PROCEEDINGS OF SCIENCE

Progress on Multi-loop Calculations

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In this contribution, we discuss the advancement made regarding multi-loop calculations within our group. In the first part, we discuss the progress made in the development of `HELAC-2LOOP`, a package for automated computations of two-loop scattering amplitudes, using the already working machinery of the `HELAC` framework. In the second part, we present recent results on two-loop five-point and three-loop four-point Feynman Integrals with one off-shell leg, using the Simplified Differential Equations approach.

Corfu Summer Institute 2021 "School and Workshops on Elementary Particle Physics and Gravity"
29 August - 9 October 2021
Corfu, Greece

*Speaker

PoS(CORFU2021)010

1. Introduction

Despite the up to now successful interpretation of the experimental data collected at the Large Hadron Collider (LHC) and the breakthrough discovery of the Higgs boson at LHC, the *Standard Model of Particle Physics* (SM) is still unable to provide a convincing explanation for a variety of issues of cosmological and astrophysical nature. The latter indicate the need of extending the SM, which shall be seen as an effective theory valid at low energies. In order to obtain glimpses of what lies beyond the SM, we need to exhaust its limits and look for inconsistencies with the high-precision experimental data of High-Luminosity LHC and future collider experiments. Thus the production of theoretical predictions of equally high precision for several SM scattering processes, with emphasis to the ones where *Quantum Chromodynamics* (QCD) is involved, has become a necessity.

Such theoretical calculations are carried out within the framework of *perturbative Quantum Field Theory*, where the scattering cross section can be calculated as a series expansion on the coupling constants of the theory at hand. The first term of this expansion corresponds to the *leading order* prediction (LO) and is followed by the *next-to-leading order* prediction (NLO), which is followed by the *next-to-next-to-leading order* prediction (NNLO) and so on. Moreover, due to the need of taming the infrared and ultraviolet divergences that become apparent in this expansion, renormalization schemes together with the *Dimensional regularization* (DR) (where the space-time dimensions are shifted to $d = 4 - 2\varepsilon$) are used. The current frontier on these computations stands at NNLO for 5-particle processes and at N^{3}LO for 4-particle ones.

For the computation of a cross section in a desired order of the perturbative expansion one needs first to compute the scattering amplitudes contributing to it. At different orders there exist amplitude contributions of different nature (virtual, real and mixed virtual-real contributions), but the most difficult to be computed appears to be, at any order, the virtual ones. The latter demand the computation of *Feynman integrals* (FI), where going beyond one loop computations is a non-trivial task. In more detail, a scattering amplitude for a specific process is constructed by collecting all the contributing *Feynman graphs*, which are generated using theory-dependent *Feynman rules*. Next the amplitude is reduced to a specific set of FI[1], the so-called *Master integrals* (MI), using techniques based at the integrand or/and integral level. Finally, the computation of the amplitude is completed with the computation of the aforementioned integrals.

There is a large advancement on the numerical calculation of FI, with the publication of automated packages, like `FIESTA5` [1] and pySecDec [2], which are implementing the method of *Sector Decomposition* [3]. Although these packages are able to provide results of high precision for several FI in the *Euclidean region*[2], this is not the case for physical regions of phase-space which are relevant for phenomenological applications. Thus the importance of analytic computation of FI, which can provide high precision results even for the physical regions, becomes apparent. The FI are usually computed analytically in the Euclidean region and afterwards they are extended in the physical regions using proper analytic continuation techniques. The modern method for the analytic computation of FI is the method of *Differential Equations* (DE) [4–7], while a lot of results have been derived via direct integration of FI in the *Feynman-Parameter representation* [8] as well.

[1]For LO there is no need for reduction, as there is not integral over loop momenta.

[2]A region of the kinematical invariants where the FI are free of branch cuts.

