

Funny business from the large N_c finite temperature crossover

Thomas DeGrand^{a,*}

^a*Department of Physics, University of Colorado,
Boulder, CO 80309, USA*

E-mail: thomas.degrand@colorado.edu

It is well known that the deconfinement transition temperature for $SU(N_c)$ gauge theory is almost independent of N_c , and the transition is first order for $N_c \geq 3$. In the real world ($N_c = 3$, light quarks) it is a crossover located far away from the pure gauge value. What happens to the transition temperature at fixed fermion mass if the number of fermion flavors is held constant ($N_f = 2$) and N_c is varied? There are multiple plausible stories, only one of which appears to be true when the systems are simulated on the lattice. I describe the physics issues which surround the question and my lattice - based answer to it.

*The 38th International Symposium on Lattice Field Theory, LATTICE2021 26th-30th July, 2021
Zoom/Gather@Massachusetts Institute of Technology*

*Speaker

The limit of QCD when the number of colors N_c becomes large is a theorist's playground for studying QCD [1–3]. It has a small lattice literature, checking its predictions for nonperturbative quantities such as masses and matrix elements. (See Refs. [4–6] for reviews.) In most cases with a lattice study, the lattice confirms its simple large N_c prediction.

Why study large N_c QCD with lattice methods? It's well known that QCD simplifies at large N_c (it basically becomes a theory of open strings). QCD also idealizes at large N_c : mesons are $q\bar{q}$ bound states, baryons are bound states of N_c quarks, and hadronic wave functions are presumed to become independent of N_c as shown by the N_c scaling of matrix elements. Most large N_c predictions are of nonperturbative quantities (even though they are often based on color counting for Feynman diagrams) and these predictions ought to be subjected to nonperturbative tests.

Large N_c lattice calculations are pretty straightforward: Simulate across N_c at fixed bare 't Hooft coupling $\lambda = g^2 N_c$ or $\beta \propto N_c^2$. Discover that this fixes the lattice spacing to be nearly equal across N_c . Measure the same observables across N_c . Scale the observable appropriately (for example, decay constants scale as $\sqrt{N_c}$) and observe curve collapse (again for example $f_{PS}/\sqrt{N_c}$ versus quark mass).

(Disclaimer: often people use some gluonic observable like the string tension or flow parameter to match lattice spacings, rather than the bare coupling. The results are the same up to $1/N_c$ corrections, which can be pushed from place to place by making different scale setting choices.)

Usually, there is a simple story for any observable at large N_c . But there is one observable which lacks a simple story – or – said better – there are at least three simple stories. That is the finite temperature crossover for QCD with a fixed number of flavors, as a function of the quark mass and N_c .

The first possibility comes from the naive large N_c expectation that gluonic degrees of freedom dominate fermionic ones in the large N_c limit. The pure gauge transition is first order for $N_c \geq 3$ and the transition temperature is nearly independent of N_c [7, 8]. As the fermion mass falls from infinity the transition becomes a crossover. Shouldn't the transition remain first order and at the same value as the pure gauge transition, with the end point pushing to ever smaller fermion mass as N_c rises?

The second scenario assumes the physics of the transition is dominated by naive chiral symmetry breaking. QCD at all N_c has an $SU(N_f) \times SU(N_f)$ symmetry which undergoes spontaneous symmetry breaking to $SU(N_f)$. The Pisarski - Wilczek analysis [9] approximates the Goldstone sector as a linear sigma model. For $N_f = 2$ the system is expected to have a second order transition at zero fermion mass, with $O(4)$ critical exponents. Second order transitions are unstable under perturbation, so the transition becomes a crossover away from $m_q = 0$. This should be true for any N_c .

The issue then is, how does the crossover temperature scale with N_c ? Linear sigma models contain one dimensionful parameter, the vacuum expectation value of the scalar field, and all dimensionful observables (the pseudoscalar decay constant f_{PS} , and the crossover temperature T_c itself) are proportional to it. We already know that $f_{PS} \propto \sqrt{N_c}$. Thus the naive prediction of the second scenario is $T_c \propto \sqrt{N_c}$ [10].

