PROCEEDINGS OF SCIENCE

# NNLO QCD corrections to $B$-meson mixing

**Vladyslav Shtabovenko**[a,*]

[a] *Institut für Theoretische Teilchenphysik, Karlsruhe Institute of Technology (KIT), 76128 Karlsruhe, Germany*

*E-mail:* v.shtabovenko@kit.edu

We report on the calculation of next-to-next-to-leading order (NNLO) QCD corrections to the width difference $\Delta\Gamma_s$ in the neutral $B$-meson mixing process $B_s^0 - \bar{B}_s^0$. These contributions represent an important step in the task of reducing the existing large perturbative errors in the theory prediction for $\Delta\Gamma_s$ and approaching the current experimental uncertainties. We explain the theoretical framework employed in this computation and point out important subtleties in the treatment of evanescent operators and the renormalization. Part of our new results is already available in the literature, while the remaining pieces are expected to be published later this year.

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PoS(EPS-HEP2021)517

## 1. Introduction

In view of current null searches for new physics at the LHC, the increasing number of anomalies in the flavor sector of the Standard Model (cf. e. g. [1] for a recent overview) hints that the much anticipated beyond the Standard Model (BSM) particles might indeed be discovered at the precision and not at the high-energy frontier. Amid the well justified efforts of the BSM community to explain such anomalies by developing suitable models and studying their implications, one should not forget about the existence of flavor precision observables and their importance for our present understanding of the SM. The word "precision" indicates that such observables are equally well accessible to experimental measurements and theoretical calculations, which creates a friendly competition between these two particle physics communities.

One of such observables is the width difference $\Delta\Gamma_s$ that arises in the $B_s^0 - \bar{B}_s^0$ oscillations. This time-dependent process can be described via

$$i\frac{d}{dt}\begin{pmatrix}|B_s^0(t)\rangle \\ |\bar{B}_s^0(t)\rangle\end{pmatrix} = \left(\hat{M} - \frac{i}{2}\hat{\Gamma}\right)\begin{pmatrix}|B_s^0(t)\rangle \\ |\bar{B}_s^0(t)\rangle\end{pmatrix}, \tag{1}$$

where $\hat{M}$ and $\hat{\Gamma}$ are Hermitian matrices. In the absence of mixing $\hat{M}$ and $\hat{\Gamma}$ would have no off-diagonal elements, while their diagonal entries would correspond to the $B_s^0$ meson mass and width respectively. Yet by exchanging $W$-bosons in box diagrams, $b$- and $\bar{s}$-quarks can turn into $\bar{b}$ and $s$, which corresponds to the flavor eigenstate $B_s^0$ transforming into its antiparticle and back via weak interactions. This loop-suppressed flavor changing neutral current induces nonzero values of $M_{12}$ and $\Gamma_{12}$. Solving Eq. (1) one arrives at two mass eigenstates $|B_L\rangle$ (lighter, almost CP-even) and $|B_H\rangle$ (heavier, almost CP-odd)

$$|B_{L/H}\rangle = p\,|B_s^0\rangle \pm q\,|\bar{B}_s^0\rangle, \qquad \text{with } p^2 + q^2 = 1. \tag{2}$$

The mass and lifetime differences between these eigenstates are given by

$$\Delta m_s \equiv M_H - M_L = 2|M_{12}|, \quad \Delta\Gamma_s \equiv \Gamma_L - \Gamma_H \approx 2|\Gamma_{12}| \tag{3}$$

The width difference $\Delta\Gamma_s$ is of limited sensitivity to new physics and would deviate from its SM value only if BSM particles contributing through loops are light and weakly coupled to SM fields. In fact, $\Delta m_s$ and $\Delta\Gamma_s$ are complementary probes of new physics because of the former quantity probing effects of heavy (multi-TeV-mass) particles.