In this proceeding, we focus on the progress made within our group on two steps of the above described procedure of precision computations. In the first section, after quickly reviewing `HELAC-1LOOP` which is an automated package for computing 1-loop scattering amplitudes, we mention the upgrades made, and the ones needed to be made, in order to enable the HELAC framework to construct and compute generic two-loop QCD scattering amplitudes. In the second section, we briefly describe the DE method and we quote some recent results on two-loop 5-point and three-loop 4-point massless FI with one external massive particle, using the *Simplified Differential Equation approach* (SDE) [9]. In the last section we conclude, mentioning future work on the same direction.

2. Scattering Amplitudes

Any l-loop n-point QCD scattering amplitude $\mathcal{M}_{l-loop}$ can be decomposed into a color factor and a kinematic-dependent part, written schematically as

$$M_{l-loop} = \sum C_l \mathcal{A}_{l-loop} \tag{1}$$

where C_l is the color factor of the corresponding amplitude, and $\mathcal{A}_{l-loop}$ is the color-stripped amplitude, which can be constructed using color-stripped Feynman rules. For the computation of C_l there exist different color representations, with the most famous one being the fundamental representation, where the C_l is expressed in terms of traces of the $SU(3)$ generators t^a_{ij}. Within the `HELAC` framework the *Color-Connection (Color-Flow) representation* [10, 11] is used, where the amplitude is contracted with a t^a_{ij} matrix over the adjoint index a for every gluon. This results to C_l being equal to a product of delta functions with fundamental and anti-fundamental indices.

2.1 `HELAC-1LOOP` in a nutshell

Considering now the 1−loop case, the general form of a n−point color-stripped amplitude is

$$\mathcal{A}_{1-loop} = \int \frac{\mu^{(4-d)} d^d k}{(2\pi)^d} A_{1-loop} = \sum_{I\subset\{1,\ldots,n\}} \int \frac{\mu^{(4-d)} d^d k}{(2\pi)^d} \frac{N_I(k, p_1, \ldots, p_{n-1}, \gamma^\mu, \epsilon^\mu)}{\prod_{i\in I} D_i} . \tag{2}$$

In the above expression, A_{1-loop} is the 1−loop amplitude integrand, N_I is the numerator depending on gamma matrices, polarization vectors and loop and external momenta, and $D_i = (k + p_i)^2 - m_i^2$ are the scalar propagators with m_i being the mass of the propagating particle. The d−dimensional loop momentum and can be decomposed as

$$k = \bar{k} + k^* \quad \text{with} \quad \bar{k} : 4-\text{dimensional} \quad \text{and} \quad k^* : \varepsilon-\text{dimensional} \tag{3}$$

and the computation of $\mathcal{A}_{1-loop}$ at $d \to 4$ is done using the well-known formula

$$\mathcal{A}_{1-loop} = \sum_i d_i\,\text{Box}_i + \sum_i c_i\,\text{Triangle}_i + \sum_i b_i\,\text{Bubble}_i + \sum_i a_i\,\text{Tadpole}_i + R_1 + R_2 + O(\varepsilon) \tag{4}$$

where Box, ..., Tadpole refer to the 1-loop MI with 4, ..., 1 external legs, R_1 is the rational term originating from the reduction process of a 4−dimensional numerator (can be computed via three

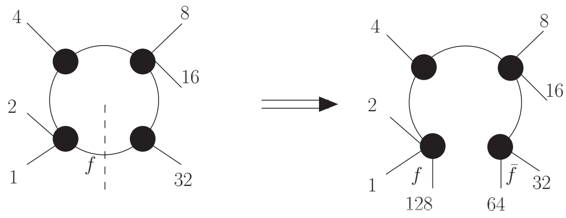

Figure 1: Cutting a propagator results to a $n + 2$ tree-level amplitude, which can be calculated by HELAC. The two cut particles have flavors f and $\bar{f}$, respectively, and obtain the usual HELAC notation (2^n and $2^{(n+1)}$).

extra scalar integrals [12]), and R_2 is the rational term originating from the explicit dependence of the numerator on ε (can be reproduced by tree−level special Feynman rules [13]).

