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Critical Behaviour in the Single Flavor Planar Thirring Model

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We report results of simulations of the $2+1d$ Thirring model with N fermion flavors, defined on a lattice using domain wall fermions. This approach is devised to respect as far as possible the underlying $\mathrm{U}(2N)$ symmetry of the continuum model, expected to be recovered in the limit wall separation $L_s \to \infty$. For $N = 1$ there is a symmetry-breaking phase transition associated with bilinear condensation at strong fermion self-interaction, which is a plausible location for a quantum critical point. Fits to a renormalisation group-inspired equation of state yield critical exponents distinct from those obtained using a version of the model defined using staggered fermions.

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1. Introduction

This presentation concerns the Thirring model in $d = 2 + 1$ spacetime dimensions. The model is a covariant quantum field theory of self-interacting fermions described by a Lagrangian density

$$\mathcal{L} = \bar{\psi}_i(\partial\!\!\!/ + m)\psi_i + \frac{g^2}{2N}(\bar{\psi}_i\gamma_\mu\psi_i)^2. \tag{1}$$

The index $\mu = 0, 1, 2$ runs over spacetime dimensions and $i = 1, \ldots N$ over autonomous fermion flavors. A reducible spinor representation is chosen so that the γ-matrices are 4×4, implying the existence of *two* further matrices γ_3, γ_5 which anticommute with the kinetic operator in (1). The interaction, with coupling strength g^2 of dimension -1, is a contact term between two conserved currents, with the consequence that like charges repel, opposite charges attract.

Most applications of this and other theories of “Flatland” fermions occur in the condensed matter physics of various layered systems: nodal fermions in d-wave superconductors; spin liquid phases in Heisenberg antiferromagnets; surface states of topological insulators; and of course graphene, where low-energy electronic excitations exhibit linear dispersion around two inequivalent *Dirac points* within the first Brillouin Zone. The Thirring interaction shares the same symmetries as the electrostatic interaction between electrons and holes, which is unscreened at half-filling. For sufficiently large g^2, and/or sufficiently small N, it is speculated that the Fock vacuum is unstable with respect to formation of a particle-hole bilinear condensate

$$\langle\bar{\psi}\psi\rangle = \frac{1}{V}\frac{\partial \ln \mathcal{Z}}{\partial m} \neq 0, \tag{2}$$

with clear congruences to chiral symmetry breaking in QCD, resulting in a dynamically-generated mass gap at Dirac point. By hypothesis, the resulting semimetal-insulator transition occuring at $g_c^2(N)$ defines a *Quantum Critical Point*, whose universal properties characterise the low energy excitation spectrum [1]. If a correlation length diverges here, a new strongly-interacting quantum field theory may be defined.

2. Symmetries

On general grounds we expect the QCP to be characterised by dimensionality, the nature of the degrees of freedom, and the pattern of symmetry breaking. For $m = 0$ the Lagrangian (1) has the following invariances:

$$\psi \mapsto e^{i\alpha}\psi, \;\; \bar{\psi} \mapsto \bar{\psi}e^{-i\alpha}; \quad \psi \mapsto e^{\alpha\gamma_3\gamma_5}\psi, \;\; \bar{\psi} \mapsto \bar{\psi}e^{-\alpha\gamma_3\gamma_5}; \tag{3}$$

$$\psi \mapsto e^{i\alpha\gamma_3}\psi, \;\; \bar{\psi} \mapsto \bar{\psi}e^{i\alpha\gamma_3}; \quad \psi \mapsto e^{i\alpha\gamma_5}\psi, \;\; \bar{\psi} \mapsto \bar{\psi}e^{i\alpha\gamma_5}. \tag{4}$$

Together these rotations generate U(2). However, only (3) remains an invariance once $m, \langle\bar{\psi}\psi\rangle \neq 0$. Bilinear condensation therefore results in a symmetry breaking U($2N$) →U(N)⊗U(N). Note the mass term $m\bar{\psi}\psi$ is hermitian and invariant under a parity inversion $x_\mu \mapsto -x_\mu$. In fact, there are two other parity-invariant mass terms related by U($2N$) which are antihermitian in Euclidean metric:

$$im_3\bar{\psi}\gamma_3\psi; \quad im_5\bar{\psi}\gamma_5\psi. \tag{5}$$

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The "Haldane" mass term $m_{35}\bar{\psi}\gamma_3\gamma_5\psi$ is not parity-invariant, and hence physically inequivalent.