This implies that $\Delta\Gamma_s$ is a superb probe for our understanding of the SM and higher perturbative corrections thereto. Indeed, in the past years experimentalists have done a great job on reducing the statistical and systematic errors in their measurements [2–4] of $\Delta\Gamma_s$ and achieving per cent level precision [5] with

$$\Delta\Gamma_s^{\text{exp}} = (0.085 \pm 0.004)\ \text{ps}^{-1} \tag{4}$$

as compared to the current theory predictions [6–11]

$$\Delta\Gamma_s^{\overline{\text{MS}}} = (0.088 \pm 0.011_{\text{pert.}} \pm 0.002_{B,\tilde{B}_S} \pm 0.014_{\Lambda_{\text{QCD}}/m_b})\ \text{ps}^{-1}, \tag{5}$$

$$\Delta\Gamma_s^{\text{pole}} = (0.077 \pm 0.015_{\text{pert.}} \pm 0.002_{B,\tilde{B}_S} \pm 0.017_{\Lambda_{\text{QCD}}/m_b})\ \text{ps}^{-1}, \tag{6}$$

that suffer from large perturbative uncertainties (denoted as "pert."). The two values given in Eqs.(5) and (6) correspond to different renormalization schemes. The reduction of these uncertainties necessitates an inclusion of the missing QCD corrections to the $B_s^0 - \bar{B}_s^0$ mixing at 2- and 3-loop level, which is the main task of our project.

## 2. Calculation

In the calculation of $\Delta\Gamma_s$ we work with the $|\Delta B| = 1$ effective Hamiltonian that can be expressed using the following set of operators [12]

$$\mathcal{H}_{\text{eff}}^{|\Delta B|=1} = \frac{4G_F}{\sqrt{2}} \Bigg[ -\lambda_t^s \Big( \sum_{i=1}^{6} C_i Q_i + C_8 Q_8 \Big) - \lambda_u^s \sum_{i=1}^{2} C_i (Q_i - Q_i^u) + V_{us}^* V_{cb} \sum_{i=1}^{2} C_i Q_i^{cu} + V_{cs}^* V_{ub} \sum_{i=1}^{2} C_i Q_i^{uc} \Bigg] + \text{h.c.}, \tag{7}$$

where $V_{ij}$ stand for the CKM matrix elements and $\lambda_a^s = V_{as}^* V_{ab}$ denote products thereof, while $C_i$ are matching coefficients arising the matching between SM and $\mathcal{H}_{\text{eff}}^{|\Delta B|=1}$. The basis also includes so-called evanescent operators $E[Q_i]$ [13, 14] that are formally of order $O(\varepsilon)$ and therefore vanish in the limit $d \longrightarrow 4$. Their relevance arises from the fact that certain relations from the 4-dimensional Dirac algebra such as Fierz identities or the Chisholm identity cannot be translated to $d$ dimensions in a unique fashion. The proper handling of the evanescent operators during the renormalization and in the matching is one of the conceptual challenges accompanying this calculation. Explicit definitions of all $|\Delta B| = 1$ operators entering our matching calculations at 2 and 3 loops can be found in [15] and [16] respectively.

In quantum field theory $\Delta\Gamma_s \approx 2|\Gamma_{12}|$ is related to $\Gamma_{12}$, the absorptive part of a bilocal matrix element featuring a time-ordered product of two $|\Delta B| = 1$ effective Hamiltonians. Simplifying this quantity by means of the Heavy Quark Expansion (HQE) [17–25] we arrive at [9]

$$\Gamma_{12} = -(\lambda_c^s)^2 \Gamma_{12}^{cc} - 2\lambda_c^s \lambda_u^s \Gamma_{12}^{uc} - (\lambda_u^s)^2 \Gamma_{12}^{uu}, \tag{8}$$

with

$$\Gamma_{12}^{ab} = \frac{G_F^2 m_b^2}{24\pi M_{B_s}} \left[ H^{ab}(z) \langle B_s | Q | \bar{B}_s \rangle + \widetilde{H}_S^{ab}(z) \langle B_s | \widetilde{Q}_S | \bar{B}_s \rangle \right] + O(\Lambda_{\text{QCD}}/m_b), \tag{9}$$

