)$ the Abelian field

strength tensor. After a transformation to Euclidean spacetime, we discretize the Lagrangian on a square lattice with lattice spacing a and physical extent $L \times L_t$ with periodic boundary conditions (antiperiodic for the fermion fields in temporal direction). We use the Wilson gauge action for the gauge field $U_\mu \in U(1)$ and include a Wilson term in the fermion derivative to circumvent the fermion doubling. The chemical potentials for the two fermions are introduced by furnishing the forward and backward temporal hopping terms with factors of $e^{\pm\mu_{u,d}}$ [16], where the fermion chemical potentials $\mu_{u,d}$ are related to the isospin chemical potential μ_I via $\mu_u = -\mu_d = \frac{1}{2}\mu_I$. The resulting Euclidean lattice action decomposes into a gluonic part S_g , which contains the dimensionless inverse coupling $\beta = 1/(ag)^2$, and two fermionic parts (for each fermion flavour), such that

$$S_E[\bar{\psi}, \psi, U, \mu_I] = S_g[U] + \bar{u} M[U, \mu_I] u + \bar{d} M[U, -\mu_I] d, \quad (2)$$

where M denotes the Wilson-Dirac matrix for one fermion flavour. Integrating out the fermionic degrees of freedom in the grand-canonical partition function yields the determinant of the Wilson-Dirac matrix for each flavour,

$$\mathcal{Z}_{GC}(\mu_I) = \int \mathcal{D}U \mathcal{D}\bar{\psi} \mathcal{D}\psi e^{-S_E} = \int \mathcal{D}U e^{-S_g[U]} \det M[U, \mu_I] \det M[U, -\mu_I]. \quad (3)$$

The fugacity expansion for a single fermion flavour separates the isospin chemical potential μ_I from the determinants,

$$\det M[U, \pm\mu_I] = \sum_{n=-L/a}^{L/a} \det_n M[U] e^{\pm \frac{\mu_I}{T} \frac{1}{2} n}, \quad (4)$$

where the sum over the fermion number is restricted by the lattice volume L/a . The canonical determinants $\det_n M$ can be defined in terms of dimensionally reduced transfer matrices with fixed fermion number n and provide the projection onto the canonical sectors with n fermions [8]. Finally, from

$$\mathcal{Z}_{GC}(\mu_I, T) = \sum_{n_u, n_d} e^{\frac{\mu_I}{T} \frac{1}{2} (n_u - n_d)} \mathcal{Z}_{n_u, n_d}(T), \quad (5)$$

where the dependence on the temperature $T = 1/L_t$ is now made explicit, we obtain the canonical partition functions $\mathcal{Z}_{n_u, n_d}(T)$ given by

$$\mathcal{Z}_{n_u, n_d}(T) = \int \mathcal{D}U e^{-S_g[U]} \det_{n_u} M[U] \det_{n_d} M[U]. \quad (6)$$

The number of up and down fermions is restricted by Gauss' law. It requires that the total electric charge Q , and hence the total fermion number, has to be zero, while the total isospin charge is not restricted, i.e.,

$$Q = n_u + n_d = 0 \quad \text{and} \quad I = \frac{n_u - n_d}{2} \quad \text{arbitrary.} \quad (7)$$

Consequently, a canonical sector with n up fermions also has n antidown (or equivalently $-n$ down) fermions which may bind together to form n -meson states. The collection of all states with n up and $-n$ down fermions forms the canonical partition function $\mathcal{Z}_{n, -n}(T)$. The vacuum sector

contains flavour singlet states and meson-antimeson states with isospin $I = 0$, and is described by the partition function $\mathcal{Z}_{0,0}(T)$.