Within the automated framework of HELAC-1LOOP [14], the 4−dimensional part of the numerator, $\bar{N}(\bar{k})$, is computed by the tree-order machinary of HELAC [10], which using the Dyson-Schwinger recursion equations, calculates tree-level amplitudes for all flavor, spin and color configurations allowed by the SM couplings. For the external particles (level 1 blobs) a binary representation is used, and a generation of all topologically inequivalent partitions of $n, n-1, n-2, ..., 1$ blobs[3] attached to the loop is done. Each numerator contribution is calculated by cutting the propagator-line connecting the first and the last blob (see Figure 1) and calculating the resulted $n + 2$ tree-level amplitude without using denominators for the internal loop propagators.

As it regards the reduction of the amplitude to MI, within HELAC-1LOOP this is done at the *integrand-level* using the OPP method [12]. In this method, the 4−dimensional part of the numerator is decomposed as

$$
\begin{aligned}
\bar{N}(\bar{k}) = & \sum_{i_0<i_1<i_2<i_3}^{I} \left[d(i_0,i_1,i_2,i_3) + \tilde{d}(\bar{k},i_0,i_1,i_2,i_3)\right] \prod_{i\neq i_0,i_1,i_2,i_3}^{I} \bar{D}_i \\
& + \sum_{i_0<i_1<i_2}^{I} \left[c(i_0,i_1,i_2) + \tilde{c}(\bar{k},i_0,i_1,i_2)\right] \prod_{i\neq i_0,i_1,i_2}^{I} \bar{D}_i \\
& + \sum_{i_0<i_1}^{I} \left[b(i_0,i_1) + \tilde{b}(\bar{k},i_0,i_1)\right] \prod_{i\neq i_0,i_1}^{I} \bar{D}_i \\
& + \sum_{i_0}^{I} \left[a(i_0) + \tilde{a}(\bar{k},i_0)\right] \prod_{i\neq i_0}^{I} \bar{D}_i
\end{aligned} \tag{5}
$$

where $d_i = d(i_0,i_1,i_2,i_3)$, $c_i = c(i_0,i_1,i_2)$, $b_i = b(i_0,i_1)$, $a_i = a(i_0)$, and $\tilde{d}$, $\tilde{c}$, $\tilde{b}$, $\tilde{a}$ multiply terms that integrate to zero (*spurious terms*). The coefficients are determined by solving iteratively systems of equations, by evaluating $\bar{N}(\bar{k})$ and the right-hand-side of (5) at values of $\bar{k}$ that are solutions of

$$\bar{D}_i(\bar{k}) = 0, \quad \text{for } i = 0, ..., M-1, \quad \text{and } M = 4, ..., 1. \tag{6}$$

One starts with the determination of $\{d, \tilde{d}\}$ and ends with that of $\{a, \tilde{a}\}$ (top-down approach).

[3]The blobs can contain propagators but they are independent of $\bar{k}$. The level of the blob is equal to the number of particles that contains.