where $z \equiv m_c^2/m_b^2$. The determination of the relevant QCD corrections to the Wilson coefficients $H^{ab}(z)$ and $\widetilde{H}_S^{ab}(z)$ is the main goal of our project. The $|\Delta B| = 2$ operators appearing in Eq. (9) are defined as

$$Q = \bar{s}_i \gamma^\mu (1 - \gamma^5) b_i \, \bar{s}_j \gamma_\mu (1 - \gamma^5) b_j, \qquad \widetilde{Q}_S = \bar{s}_i (1 - \gamma^5) b_j \, \bar{s}_j (1 - \gamma^5) b_i, \tag{10}$$

with $i, j$ specifying the color indices of the quark fields. The complete $|\Delta B| = 2$ operator basis (cf. [15]) also features suitable evanescent operators as well as the operator $R_0$ whose renormalized matrix elements are $1/m_b$-suppressed, while its bare matrix elements are not [6].

In the matching between $|\Delta B| = 1$ and $|\Delta B| = 2$ effective theories we choose to treat the $s$ quark as massless and to set its external momentum to zero, while the external momentum of the $b$ quark

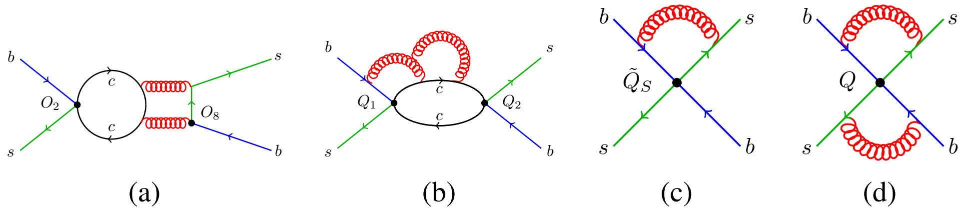


**Figure 1:** Sample $\Delta B = 1$ and $\Delta B = 2$ Feynman diagrams contributing to the process $b\bar{s} \to \bar{b}s$. Here (a) and (b) represent $\Delta B = 1$ contributions from $Q_2 \times Q_8$ (2 loops) and $Q_1 \times Q_2$ (3 loops), while (c) and (d) visualize 1- and 2-loop matrix elements of $\Delta B = 2$ operators $\tilde{Q}_s$ and $Q$.

is taken on-shell. Furthermore, at 2 loops we expand in $z$ up to $O(z)$, while the 3-loop diagrams are evaluated in the $z = 0$ limit. On the $|\Delta B| = 1$ side of the matching we calculate all possible operator insertions up to 2 loops i. e. all combinations of $Q_{1,2}$, $Q_{3-6}$ and $Q_8$ appearing in either of the two vertices. Notice that the 2-loop contribution to $Q_8 \times Q_8$ actually belongs to NNLO, but here we obtain it as a byproduct of our calculation. At 3 loops we evaluate only the $Q_{1,2} \times Q_{1,2}$ correlator. As far as the $|\Delta B| = 2$ theory is concerned, the 2-loop $|\Delta B| = 1$ contributions are matched to the 1-loop $|\Delta B| = 2$ diagrams, while the 3-loop $Q_{1,2} \times Q_{1,2}$ correlator requires us to consider 2-loop corrections to the $|\Delta B| = 2$ operators. Some of the representative Feynman diagrams visualizing the corresponding operator insertions are shown in Fig. 1.