The third scenario is older than QCD. Confining theories are expected to show an exponentially growing spectrum of resonances with mass, forming a Hagedorn spectrum [11]. The tower of resonances implies a limiting temperature T_0 and this implies a crossover temperature $T_c \sim T_0$ [12].

In the real world, the Hagedorn temperature is about 160 MeV. The extension of the story for large N_c and nonzero N_f is that the spectrum of meson resonances is basically identical across N_c . If two theories have the same spectrum, then they ought to have the same critical properties. So the third prediction is that any $N_c \neq 3$ with $N_f = 2$ will qualitatively resemble $N_c = 3$, $N_f = 2$. (Since large N_c with nonzero N_f is different from quenched QCD, which only has glueballs, its Hagedorn temperature is different.)

This year I finished a little large N_c study of $N_f = 2$ QCD with $N_c = 3, 4$, and 5 with medium heavy fermions, $(m_{PS}/m_V)^2 \sim 0.25 - 0.65$ [13]. The answer I found is that the third scenario is most correct: the three systems show nearly identical crossover behavior as a function of temperature.

All the technical details are in the paper, but in a few words, this is what I did. I used clover fermions. The chiral condensate for Wilson type fermions has a set of divergent $1/a^n$ pieces, but the temperature dependent condensate is well defined [14, 15]

$$\langle \psi \psi \rangle_{sub} = \langle \psi \psi \rangle_T - \langle \psi \psi \rangle_{T=0}. \quad (1)$$

I measured

$$\frac{3}{N_c} t_0^{3/2} \Sigma(T) = \frac{3}{N_c} t_0^{3/2} \times m_q (\Delta_{PP}(T) - \Delta_{PP}(T=0)) \quad (2)$$

where (explicitly showing the conversion from the lattice quantity computed from clover fermions to a continuum one)

$$\Delta_{PP}(T) = \hat{\Delta}_{PP}(N_t) \left(1 - \frac{3\kappa}{4\kappa_c}\right)^2. \quad (3)$$

The lattice observable is

$$\hat{\Delta}_{PP}(N_t) = \sum_{t=0}^{N_t} \sum_x \langle P(x,t) P(0,0) \rangle \quad (4)$$

where $P(x,t) = \psi(x,t) \gamma_5 \psi(x,t)$ is the pseudoscalar current. The factor of $t_0^{3/2}$ in Eq. 2 makes the observable dimensionless and the overall factor of $3/N_c$ is included to show collapse to a common curve across N_c when the condensate scales proportional to N_c as expected by large N_c counting.

Two figures from the paper show the results: the condensate itself in Fig. 1 and $\Delta\Sigma(T)/\Delta T$ in Fig. 2.

How dull can things be? There is a smooth crossover showing chiral restoration at the same temperature across N_c . The derivative in Fig. 2 was my attempt to see something with a peak. There is a peak but it is very broad. The location of the crossover is intermediate between the pure gauge result of Refs. [7, 8] and $SU(3)$ results at the physical point. There is no sign of anything first order.

So what can we make of this? There is no first order transition. Apparently, fermions are still sometimes important degrees of freedom at large N_c .

What about Pisarski - Wilczek? It's well known that pions in QCD are NOT described by a linear sigma model (see Gasser and Leutwyler, Ref. [16]). The linear sigma model is really only used to make statements about the critical properties of a system. This does not include the critical temperature, which is not universal. Only critical exponents are universal.

What about a connection with the spectrum? It's not possible to test for a Hagedorn spectrum directly (yet) for any N_c . (Perhaps one should be more modern and talk about the hadron resonance