Why is it so important to capture the global symmetries faithfully? The following argument is very far from rigorous, but illustrates the point. In the limit of large N, the Thirring model (1) is amenable to a diagrammatic analysis. The interaction at strong coupling between conserved fermion currents is mediated by a propagating vector boson, (actually a fermion – antifermion bound state) with mass M_V given by [2]

$$\frac{M_V}{m} = \sqrt{\frac{6\pi}{mg^2}}, \tag{6}$$

whose masslessness in the limit $g^2 \to \infty$ suggests equivalence to QED$_3$, an asymptotically-free theory long believed to support a conformal IR fixed point. Since the dimensionless interaction strength at this fixed point scales $\propto N^{-1}$, there should be a critical N_c below which, again, the ground state is unstable with respect to bilinear $\bar{\psi}\psi$ condensation. Since the UV limit of the Thirring model and the IR limit of QED$_3$ have the same vector propagator, it is conceivable the fixed points in these limits coincide, so that the two model share the same N_c. For an asymptotically-free theory like QED$_3$ there is an argument to constrain N_c based on counting degrees of freedom in both symmetric and broken phases [3]:

$$\#\,\text{Goldstone bosons in IR} = 2N^2 \le \frac{3}{4} \times \#\,\text{fermion degrees of freedom in UV} = 3N, \tag{7}$$

where the factor $\frac{3}{4}$ reproduces the Fermi-Dirac distribution correctly in 3 dimensions. Saturating the inequality yields $N_c \le \frac{3}{2}$ [1]. Now, many early attempts to identify N_c with lattice field theory used the staggered fermion formulation. Away from weak couplng staggered fermions support a different symmetry-breaking U(N)⊗U(N) →U(N), yielding a modified counting:

$$N^2 \le \frac{3}{4} \times 2^d \times N \Rightarrow N_c^{\rm stagg} \le 6, \tag{8}$$

In fact, numerical simulations with staggered fermions find $N_c^{\rm stagg} = 3.4(1)$ [5].

3. Domain Wall Fermions

The arguments of the previous section highlight the importance of choosing a lattice regularisation which respects the global symmetries of the target theory as much as possible. We have chosen domain wall fermions (DWF) formulated in 2+1+1d, which recover U($2N$) symmetry in the limit that the separation L_s of two domain walls along a fictitious third dimension x_3 grows large – though it is not clear *a priori* how "large" that means. Fermion fields in the 2+1d target space are supposedly localised on the walls and are identified in terms of 2+1+1d fields $\Psi, \bar{\Psi}$ via

$$\psi(x) = P_-\Psi(x,1) + P_+\Psi(x,L_s); \quad \bar{\psi}(x) = \bar{\Psi}(x,L_s)P_- + \bar{\Psi}(x,1)P_+. \tag{9}$$

with projectors $P_\pm = \frac{1}{2}(1 \pm \gamma_3)$.

The basic setup was introduced in the context of quenched QED$_3$ in Ref. [6], where it was noted that U($2N$) restoration occurs most rapidly for condensates corresponding to the mass terms

[1] A treatment of QED$_3$ based on an F-theorem which does not assume free-field dynamics predicts $N_c < 4.4$ [4]

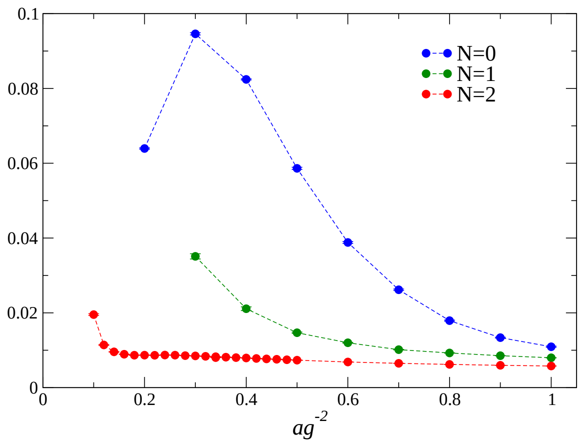

Figure 1: Bilinear condensate $i\langle\bar{\psi}\gamma_3\psi\rangle$ vs. ag^{-2} on $12^3 \times 16$ ($16^3 \times 16$ for $N = 0$) with $ma = 0.01$.

(5). Henceforth we quote results for $\langle i\bar{\psi}\gamma_3\psi\rangle$, which is formed from 2+1+1d propagators connected to opposite walls. Further details of both the lattice formulation of the Thirring model used in this study (the so-called *bulk* variant) and the simulation algorithm can be found in [7].