## 3. Technical details

To carry out the analytic evaluation of the Feynman diagrams on both sides of the matching we make use of our well tested in-house calculational setup. We generate the required Feynman graphs using QGRAF [26] and employ q2e/exp [27, 28] or tapir [29] to insert Feynman rules and identify the occurring integral topologies. The resulting amplitudes are then processed with the aid of the FORM-based [30] calc framework. For cross checks of the results obtained from single diagrams we also employ FeynRules [31], FeynArts [32] and FeynCalc [33–35]. The latter, in conjunction with Fermat [36] is also used to derive tensor integral reduction formulas [37] that are used in our FORM code. Alternatively, we also employ a set of suitable Dirac and color projectors. FIRE [38] and LiteRed [39] allow us to IBP-reduce [40, 41] the occurring loop integrals to a small set of master integrals, many of which have already been calculated in the past [42, 43]. Most of the on-shell 3-loop master integrals, however, appear to be new and need to be calculated from scratch. This is done using FeynCalc, HyperInt [44], HyperLogProcedures [45] and PolyLogTools [46], so that at the end of the day we are able to obtain explicit analytic results for all integrals occurring in this matching calculation. We also cross check these results numerically using pySecDec [47–49] and FIESTA [50].

## 4. Renormalization and matching

We renormalize the bare $|\Delta B| = 1$ and $|\Delta B| = 2$ amplitudes in the $\overline{\text{MS}}$ scheme. Notice that in addition to the QCD renormalization constants we also need to take into account the renormalization of $|\Delta B| = 1$ and $|\Delta B| = 2$ operators. The generic operator renormalization matrix is of the form

$$Z = \begin{pmatrix} Z_{QQ} & Z_{QE} \\ Z_{EQ} & Z_{EE} \end{pmatrix}, \tag{11}$$

where the submatrices $Z_{ij}$ indicate mixing between different operator subclasses. For example, $Z_{EQ}$ encodes the mixing of evanescent into physical operators at order $O(\alpha_s)$. The $|\Delta B| = 1$ renormalization matrix is readily available in the literature [51] and is sufficient for NNLO accuracy. As far as $Z^{|\Delta B|=2}$ is concerned, the situation is less favorable, where mostly only $Z_{QQ}$ can be found in the literature. For this reason we choose to determine $Z^{|\Delta B|=2}$ tailored to our operator basis at 2 loops in a separate calculation.

To ensure the correctness of our matching calculation, at 2 loops we not only regularize UV and IR divergences dimensionally, but also employ a finite gluon mass as an infrared regulator. The latter makes the evaluation of the amplitudes and the calculation of the master integrals somewhat more involved but leads to significant simplifications in the matching. In particular, upon the UV renormalization our $|\Delta B| = 1$ and $|\Delta B| = 2$ amplitudes are free of $\varepsilon$ poles, so that one can safely take the limit $d \to 4$, where the matrix elements of evanescent operator vanish.

If we choose to work with massless gluons and therefore use the same $\varepsilon$ as our UV and IR regulator, then the renormalized amplitudes on both sides of the matching still contain IR poles and the contributions of evanescent operators must be kept. In this case the matching should be carried out according to the prescriptions outlined in [7]. This way all IR poles cancel and at 2-loops we obtain the same matching coefficients as in the calculation with massive gluons. At 3 loops we only work with massless gluons and using the method from [7] we observe an explicit cancellation of all IR poles in the matching.

## 5. Results

Due to the large number of new matching coefficients obtained in the course of this project, we summarize the obtained results in Table 1, which also indicates the previous status quo from the literature. Since our 2- and 3-loop results are of $O(z)$ and $O(z^0)$ respectively, it is understood that when comparing to the literature we also need to expand the relevant expressions in $z$. Under these conditions we confirm all the existing literature results, including the fermionic part of the 3-loop correlator $Q_{1,2} \times Q_{1,2}$ computed in [10]. The final NNLO theory prediction for the width difference is still work in progress, due to additional checks required to ensure that our treatment of the $1/m_b$-suppressed $R_0$ operator at 3 loops is correct. As far as the 2-loop contributions are concerned, explicit results for the $Q_{1,2} \times Q_{3-6}$ diagrams have already been published in [15], while the remaining 2-loop matching coefficients are expected to appear soon [53]. The 3-loop result together with the updated NNLO theory prediction are also in preparation [16].

