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Electroweak radiative corrections to the pion and kaon semi-leptonic decays from lattice QCD

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This work summarizes the lattice QCD calculations of the axial γW-box diagrams relevant for the semi-leptonic decays of pions and kaons. Using the method combining lattice data at small Q^2 and perturbative calculation at large Q^2, the γW-box contributions to the pion decays at the physical point and the kaon decays at the flavor SU(3) limit are calculated with the total uncertainty controlled at the level of $\sim 1\%$. These results could be used to determine the low energy constants for chiral perturbation theory with its uncertainty reduced significantly. This work could be generalized to the cases of the free and bound neutron decays, which plays an important role in the determination of the CKM matrix element $|V_{ud}|$, which is essential to the unitarity test of the CKM matrix.

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1. Introduction

The Cabibbo-Kobayashi-Maskawa (CKM) matrix is a 3×3 unitary matrix that describes the strength of the weak interactions of quarks. The test of the CKM unitarity plays a key role in the search of new physics beyond the Standard Model. As quoted in the 2021 review by the Particle Data Group[1], there exists a $2 \sim 3\sigma$ deviation from unitarity in the first row of CKM matrix elements,

$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 0.9984(6)_{V_{ud}}(4)_{V_{us}}. \tag{1}$$

Since $|V_{ub}|$ is negligible, $|V_{ud}|$ and $|V_{us}|$ dominate the uncertainty. $|V_{ud}|$ used in the unitarity test is from superallowed β decays. The uncertainty of V_{ud} becomes larger compared to the 2020 PDG review because of the update of the radiative correction (RC) term which depends on both nuclear structure (NS) and the energy of the electron. In the 2020 review, the deviation mentioned above was 3σ due to the smaller uncertainty of V_{ud}.

The reason for the deviation in PDG 2020 is from the analysis of the RCs. Previously, the RCs for the nuclear β decays is calculated by Marciano and Sirlin[2] with the vector meson dominance providing the information of the intermediate distances. To reduce the uncertainty of RCs, Chien-Yeah Seng et. al.[3] adopted a dispersive analysis and a $\sim 3\sigma$ deviation from CKM unitarity was found due to the update of $|V_{ud}|$.

In addition to the nuclear and nucleon beta decays, the semi-leptonic decays of mesons can also be used to determine the CKM matrix elements. For example, $|V_{ud}|$ can be extracted from pion decays while $|V_{us}|$ from kaon decays. The master formula of the decay rate of $\pi_{\ell 3}$ [4] and $K_{\ell 3}$ [1] reads

$$\Gamma_{\pi\ell 3} = \frac{G_F^2 m_\pi^5}{64\pi^3}(1+\delta_\pi)\,|V_{ud}|^2 \left|f_+^\pi(0)\right|^2 I_\pi, \tag{2}$$

$$\Gamma_{K\ell 3} = \frac{G_F^2 m_K^5}{192\pi^3}\,(1+\delta_K)\,|V_{us}|^2\,|f_+^K(0)|^2 I_K^\ell S_{\text{EW}} C^2. \tag{3}$$

where $G_F = 1.1663787(6) \times 10^{-5}\text{GeV}^{-2}$ is the Fermi's constant, m_H is the mass of the initial hadrons, $f_+^H(0)$ is the form factor at zero momentum, δ_H is the RCs, I_H is the phase-space integral $(H = \pi, K)$. S_{EW} is the short-distance electroweak factor, C is the Clebsch-Gordan coefficient with $C = 1$ for $K_{\ell 3}^0$ and $C = \frac{\sqrt{2}}{2}$ for $K_{\ell 3}^+$.

Since in the intermediate-distance region ($0.1\text{GeV}^2 \lesssim Q^2 \lesssim 1\text{GeV}^2$) the QCD contribution is nonperturbative, a first-principle lattice-QCD study of the γW-box diagrams is appealing. In the recent years, lattice QCD studies play an increasingly important role in high-precision flavor physics [5]. The research horizon has been extended to include the quantities which are very difficult to compute on the lattice. The examples involve long-distance contributions to rare kaon decays [6–11] and electromagnetic corrections to the leptonic and semi-leptonic decays [12–20]. In this proceeding, we focus on the lattice-QCD calculation to determine the γW-box corrections to semi-leptonic decays with controlled uncertainties[21, 22].