In the canonical formalism, it is now straightforward to examine multi-meson states and their ground-state energies $E_{n\pi}$ by taking the free energy difference between the corresponding canonical sector and the vacuum and extrapolating it to zero temperature

$$E_{n\pi} = - \lim_{T \rightarrow 0} T \log \left(\frac{\mathcal{Z}_{+n,-n}(T)}{\mathcal{Z}_{0,0}(T)} \right). \quad (8)$$

We have explicitly checked in the 1-, 2- and 3-meson sectors that the ground-state energies coincide with the direct measurements of the corresponding energies extracted from correlators formed with π^+ , $\pi^+\pi^+$ and $\pi^+\pi^+\pi^+$ operators.

3. Isospin $I = 1$ sector and finite-volume corrections

The lightest particles in the massive 2-flavour Schwinger model form the mass-degenerate meson (or pion) triplet $|\pi\rangle = \{|\pi^-\rangle, |\pi^0\rangle, |\pi^+\rangle\}$. Within this triplet, the state with maximal z -component of the isospin is built from an up and antidown fermion, $|\pi^+\rangle = |u\bar{d}\rangle$. It can be identified as the ground state of the isospin $I = 1$ sector described by the canonical partition function in eq. (6) with $\{n_u, n_d\} = \{1, -1\}$. Hence, the ground-state energy of the 1-meson sector, i.e., the mass of the pion, is determined by eq. (8) using $n = 1$. We use this prescription to compute the pion mass $m_\pi(L)$ for different volumes L , as illustrated in figure 1, where we show the behaviour of the pion mass towards zero temperature, i.e., $L_t \rightarrow \infty$. It is governed by contributions from excited states in the $I = 1$ sector and the vacuum sector. Corresponding fits to the data are shown

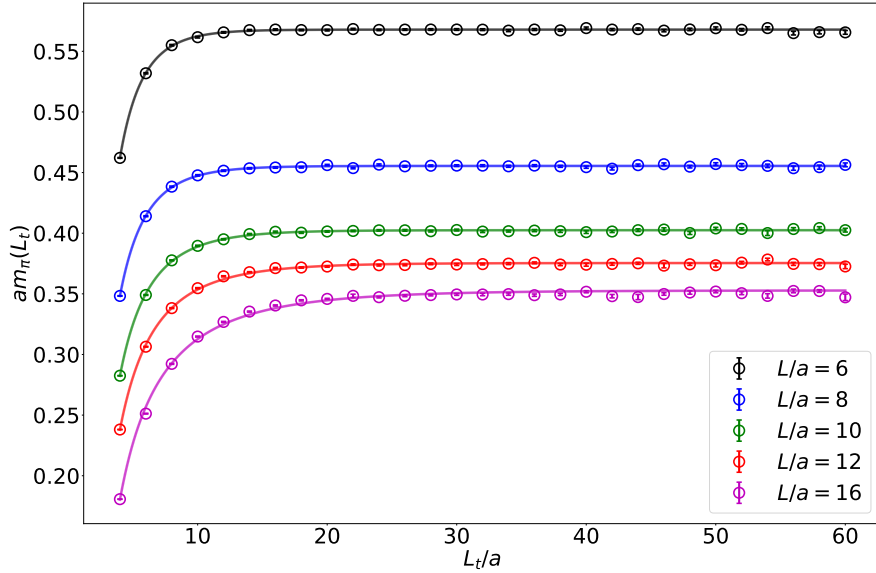


Figure 1: Temperature dependence of the pion mass for different volumes at fixed lattice spacing $\beta = 5.0$. The lines with error bands represent fits including excited state contributions from the $I = 1$ and the vacuum sector.