To highlight the relevance of our computation, let us observe that alone the complete (i. e. not just its fermionic piece) 2-loop contribution $Q_{1,2} \times Q_{3-6}$ leads to a significant relative shift of $\Delta\Gamma_s$ as compared to the 1-loop result. This can be seen from building the ratio between full $\Delta\Gamma_s$ and the $Q_{1,2} \times Q_{3-6}$ piece only. For the width difference incorporating contributions listed in Table 1



PoS(EPS-HEP2021)517

| Contribution | Literature result | This work |
|---|---|---|
| $Q_{1,2} \times Q_{3-6}$ | 2 loops, $z$-exact, $n_f$-part only [11] | 2 loops, $O(z)$, full |
| $Q_{1,2} \times Q_8$ | 2 loops, $z$-exact, $n_f$-part only [11] | 2 loops, $O(z)$, full |
| $Q_{3-6} \times Q_{3-6}$ | 1 loop, $z$-exact, full [52] | 2 loops, $O(z)$, full |
| $Q_{3-6} \times Q_8$ | 1 loop, $z$-exact, $n_f$-part only [11] | 2 loops, $O(z)$, full |
| $Q_8 \times Q_8$ | 1 loop, $z$-exact, $n_f$-part only [11] | 2 loops, $O(z)$, full |
| $Q_{1,2} \times Q_{1,2}$ | 3 loops, $O(\sqrt{z})$, $n_f$-part only [10] | 3 loops, $O(z^0)$, full |

**Table 1:** Overview of the existing and new results required for the NNLO theory prediction of $\Delta\Gamma_s$ that were considered in this work. With "$n_f$-part only" we signify that the corresponding literature result provides only fermionic contributions, while "full" means that both fermionic and nonfermionic pieces are included.

of [15] and the 1-loop result for $Q_{1,2} \times Q_{3-6}$ we find

$$\frac{\Delta\Gamma_s^{p,12\times 36,\alpha_s^0}}{\Delta\Gamma_s} = 7.6\% \quad \text{(pole)}, \qquad \frac{\Delta\Gamma_s^{p,12\times 36,\alpha_s^0}}{\Delta\Gamma_s} = 6.1\% \quad (\overline{\text{MS}}), \tag{12}$$

while the inclusion of the 2-loop $Q_{1,2} \times Q_{3-6}$ piece yields

$$\frac{\Delta\Gamma_s^{p,12\times 36,\alpha_s}}{\Delta\Gamma_s} = 0.3\% \quad \text{(pole)}, \qquad \frac{\Delta\Gamma_s^{p,12\times 36,\alpha_s}}{\Delta\Gamma_s} = 1.4\% \quad (\overline{\text{MS}}). \tag{13}$$

Here we would like to refer to [15] for explicit values of all numerical parameters entering this comparison. The notions "$\overline{\text{MS}}$" and "pole" concern the treatment of the $m_b^2$ prefactor in Eq. (9). The former means that it is evaluated in the $\overline{\text{MS}}$ scheme, while the latter implies the usage of the on-shell scheme. Notice that even in the pole scheme all quantities except for the $m_b^2$ prefactor are handled in the $\overline{\text{MS}}$ scheme.

## 6. Summary

In our quest to improve theory prediction for the width difference $\Delta\Gamma_s$ in $B_s^0 - \bar{B}_s^0$ oscillations we addressed the problem of uncalculated QCD corrections at 2- and 3-loop accuracy. The evaluation of these corrections is a crucial step required to achieve a significant reduction of the existing perturbative uncertainties. In our matching calculation between $|\Delta B| = 1$ and $|\Delta B| = 2$ effective theories we were able to obtain fully analytic results for all of the required contributions by expanding the Feynman diagrams in the ratio $z \equiv m_c^2/m_b^2$. Our final results are valid up to $O(z)$ at 2 loops and $O(z^0)$ at 3 loops, while the inclusion of higher orders in $z$ is planned for future iterations of this work. The first part of our results was made public in [15], while the formulas addressing the remaining 2- and 3-loop contributions will appear in subsequent publications [16, 53]. In [16] we also intend to provide a new theory update on the value $\Delta\Gamma_s$ featuring NNLO accuracy and reduced theoretical errors.

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