2. Theoritical Analysis

There are two representations of the analysis in the RCs. One is derived by Sirlin [23], the other is based on chiral perturbation theory (ChPT) [20]. Let's first introduce Sirlin's representation.

2.1 Sirlin's representation

Sirlin's representation is based on the current algebra, which describes the equal-time commutation relations of the current density operators. For the weak and the electromagnetic currents, it satisfied that

$$\begin{aligned}
\left[J^0_W(\vec{x},t), J^\mu_Z(\vec{y},t)\right] &= \cos^2\theta_W J^\mu_W(\vec{x},t)\delta^3(\vec{x}-\vec{y}),\\
\left[J^0_W(\vec{x},t), J^\mu_\gamma(\vec{y},t)\right] &= J^\mu_W(\vec{x},t)\delta^3(\vec{x}-\vec{y}),\\
\left[J^0_W(\vec{x},t), J^{\mu\dagger}_W(\vec{y},t)\right] &= -J^\mu_3(\vec{x},t)\delta^3(\vec{x}-\vec{y}) + \text{ S.T. },\\
J^\mu_3 \equiv \bar{\psi}_L\gamma^\mu C_3\psi_L &= 2\left(\sin^2\theta_W J^\mu_\gamma + J^\mu_Z\right),
\end{aligned} \tag{4}$$

where S.T. is a c-number called "Schwinger term".

Using current algebra, the RCs for the β decays could be parameterized as [23]

$$\delta = \frac{\alpha_e}{2\pi}\left[\bar{g} + 3\ln\frac{m_Z}{m_p} + \ln\frac{m_Z}{m_W} + \tilde{a}_g\right] + \delta^{\text{QED}}_{\text{HO}} + 2\Box^{VA}_{\gamma W}, \tag{5}$$

where $\Box^{VA}_{\gamma W}$ is the RC term associated with the axial part of the γW-box diagrams, which could be obtained with the master formula

$$\Box^{VA}_{\gamma W}\Big|_H = \frac{3\alpha_e}{2\pi}\int\frac{dQ^2}{Q^2}\frac{m_W^2}{m_W^2+Q^2}M_H\left(Q^2\right), \tag{6}$$

where $Q^2 = -q^2$ is the spacelike four-momentum square, M_H $(H = \pi, K)$ is the hadronic function defined by [21]

$$M_H\left(Q^2\right) = -\frac{1}{6}\frac{1}{F^H_+}\frac{\sqrt{Q^2}}{m_H}\int d^4x\,\omega(t,\vec{x})\epsilon_{\mu\nu\alpha 0}x_\alpha\mathcal{H}^{VA}_{\mu\nu}(t,\vec{x}), \tag{7}$$

$$\omega(t,\vec{x}) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\cos^3\theta d\theta}{\pi}\frac{j_1\left(\sqrt{Q^2}|\vec{x}|\cos\theta\right)}{|\vec{x}|}\cos\left(\sqrt{Q^2}t\sin\theta\right). \tag{8}$$

According to the current algebra analysis of Sirlin, for the process of the beta decays, if the hadrons of the initial and the final states have the same masses, the contribution for RCs from γW-box diagram is the only term sensitive to the non-perturbative hadronic effects, so it dominates the theoretical uncertainty.

For pion decays, this condition is satisfied naturally. However, for kaon decays, this condition couldn't be satisfied. To avoid the full calculation of RCs, including some contributions of five-point functions, we could combine the Sirlin's representation with the analysis of ChPT.

2.2 ChPT

ChPT is an effective field theory which involves the low-energy degrees of freedom like mesons, rather than quarks and gluons. To make the effective theory predictable, low energy constants (LECs) are needed. LECs associated with the semi-leptonic decays are X_1 and $\tilde{X}_6^{\text{phys}}$[20].