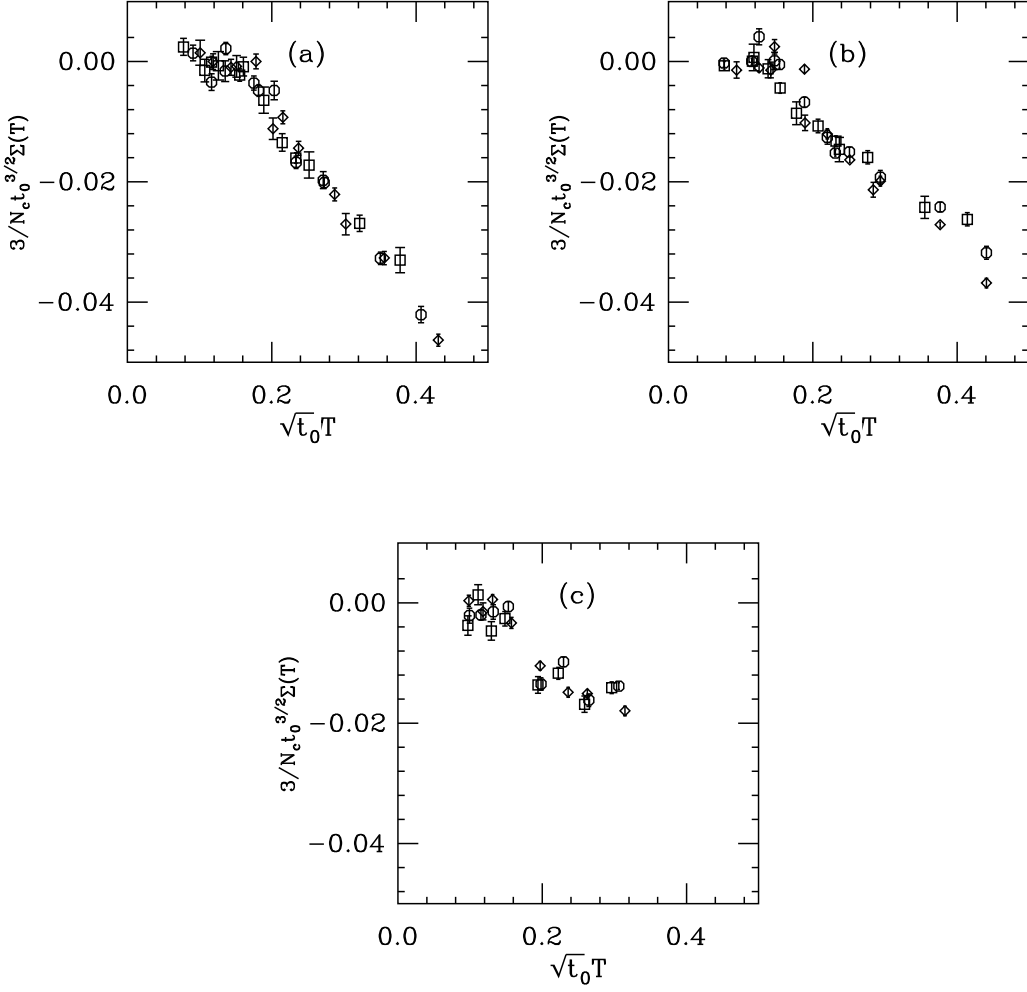


Figure 1: The temperature dependent condensate, rescaled by $3/N_c$, as a function of temperature, in appropriate units of t_0 . Squares, octagons, and diamonds label $N_c = 3, 4,$ and 5 . (a) $(m_{PS}/m_V)^2 \sim 0.63$; (b) $(m_{PS}/m_V)^2 \sim 0.5$; (c) $(m_{PS}/m_V)^2 \sim 0.25$.

gas model – a sum over all the resonances in the Review of Particle Properties.) But basically all mesons created with an interpolating field $\psi\Gamma\psi$ mesons (S-wave and P-wave mesons) are known to have an N_c –independent spectrum. Furthermore, the scaling of the pseudoscalar decay constant $f_{PS} \propto \sqrt{N_c}$ means that the amplitude for pion scattering $A(\pi\pi \rightarrow \pi\pi) \propto 1/N_c$: pions (like other hadrons) don’t interact at large N_c . In addition, large N_c scaling says that the vector meson decay constant $f_V \sim \langle \gamma | \rho \rangle \propto \sqrt{N_c}$ and vector dominance says $g_{\rho\pi\pi} \propto 1/\langle \gamma | \rho \rangle \propto 1/\sqrt{N_c}$. (For a textbook discussion, see Ref. [17], being careful with conventions defining f_V .) Again, hadrons don’t interact at large N_c . So what else is left but the density of states?