in figure 1 as lines with (barely visible) error bands. In the canonical formalism excited states contribute to the free energy differences with amplitudes given solely by their degeneracies, see, e.g., Ref. [11]. This is in contrast to traditional spectroscopy with correlation functions, where the excited state contributions depend on the overlap of the pion operators with the pion wave function. The results can now be used to investigate finite-volume effects. They arise when the wave function of the pion overlaps at the boundaries of the box and therefore interacts with itself. This leads to an increase of the pion mass for small volumes. Lüscher appropriately called these kinds of effects "interactions around the world" and provided a formula that can be used to describe these finite-volume effects [17]. In the case of a two-dimensional quantum field theory, one has

$$m_\pi(L) = m_\pi + \frac{1}{\sqrt{m_\pi L}} \left(\frac{F(0)}{\sqrt{2\pi}4m_\pi} \right) e^{-m_\pi L} + \left(\frac{\lambda^2}{4\sqrt{3}m_\pi^3} \right) e^{-\frac{\sqrt{3}}{2}m_\pi L}. \quad (9)$$

Here, $m_\pi = \lim_{L \rightarrow \infty} m_\pi(L)$ denotes the infinite-volume pion mass, $F(0)$ the forward scattering amplitude, and λ some effective 3-meson coupling.

In figure 2 we show the relative finite-volume effects in the pion mass $\delta m_\pi = (m_\pi(L) - m_\pi)/m_\pi$

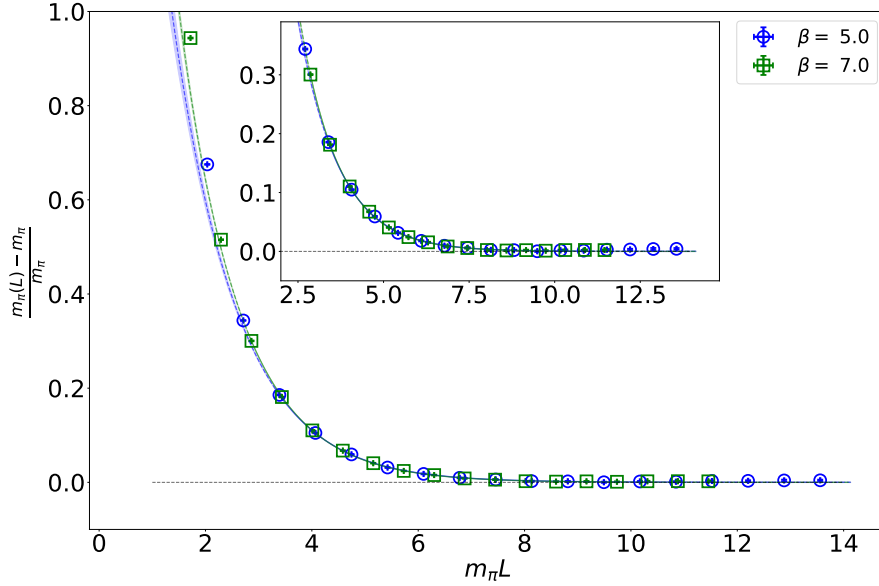


Figure 2: Volume dependence of the pion mass $m_\pi(L)$ for two different lattice spacings $\beta = 5.0$ and 7.0 with the infinite-volume pion mass fixed at $m_\pi \sqrt{\beta} \sim 0.7580$. Shown are the relative finite-volume corrections. The lines with error bands represent Lüscher's finite-volume formula in eq. (9).

at two different lattice spacings $\beta \in \{5.0, 7.0\}$. We use Lüscher's ansatz to describe the finite-volume effects using m_π , $F(0)$ and the effective 3-meson coupling λ as fit parameters. The ansatz allows us to describe the measurements down to small volumes $m_\pi L \gtrsim 3.0$. In order to do so, we need to include the term related to the effective three-meson coupling. While G-parity forbids a 3-pion coupling, an effective 3-meson coupling can apparently not be excluded. For the measurements presented here, we kept the infinite-volume pion mass fixed in physical units, i.e.,

$m_\pi\sqrt{\beta} = m_\pi/g \sim 0.7580$, in order to estimate lattice artefacts. Our results indicate that the artefacts are very well under control, even for small volumes. It is interesting to note that the data for the relative finite-volume corrections δm_π obtained at different pion masses also fall onto the same curve, emphasizing the universal character of the corrections given by eq. (9).