Previously, LECs of ChPT are obtained from some model-dependent analysis, like the minimal resonance model. Since it's hard to estimate the uncertainty, Ref.[24] assigned LECs with a uncertainty of 100%.

$$X_1 = -3.7(3.7)\times 10^{-3}, \quad \tilde{X}_6^{\text{phys}} = 10.4(10.4)\times 10^{-3}. \tag{9}$$

The relationship between RCs and LECs could be derived as

$$\delta^{\ell}_{K^\pm} = 2e^2\left[-\frac{8}{3}X_1 - \frac{1}{2}\tilde{X}_6^{\text{phys}}\ (M_\rho)\right] + \cdots, \tag{10}$$

$$\delta^{\ell}_{K^0} = 2e^2\left[\frac{4}{3}X_1 - \frac{1}{2}\tilde{X}_6^{\text{phys}}\ (M_\rho)\right] + \cdots, \tag{11}$$

$$\delta^{\ell}_{\pi^\pm} = 2e^2\left[-\frac{2}{3}X_1 - \frac{1}{2}\tilde{X}_6^{\text{phys}}\ (M_\rho)\right] + \cdots. \tag{12}$$

The neutral kaon decay mode is free from the $\pi^0 - \eta$ mixing, so it is chosen to extract LECs. Together with the process of π_{e3} decays, we can finally derive the LECs mentioned above.

The necessary condition to use Sirlin's representation is the hadrons H_i and H_f having nearly the same masses. So firstly, we can calculate the RCs in the flavor SU(3) limit, which satisfies $m_K = m_\pi$.

Obviously, the result from this unphysical setup couldn't give the physical result, however, LECs of the ChPT could be obtained. Since the LECs don't depend on the quark masses, one can calculate the physical RCs using ChPT. Inserting the result of γW-box contribution into the equation, the LECs are calculated with the uncertainty under control.

3. Lattice Setup

The information of the ensembles is shown in Table 1. We use five gauge ensemble with 2 + 1-flavor domain wall fermion. Each ensemble is set at the physical pion mass. Here 48I and 64I use the Iwasaki gauge action in the simulation (denoted as Iwasaki in this work) while the other three ensembles use Iwasaki+DSDR action (denoted as DSDR). The flavor SU(3) limit is achieved by taking the strange quark mass to be the same as the light quarks'.

Ensemble		m_π [MeV]	L	T	a^{-1} [GeV]
DSDR	24D	141.2(4)	24	64	1.015
	32D	141.4(3)	32	64	1.015
	32D-fine	143.0(3)	32	64	1.378
Iwasaki	48I	135.5(4)	48	96	1.730
	64I	135.3(2)	64	128	2.359

Table 1: Information of ensembles used in this work. For each ensemble we list the pion mass m_π, the spatial and temporal extents, L and T, and the inverse of lattice spacing a^{-1}.

4. Numerical Results

The result of this proceeding is published in Ref.[21, 22]. Here we just give some brief introduce. For more detailed results, you can search for the published works.