4. Results

Fig. 1 presents results for the bilinear condensate as a function of dimensionless inverse coupling ag^{-2} for exploratory runs with $L_s = 16$ [7], which enable a comparison between quenched $N = 0$, $N = 1$ obtained using an RHMC algorithm, and $N = 2$ using a much cheaper HMC algorithm. As might be expected there is a clear hierarchy of condensation scales as the coupling gets stronger, confirming our suspicion that this system is very sensitive to N. However, the U($2N$)-symmetric limit first requires $L_s \to \infty$. For $N = 1$ we have performed systematic studies of system sizes $12^3, 16^3 \times L_s$ with $L_s = 8, 16, \ldots, 48$ [8] and are currently accumulating data on 16^3 with $L_s = 64, 80$. Empirically the extrapolation is well-described by

$$\langle\bar{\psi}\psi\rangle_\infty - \langle\bar{\psi}\psi\rangle_{L_s} = A(g^2, m)e^{-\Delta(g^2,m)L_s}, \tag{10}$$

as shown across a range of couplings in Fig. 2 using our latest data. The extrapolation (10) is particularly important at larger couplings, where fortunately it is also easier to fit (difficulties fitting (10) at weaker couplings are reflected in the large error bars in Fig. 4 below). The decay constant Δ is shown in Fig. 3. For weak couplings Δ is approximately m-independent but once $ag^{-2} \leq 0.36$ the behaviour alters to $\Delta \propto m$. Fig. 3 graphically illustrates the challenge of finding the U(2) limit in the strong coupling and massless limits; we are truly stress-testing DWF. Further aspects of the approach to U(2) symmetry as $L_s \to \infty$ are discussed in [8].

In order to identify a possible QCP, we need to identify a critical coupling g_c^{-2} such that in the limit $m \to 0$ U(2) symmetry spontaneously breaks for $g^{-2} < g_c^{-2}$. Since direct estimates of the $\langle\bar{\psi}\psi\rangle$ order parameter are not available for $m = 0$, our strategy is to accumulate data for

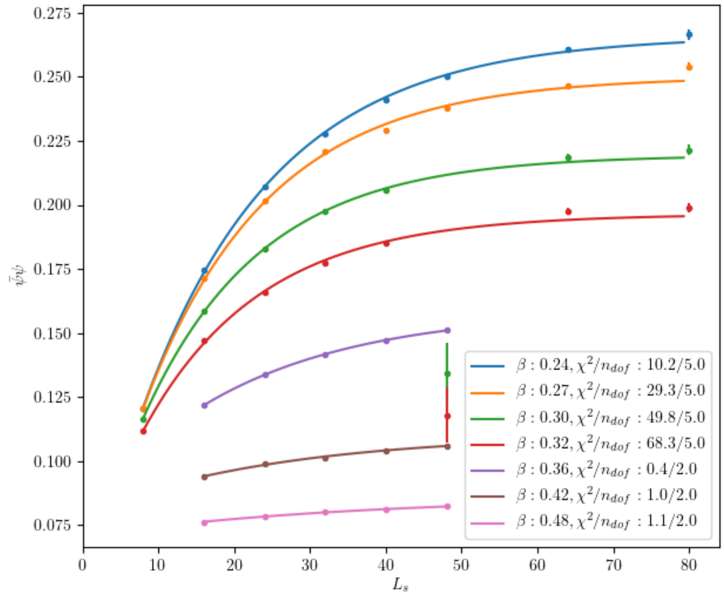

Figure 2: Bilinear condensate $i\langle\bar\psi\gamma_3\psi\rangle$ on $16^3 \times L_s$ with $ma = 0.05$. $\beta \equiv ag^{-2}$.

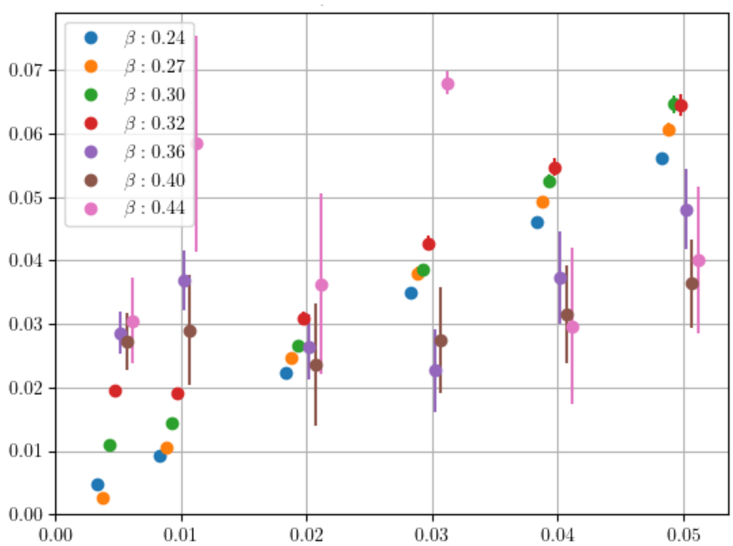

Figure 3: The decay constant $\Delta(g^2, m)$.