4. Isospin $I = 2$ sector and scattering phase shifts

The ground-state energies in the isospin $I = 2$ sector, corresponding to the energies of the 2-meson (or 2-pion) states, are obtained from eq. (8) with $n = 2$. The results for the relative finite-volume corrections $\delta E_{2\pi} = (E_{2\pi}(L) - E_{2\pi})/E_{2\pi}$ are depicted in figure 3 as a function of the volume. The volume dependence of the 2-pion ground-state energies is of particular interest in the

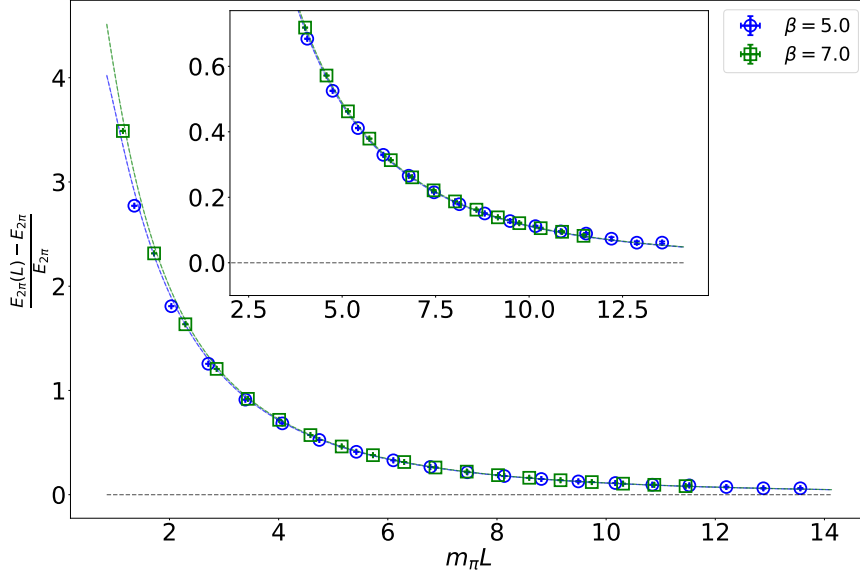


Figure 3: Volume dependence of the two-pion energy $E_{2\pi}(L)$ for two different lattice spacings $\beta = 5.0$ and 7.0 with the infinite-volume pion mass fixed at $m_\pi\sqrt{\beta} \sim 0.7580$. Shown are the relative finite-volume corrections. The lines with error bands represent fits resulting from an effective ansatz for the scattering phase shift.

context of scattering. Consider a situation where one has two pions in a box of size L with equal masses m_π and momenta p_1, p_2 . The continuum dispersion relation for such a state in the center of mass frame ($P = p_1 + p_2 = 0$) reads

$$E_{2\pi} = 2\sqrt{m_\pi^2 + k(L)^2}, \quad (10)$$

where $\pm k(L)$ denote the volume-dependent momenta of the two pions in the finite box. These momenta are determined by the scattering phase shift $\delta(L)$ which needs to be introduced due to the boundary conditions. The phase shifts are described by the intriguingly simple quantization