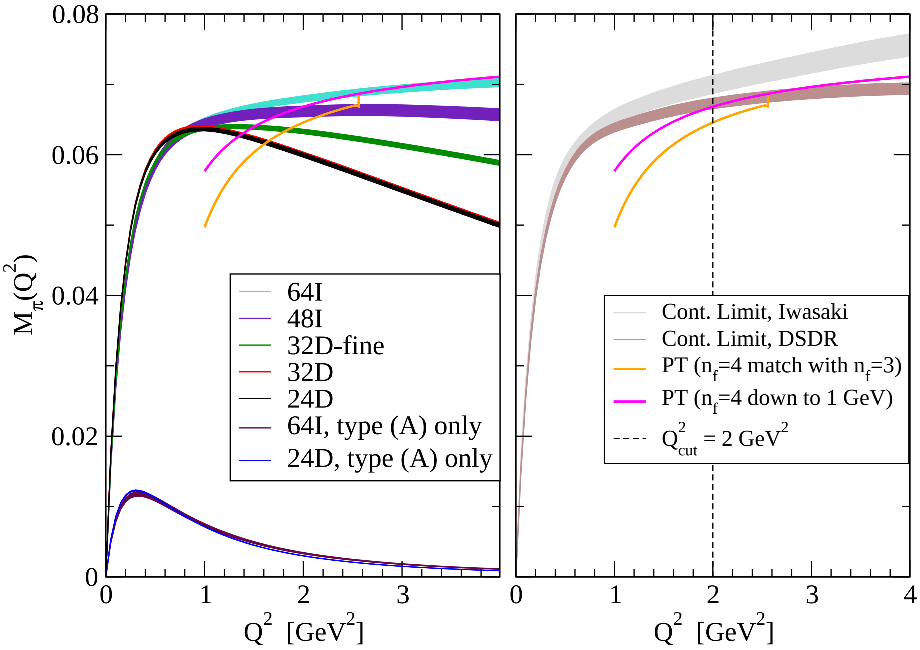

Figure 1: $M_\pi(Q^2)$ as a function of Q^2. The curves labeled by64I, 48I, 32D, 32D-fine and 24D are lattice results, while the curves labeled by PT are from the perturbation calculation. We also calculate the result from a particular type of Feymann diagram, denoted as type (A) here, to estimate the size of higher twist.

The lattice data and perturbative calculation for pion decays is shown in Fig.1. We can find that for small Q^2, lattice data are consistent with each other. When the Q^2 gets larger, the lattice discretization effects dominate the uncertainties and we should take perturbation calculation. So does the result in kaon decays. Choosing the $Q^2_{\text{cut}} = 2\text{GeV}^2$, the RCs of these two processes could be obtained as

$$\square^{VA}_{\gamma W}\Big|_{\pi} = 2.830(11)_{\text{stat}}(28)_{\text{syst}} \times 10^{-3} \tag{13}$$

$$\square^{VA}_{\gamma W}\Big|_{K^0,\text{SU}(3)} = 2.437(20)_{\text{stat}}(39)_{\text{syst}} \times 10^{-3} \tag{14}$$

On the other hand, the box terms can also be represented in the ChPT. By combining the Eq. (5) and Eq.(10) together with Eq.(12), the relations between the axial γW-box contribution and LECs are

$$-\frac{8}{3}X_1 + \bar{X}_6^{\text{phys}}(M_\rho) = -\frac{1}{2\pi\alpha}\left(\square^{VA}_{\gamma W}\Big|_{K^0,\text{SU}(3)} - \frac{\alpha}{8\pi}\ln\frac{M_W^2}{M_\rho^2}\right) + \frac{1}{8\pi^2}\left(\frac{5}{4} - \tilde{a}_g\right) \tag{15}$$

$$\frac{4}{3}X_1 + \bar{X}_6^{\text{phys}}(M_\rho) = -\frac{1}{2\pi\alpha}\left(\square^{VA}_{\gamma W}\Big|_{\pi} - \frac{\alpha}{8\pi}\ln\frac{M_W^2}{M_\rho^2}\right) + \frac{1}{8\pi^2}\left(\frac{5}{4} - \tilde{a}_g\right) \tag{16}$$

where $\bar{X}_6^{\text{phys}}$ is equivalent to $\tilde{X}_6^{\text{phys}}$ up to the pQCD corrections,

$$\bar{X}_6^{\text{phys}}(M_\rho) \equiv \tilde{X}_6^{\text{phys}}(M_\rho) + \left(X_6^{\text{phys}}\right)_{\alpha_s} \tag{17}$$

For the π_{e3} decays, higher-order QED effects $\delta_{\text{HO}}^{\text{QED}}$ and γW-box contribution $\square_{\gamma W}^{VA}$ dominate the uncertainty of δ. The uncertainty of axial γW-box contribution used to be estimated with the LECs of ChPT before. With $\square_{\gamma W}^{VA}$ obtained from lattice method, it's possible to almost completely remove the dominant uncertainty from the LECs,

$$\delta = 0.0334(10)_{\text{LEC}}(3)_{\text{HO}} \to 0.0332(1)_{\gamma W}(3)_{\text{HO}} \tag{18}$$

Using the new RCs, the $|V_{ud}|$ obtained from the pion decays could updated as

$$|V_{ud}| = 0.9740(28)_{\text{exp}}(1)_{\text{th}}, \tag{19}$$

which leads to a reduction of the total uncertainty by a factor of 3.

For the physical $K_{\ell 3}$ decays, since $m_K \neq m_\pi$, Sirlin's representation couldn't be used directly, contributions of other diagrams should be considered in the analysis of RCs of $K_{\ell 3}$ decays, like the contribution of the five-point correlation function. To avoid this difficulty, a calculation of the γW-box corrections in the flavor SU(3) limit could be taken firstly, in which the strange quark mass is tuned down to be the same as the light quark mass, so that $m_K = m_\pi$.