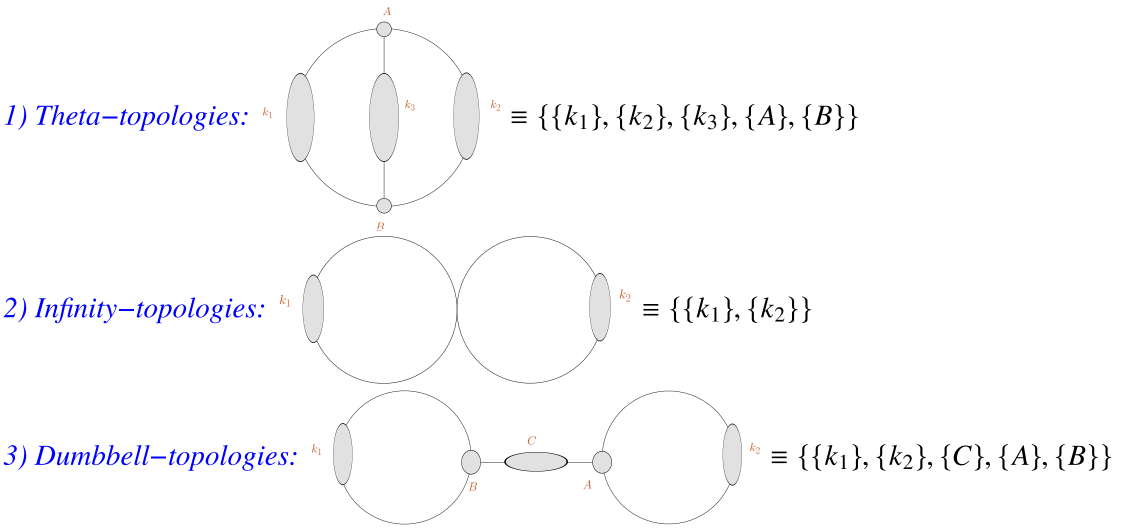

Figure 2: The three graph grand-topologies that contribute to the 2−loop amplitudes. The sublists represent the incoming particles to the corresponding loop-lines (k_1, k_2, k_3), the internal line (C) and the vertex-points (A, B).

2.2 `HELAC-2LOOP` in construction

Focusing now on the 2−loop case, a n−particle color−stripped amplitude has the following form

$$\mathcal{A}_{2\text{-}loop} = \int \frac{d^d k_1 d^d k_2}{(2\pi)^{2d} \mu^{2(d-4)}} A_{2\text{-}loop} = \sum_{I \subseteq T} \int \frac{d^d k_1 d^d k_2}{(2\pi)^{2d} \mu^{2(d-4)}} \frac{N_I(k_1, k_2, p_1, \ldots, p_{n-1}, \gamma^\mu, \epsilon^\mu)}{\prod_{\{i_1,i_2,i_3\} \in I} D_{i_1}(k_1) D_{i_2}(k_2) D_{i_3}(k_1, k_2)} \tag{7}$$

with T being the set containing all the 2−loop graph topologies of the process at hand. Driven by the 1−loop case we expect that at two loops, the amplitude at $d \to 4$ can be cast in the following form

$$\mathcal{A}_{2\text{-}loop} = \sum_i c_i(\mathbf{s}) F_i(\mathbf{s}, \varepsilon) + R_1^{2\text{-}loop}(\mathbf{s}, \varepsilon) + R_2^{2\text{-}loop}(\mathbf{s}, \varepsilon) + O(\varepsilon) \tag{8}$$

where F_i is a basis of 2−loop MI, $\mathbf{s}$ are Mandelstam variables or/and internal masses, and $\{R_1^{2\text{-}loop}, R_2^{2\text{-}loop}\}$ are the 2−loop generalization of the 1−loop rational terms. Considering the MI, a general basis for any process at hand (as in the 1−loop case) is still undeterminded while at the same time a lot of progress needs to be done on the computation of MI that contribute to such a basis. For what concerns the rational terms, a lot of progress has been done for the computation of $R_2^{2\text{-}loop}$ [15–17], but the same doesn't hold true for the $R_1^{2\text{-}loop}$ term.

At two loops and within the SM, the amplitude receives contributions from three different graph grand-topologies, which we name *Theta*, *Infinity* and *Dumbbell topologies*, and we depict them on Figure 2. Within `HELAC-2LOOP`, we aim on the construction of the 4−dimensional part of the numerator using the already working approach of `HELAC-1LOOP`, meaning cutting the graph topologies in two (appropriately chosen) propagator-lines and computing, using `HELAC`, the resulting $n + 4$ tree-level amplitude without using denominators for the loop propagators.

For this reason we have created a `Fortran`-based generator (`GENTOOLS`), which generates all the 2−loop graph topologies in a five list format[4] using the blob-binary representation. The algorithm of `GENTOOLS` goes as follows. In the beginning, generates all the possible combinations of the higher level blobs into the lists (lower-topologies). Then creates the higher topologies by taking all the possible splittings of the blobs. This step is performed till we arrive to graph topologies containing only level 1 blobs[5]. We note that the described procedure of generating loop topologies leads to possible double countings, which are removed making use of symmetries inherent in each of the three graph grand-topologies (see [18] for more details).