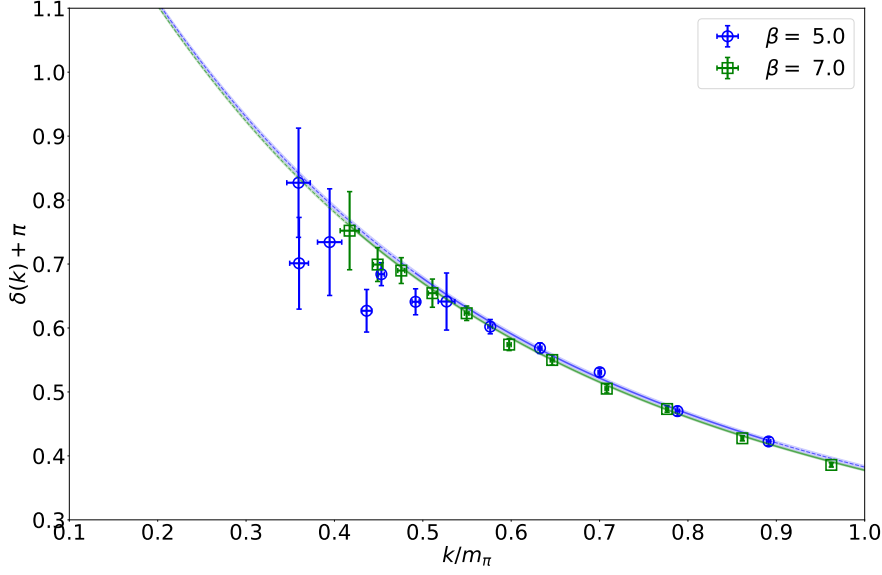


Figure 4: Scattering phase shift $\delta(k)$ for two different lattice spacings $\beta = 5.0$ and 7.0 with the infinite-volume pion mass fixed at $m_\pi\sqrt{\beta} \sim 0.7580$. The lines with error bands represent fits with an effective ansatz motivated by analytical results from the Sine-Gordon model.

condition

$$\delta(k(L)) = -\frac{k(L)L}{2} \equiv \delta(L) \quad (11)$$

as shown by Lüscher in [18]. If the scattering phase shift $\delta(k)$ is known, one can construct the allowed relative momenta k and compute the 2-pion ground-state energy for arbitrary volumes. Conversely, one can determine the scattering phase shifts from the 2-pion energies by using the dispersion relation eq. (10) in combination with the quantization condition eq. (11). In order to (partially) account for lattice artefacts, we use the bosonic lattice dispersion relation

$$E_{2\pi}(L) = 2 \cosh^{-1}(\cosh(m_\pi) + 1 - \cos(k(L))) \quad (12)$$

instead of the continuum one in eq. (11). The results are shown in figure 4 for two lattice spacings $\beta = 5.0$ and 7.0 and infinite-volume pion mass fixed at $m_\pi\sqrt{\beta} \sim 0.7580$. The scattering phase shift can be fitted using an effective ansatz motivated by the analytical result from the Sine-Gordon model.¹ In this way, we obtain a heuristic description of the scattering phase shift $\delta(k)$ for arbitrary k . The results of the fits are shown in figures 3 and 4 by the lines with error bands.

5. Isospin $I = 3$ sector and 3-particle quantization conditions

Next, we consider three pions in a finite box of size L and determine the ground-state energies in the corresponding isospin $I = 3$ sector from eq. (8) with $n = 3$. In figure 5 we show the relative

¹In the strong-coupling limit, the 2-flavour Schwinger model goes over to the Sine-Gordon model.