By using this unphysical setup, combined with the results of the π_{e3} decays, LECs of the ChPT could be obtained. Then, by using ChPT, one can calculate the physical RCs. Inserting the result of γW-box contribution into the equation, the LECs are calculated with the uncertainty under control.

$$X_1 = -2.2(4) \times 10^{-3}, \quad \bar{X}_6^{\text{phys}} = 16.9(7) \times 10^{-3} \tag{20}$$

Putting the new LECs into Eq. (10) and Eq.(11), the physical RCs of $K_{\ell 3}$ decays is updated as (in units of %)

$$\begin{aligned}
\delta_{K^0}^{e} &= 0.99(19)_{e^2p^4}(11)_{\text{LEC}} && \to && 1.00(19), \\
\delta_{K^0}^{\mu} &= 1.40(19)_{e^2p^4}(11)_{\text{LEC}} && \to && 1.41(19), \\
\delta_{K^\pm}^{e} &= 0.10(19)_{e^2p^4}(16)_{\text{LEC}} && \to && -0.01(19), \\
\delta_{K^\pm}^{\mu} &= 0.02(19)_{e^2p^4}(16)_{\text{LEC}} && \to && -0.09(19).
\end{aligned} \tag{21}$$

where the uncertainty from the LECs are almost negligible now. The dominate uncertainty arises from higher order terms in ChPT expansion, denoted as (e^2p^4) here.

5. Conclusions

The axial γW-box contribution $\square_{\gamma W}^{VA}$ is the only term sensitive to hadronic scales in RCs, so its determination requires non-perturbative treatment. In this work, we perform the lattice QCD calculation of the $\square_{\gamma W}^{VA}$ in both pion and kaon semi-leptonic decays.

For the π_{e3} decays, we can adpot Sirlin's representation in Eq.(5) and get the RCs δ, which leads to an uncertainty reduction of the theoretical prediction of $|V_{ud}|$ from pion semi-leptonic decays by a factor of 3. The uncertainty of the experimental results for the pion decays is still so large that this result cannot test the CKM unitarity. However, this technique could be generalized to the calculation of the nucleon decays, like free and bound neutron decays, which can provide more precise determination of $|V_{ud}|$ and test the CKM unitarity.

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For the $K_{\ell 3}$ decays, by matching the results from Sirlin's representation in flavor SU(3) limit and ChPT, we can obtain the LECs with segnificantly reduced error, which further gives the physical RCs. The uncertainty from LECs is almost removed, and now the residual uncertainty is dominated by higher terms in the power counting of the ChPT.

References

[1] Particle Data Group collaboration, *Review of Particle Physics*, *PTEP* **2020** (2020) 083C01.

[2] W.J. Marciano and A. Sirlin, *Improved calculation of electroweak radiative corrections and the value of V(ud)*, *Phys. Rev. Lett.* **96** (2006) 032002 [hep-ph/0510099].

[3] C.-Y. Seng, M. Gorchtein, H.H. Patel and M.J. Ramsey-Musolf, *Reduced Hadronic Uncertainty in the Determination of* V_{ud}, *Phys. Rev. Lett.* **121** (2018) 241804 [1807.10197].

[4] D. Pocanic et al., *Precise measurement of the pi+ —> pi0 e+ nu branching ratio*, *Phys. Rev. Lett.* **93** (2004) 181803 [hep-ex/0312030].

[5] Y. Aoki et al., *FLAG Review 2021*, 2111.09849.

[6] RBC, UKQCD collaboration, *Prospects for a lattice computation of rare kaon decay amplitudes II* $K \to \pi\nu\bar{\nu}$ *decays*, *Phys. Rev. D* **93** (2016) 114517 [1605.04442].

[7] Z. Bai, N.H. Christ, X. Feng, A. Lawson, A. Portelli and C.T. Sachrajda, *Exploratory Lattice QCD Study of the Rare Kaon Decay* $K^+ \to \pi^+\nu\bar{\nu}$, *Phys. Rev. Lett.* **118** (2017) 252001 [1701.02858].

[8] Z. Bai, N.H. Christ, X. Feng, A. Lawson, A. Portelli and C.T. Sachrajda, $K^+ \to \pi^+\nu\bar{\nu}$ *decay amplitude from lattice QCD*, *Phys. Rev. D* **98** (2018) 074509 [1806.11520].