There is a lattice test one could do: compare the trace anomaly $(\epsilon - 3P)/T^4$ for $T < T_c$ (ϵ is the energy density, P is the pressure) to a hadron resonance gas model, or just compare one N_c to another. Results from all N_c ’s should match.

Let’s summarize:

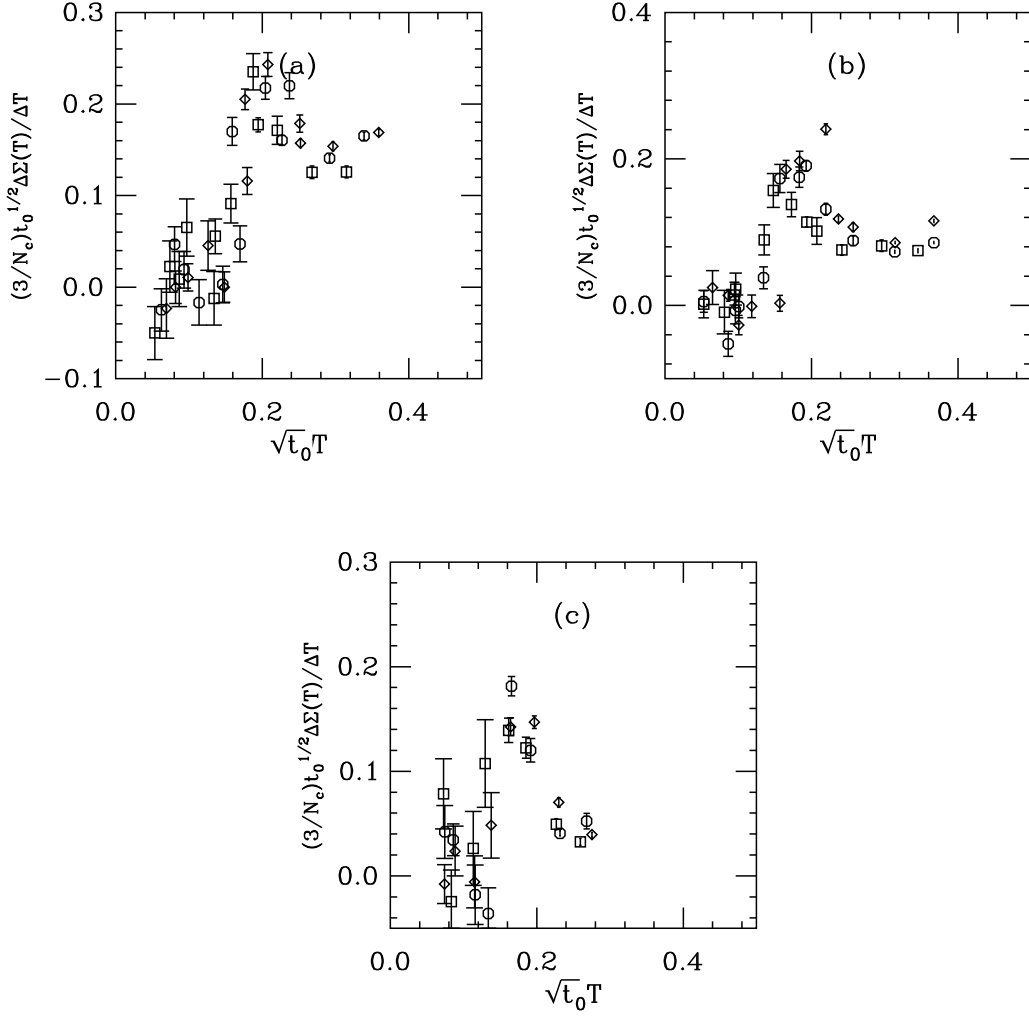


Figure 2: $\Delta\Sigma(T)/\Delta T$, rescaled by $3/N_c$, as a function of temperature, in appropriate units of t_0 . Squares, octagons, and diamonds label $N_c = 3, 4,$ and 5 . (a) $(m_{PS}/m_V)^2 \sim 0.63$; (b) $(m_{PS}/m_V)^2 \sim 0.5$; (c) $(m_{PS}/m_V)^2 \sim 0.25$.