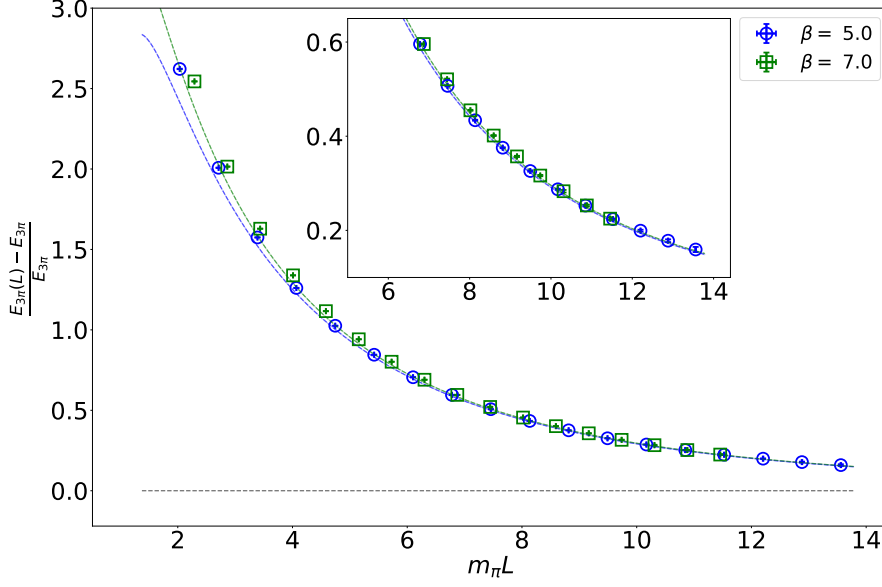


Figure 5: Finite-volume dependence of the 3-pion ground-state energies $E_{3\pi}(L)$ at two different lattice spacings $\beta = 5.0$ and 7.0 with the infinite-volume pion mass fixed at $m_\pi \sqrt{\beta} \sim 0.7580$. Shown are the relative finite-volume corrections. The lines and error bands correspond to the predictions based on the 3-particle quantization conditions eqs. (14) and the scattering phase shift obtained in Sec. 4.

finite-volume corrections $\delta E_{3\pi} = (E_{3\pi}(L) - E_{3\pi})/E_{3\pi}$ as a function of the volume. The bosonic lattice dispersion relation for the energy of a 3-pion state reads

$$E_{3\pi}(L) = \sum_{i=1,2,3} \cosh^{-1}(\cosh(m_\pi) + 1 - \cos(p_i(L))), \quad (13)$$

where the $p_i(L)$ denote the volume-dependent momenta of the three pions. Following the work in [5, 6] the momenta of the three pions are determined by the 3-particle quantization conditions based on the scattering phase shift. These quantization conditions are valid in a nonrelativistic setup and under the assumption that only short-ranged 2-particle interactions are present, i.e., 3-particle interactions arise only from a sequence of subsequent 2-particle interactions. Of course, it is not clear to what extent these assumptions are fulfilled in the 2-flavour Schwinger model we consider here. In the center of mass frame, where $P = p_1 + p_2 + p_3 = 0$, the 3-particle quantization conditions read

$$\cot(\delta(-q_{31}) + \delta(q_{12})) + \cot\left(\frac{p_1 L}{2}\right) = 0, \quad \cot(\delta(-q_{23}) + \delta(q_{12})) - \cot\left(\frac{p_2 L}{2}\right) = 0, \quad (14)$$

with $q_{ij} = (p_i - p_j)/2$. These equations can now be solved using the previously determined scattering phase shift $\delta(k)$, yielding the momenta $p_i(L)$ and subsequently the 3-pion energy through the bosonic dispersion relation in eq. (13). In this way, we obtain predictions for the 3-pion ground-state energies and the corresponding relative finite-volume corrections based on the quantization conditions and the scattering phase shift. In figure 5 we present the results of this exercise together

with our direct determinations using the free energy differences. Shown are the relative finite-volume corrections. The comparison demonstrates nice agreement down to surprisingly small volumes.

6. Summary

In these proceedings, we reported some results concerning finite-volume effects and meson-scattering in the 2-flavour Schwinger model. Using the canonical formalism we determined the ground-state energies in the sectors with fixed fermion numbers and hence with fixed isospin. These energies are the energies of the corresponding multi-pion states. In this way, we extracted the pion mass $m_\pi(L)$, as well as the 2- and 3-pion ground-state energies $E_{2\pi}(L)$, $E_{3\pi}(L)$ as a function of the spatial volume L . The infinite-volume pion mass m_π and the 2-pion ground-state energy were then used to compute the scattering phase shift $\delta(k(L))$ for each volume. The momentum dependence of the phase shift can be well described in terms of a heuristic ansatz inspired by the Sine-Gordon model. Using the infinite-volume pion mass, the scattering phase shift and the 3-particle quantization conditions from [5, 6], the 3-pion ground-state energies can be predicted. The comparison between these predictions and our direct measurements shows very good agreement down to rather small volumes.

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