[9] RBC, UKQCD collaboration, *Lattice QCD study of the rare kaon decay* $K^+ \to \pi^+\nu\bar{\nu}$ *at a near-physical pion mass*, *Phys. Rev. D* **100** (2019) 114506 [1910.10644].

[10] RBC, UKQCD collaboration, *Prospects for a lattice computation of rare kaon decay amplitudes:* $K \to \pi\ell^+\ell^-$ *decays*, *Phys. Rev. D* **92** (2015) 094512 [1507.03094].

[11] N.H. Christ, X. Feng, A. Juttner, A. Lawson, A. Portelli and C.T. Sachrajda, *First exploratory calculation of the long-distance contributions to the rare kaon decays* $K \to \pi\ell^+\ell^-$, *Phys. Rev. D* **94** (2016) 114516 [1608.07585].

[12] N. Carrasco, V. Lubicz, G. Martinelli, C.T. Sachrajda, N. Tantalo, C. Tarantino et al., *QED Corrections to Hadronic Processes in Lattice QCD*, *Phys. Rev.* **D91** (2015) 074506 [1502.00257].

[13] V. Lubicz, G. Martinelli, C.T. Sachrajda, F. Sanfilippo, S. Simula and N. Tantalo, *Finite-Volume QED Corrections to Decay Amplitudes in Lattice QCD*, *Phys. Rev.* **D95** (2017) 034504 [1611.08497].

[14] D. Giusti, V. Lubicz, G. Martinelli, C.T. Sachrajda, F. Sanfilippo, S. Simula et al., *First lattice calculation of the QED corrections to leptonic decay rates*, *Phys. Rev. Lett.* **120** (2018) 072001 [1711.06537].

[15] N.H. Christ, X. Feng, J. Lu-Chang and C.T. Sachrajda, *Electromagnetic corrections to leptonic pion decay from lattice QCD using infinite-volume reconstruction method*, *PoS* **LATTICE2019** (2020) 259.

[16] C.-Y. Seng, X. Feng, M. Gorchtein and L.-C. Jin, *Joint lattice QCD–dispersion theory analysis confirms the quark-mixing top-row unitarity deficit*, *Phys. Rev. D* **101** (2020) 111301 [2003.11264].

[17] A. Desiderio et al., *First lattice calculation of radiative leptonic decay rates of pseudoscalar mesons*, *Phys. Rev. D* **103** (2021) 014502 [2006.05358].

[18] R. Frezzotti, M. Garofalo, V. Lubicz, G. Martinelli, C.T. Sachrajda, F. Sanfilippo et al., *Comparison of lattice QCD+QED predictions for radiative leptonic decays of light mesons with experimental data*, 2012.02120.

[19] X.-Y. Tuo, X. Feng, L.-C. Jin and T. Wang, *Lattice QCD calculation of $K \to \ell\nu_\ell\ell'^+\ell'^-$ decay width*, 2103.11331.

[20] C.-Y. Seng, X. Feng, M. Gorchtein, L.-C. Jin and U.-G. Meißner, *New method for calculating electromagnetic effects in semileptonic beta-decays of mesons*, *JHEP* **10** (2020) 179 [2009.00459].

[21] X. Feng, M. Gorchtein, L.-C. Jin, P.-X. Ma and C.-Y. Seng, *First-principles calculation of electroweak box diagrams from lattice QCD*, *Phys. Rev. Lett.* **124** (2020) 192002 [2003.09798].

[22] P.-X. Ma, X. Feng, M. Gorchtein, L.-C. Jin and C.-Y. Seng, *Lattice QCD calculation of the electroweak box diagrams for the kaon semileptonic decays*, *Phys. Rev. D* **103** (2021) 114503 [2102.12048].

[23] A. Sirlin, *Current Algebra Formulation of Radiative Corrections in Gauge Theories and the Universality of the Weak Interactions*, *Rev. Mod. Phys.* **50** (1978) 573.

[24] V. Cirigliano, M. Giannotti and H. Neufeld, *Electromagnetic effects in K(l3) decays*, *JHEP* **11** (2008) 006 [0807.4507].