My numerics were not very high quality but the effect was so obvious, it didn't matter.

It might be interesting to do large N_c thermodynamics "right." This probably means using staggered fermions since the volume scaling for thermodynamics is so fierce. A lot of the continuum quark gluon plasma phenomenology is large N_c based (AdS/CFT certainly is) and some predictions might be checked by going to large N_c on the lattice.

Acknowledgments

I am grateful to Rob Pisarski for a conversation about large N_c expectations for QCD thermodynamics. My computer code is based on the publicly available package of the MILC collaboration [18]. The version I use was originally developed by Y. Shamir and B. Svetitsky. This material is based upon work supported by the U.S. Department of Energy, Office of Science, Office of High

Energy Physics under Award Number DE-SC-0010005. Some of the computations for this work were also carried out with resources provided by the USQCD Collaboration, which is funded by the Office of Science of the U.S. Department of Energy using the resources of the Fermi National Accelerator Laboratory (Fermilab), a U.S. Department of Energy, Office of Science, HEP User Facility. Fermilab is managed by Fermi Research Alliance, LLC (FRA), acting under Contract No. DE-AC02-07CH11359.

References

- [1] G. 't Hooft, Nucl. Phys. B **72**, 461 (1974). doi:10.1016/0550-3213(74)90154-0
- [2] G. 't Hooft, Nucl. Phys. B **75**, 461 (1974). doi:10.1016/0550-3213(74)90088-1
- [3] E. Witten, Nucl. Phys. B **160**, 57-115 (1979) doi:10.1016/0550-3213(79)90232-3
- [4] B. Lucini and M. Panero, Phys. Rept. **526**, 93 (2013) doi:10.1016/j.physrep.2013.01.001 [arXiv:1210.4997 [hep-th]].
- [5] M. Garcia Perez, Proc. Sci., LATTICE2019 (2020) 276 arXiv:2001.10859 [hep-lat].
- [6] P. Hernández and F. Romero-López, Eur. Phys. J. A **57**, no.2, 52 (2021) doi:10.1140/epja/s10050-021-00374-2 [arXiv:2012.03331 [hep-lat]].
- [7] B. Lucini, M. Teper and U. Wenger, JHEP **0502**, 033 (2005) doi:10.1088/1126-6708/2005/02/033 [hep-lat/0502003].
- [8] B. Lucini, A. Rago and E. Rinaldi, Phys. Lett. B **712**, 279 (2012) doi:10.1016/j.physletb.2012.04.070 [arXiv:1202.6684 [hep-lat]].
- [9] R. D. Pisarski and F. Wilczek, Phys. Rev. D **29**, 338 (1984). doi:10.1103/PhysRevD.29.338
- [10] Thanks to Rob Pisarski for making this point.
- [11] R. Hagedorn, Nuovo Cim. A **56**, 1027 (1968). doi:10.1007/BF02751614
- [12] N. Cabibbo and G. Parisi, Phys. Lett. **59B**, 67 (1975). doi:10.1016/0370-2693(75)90158-6
- [13] T. DeGrand, Phys. Rev. D **103**, no.9, 094513 (2021) doi:10.1103/PhysRevD.103.094513 [arXiv:2102.01150 [hep-lat]].
- [14] S. Borsanyi *et al.*, JHEP **1208**, 126 (2012) doi:10.1007/JHEP08(2012)126 [arXiv:1205.0440 [hep-lat]].
- [15] S. Borsanyi *et al.*, Phys. Rev. D **92**, no. 1, 014505 (2015) doi:10.1103/PhysRevD.92.014505 [arXiv:1504.03676 [hep-lat]].
- [16] J. Gasser and H. Leutwyler, Annals Phys. **158**, 142 (1984) doi:10.1016/0003-4916(84)90242-2
- [17] R. P. Feynman, "Photon-hadron interactions," Reading, Benjamin, 1972.
- [18] https://github.com/milc-qcd/milc_qcd/