$ma = 0.005, 0.01, 0.02, \ldots, 0.05$ and fit the whole set to a renormalisation group-inspired equation of state [9]

$$m = A(g^{-2} - g_c^{-2})\langle\bar\psi\psi\rangle^{\delta-1/\beta} + B\langle\bar\psi\psi\rangle^{\delta}. \tag{11}$$

The latest simulations are aimed at straddling the critical coupling identified in [8] with wall separations up to and including $L_s = 80$. Our fit to the order parameter data extrapolated to $L_s \to \infty$ is shown in Fig. 4, along with the fitted curve in the $m \to 0$ limit. The critical parameters

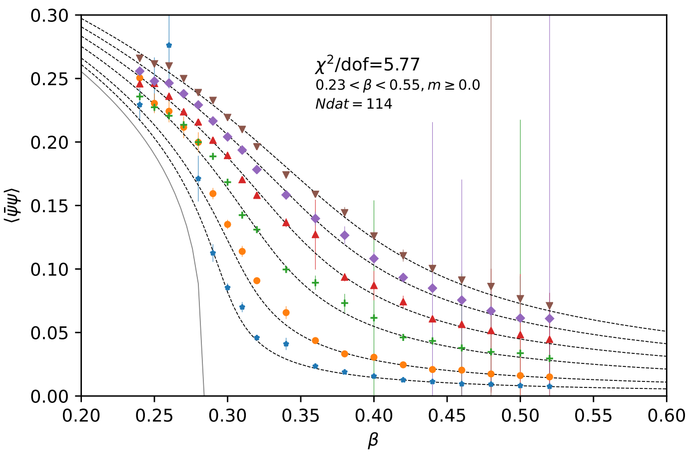

Figure 4: Equation of state fit for $N = 1$ on 16^3 in the $L_s \to \infty$ limit.

are

$$\beta_c \equiv ag_c^{-2} \quad = \quad 0.283(1); \tag{12}$$

$$\delta \quad = \quad 4.17(5); \qquad \beta = 0.320(5), \tag{13}$$

compatible with the fit based on data taken exclusively in the symmetric phase reported in [8]. Using hyperscaling the critical exponents (13) can be related to those more usually extracted from orthodox finite-volume scaling studies:

$$\nu = 0.55(1); \qquad \eta = 0.16(1). \tag{14}$$

5. Discussion

The success of the fit to (11) is strong evidence for spontaneous U(2) symmetry breaking and the existence of a QCP for the $N = 1$ Thirring model. Absence of symmetry breaking for $N = 2$ [7] as illustrated in Fig. 1 leads to the conclusion

$$1 < N_c < 2. \tag{15}$$

By way of contrast, investigations of the Thirring model using $N = 1$ staggered fermion flavors [9] reveal distinct critical exponents (see also [10]):

$$\delta = 2.75(9); \quad \beta = 0.57(2); \quad \nu = 0.71(3); \quad \eta = 0.60(4), \tag{16}$$

and as already observed in Sec. 2, the critical $3 < N_c^{\rm stagg} < 4$ for staggered fermions is considerably larger. It seems plausible that the phase transitions probed by numerical simulations of DWF and staggered fermions lie in different universality classes, and define distinct QCPs. This is perfectly consistent with the remarks made in Sec. 2. It is possible to formulate a Thirring model with U(N)⊗U(N) symmetry using Kähler-Dirac fermions in which spinor and taste degrees of freedom are entangled, which may correspond to the continuum limit of the staggered model [11] – there is no reason to expect "taste symmetry restoration" away from weak coupling.

Before concluding, we should note that other non-perturbative lattice studies with explicit U($2N$) symmetry have been performed using fermions with a kinetic term employing the SLAC derivative [12]. The results differ significantly, in particular $N_c^{\rm SLAC} \approx 0.8$ implying there is no QCP corresponding to a local unitary quantum field theory. Clearly the question of defining suitable regularisations of QFTs away from weak coupling is delicate: different schemes may fall in the basin of attraction of different fixed points.

In future work we plan to switch attention to two-point correlation functions, focussing on both Goldstone and non-Goldstone bound states, as well as the fermion propagator iteslf, which in principle enables the extraction of a further exponent η_ψ.

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