Having a specific process at hand, the aforementioned graph topologies are dressed with flavor using SM Feynman rules, cut in a k_1 and a k_2 loop-line, and then dressed with color. After color and flavor dressing, the tree-level `HELAC` machinery is employed which for every color configuration constructs a numerator contributing to the skeleton of the amplitude. After color and flavor dressing, the tree-level `HELAC` machinery is employed to compute recursively the numerator for each contributing configuration, using Dyson-Schwinger equations.

Concerning the amplitude reduction at two loops, our intention is to combine new methods on this direction [19–32] together with an extension of the OPP method at two loops. Within a two-loop OPP−like method, the coefficients of the master integrals in (8) could be determined at the *integrand level* by the 4−dimensional part of the amplitude via following a top-down approach, starting by the determination of the top-sector coefficients and going down till the lowest sectors coefficients. This means that an expression of the following form would hold true

$$A_{2-loop} = \frac{\bar{N}_I(\bar{k}_1, \bar{k}_2, p_1, \dots, p_{n-1}, \gamma^\mu, \epsilon^\mu)}{\prod_{\{i_1,i_2,i_3\}\in I} \bar{D}_{i_1}(\bar{k}_1)\bar{D}_{i_2}(\bar{k}_2)\bar{D}_{i_3}(\bar{k}_1, \bar{k}_2)} = \sum_i c_i(\mathbf{s}) I_i + \sum_j \tilde{c}_j(\mathbf{s}) S_j \tag{9}$$

where I_i are the *master integrands* that will integrate to the *master integrals* F_i and S_j are the *spurious* terms that will integrate to zero. For the application of such an approach, a complete basis of master integrands and spurious terms is needed, which for the moment is still missing.

3. Feynman Integrals

As we have already mentioned, the modern approach for computing FI is using the DE method [4–7]. Within this method, for specific kinematic processes we define families of integrals which are of the following form

$$F_{\alpha_1,\dots,\alpha_N} = \int \left(\prod_{i=1}^{L} \frac{d^d k_i}{i\pi^{d/2}} \right) \frac{1}{D_1^{\alpha_1} \dots D_N^{\alpha_N}} \tag{10}$$

with α_i arbitrary integers, L the number of loop momenta (k_i), $E+1$ the number of external momenta (p_i) and $N = L(L+1)/2 + LE$ the number of linear scalar independent propagators, $D_a = (k_i + p_j)^2 - m_a^2$, of the family. For $L > 1$, there exists a set of propagators coming from *irreducible scalar products*[6] for which the indices α_j are negative, meaning that these propagators can appear only in the numerator.

[4] For the *Infinity* topologies a two list format is used.

[5] These hold true for all but the blobs that are attached to the vertices A and B, which can not be splitted.

[6] These are products of loop momenta with external momenta that can not be written in terms of the propagators of the top-sector Feynman graph

The fact that the total derivatives vanish within DR gives rise to the *Integration by Parts Relations* (IBP) [27]

$$\int \prod_{i=1}^{L} \frac{d^d k_i}{i\pi^{d/2}} \frac{\partial}{\partial k_i} \left(\frac{l_j}{D_1^{\alpha_1} \dots D_N^{\alpha_N}} \right) = 0 \quad \text{with} \quad l_j = k_j \text{ or } p_j \,. \tag{11}$$

which imply the existence of a finite basis of integrals [33], the *Master integrals* (MI). There is a freedom in the choice of this basis, and any FI of the family can be expressed as a linear combination of MI with some algebraic coefficients of $s_{ij} = (p_i + p_j)^2$, m_a^2 and ε. For the IBP reduction to MI there exist modern automated packages implementing Laporta's algorithm [28], such as `FIRE6` [34] and `KIRA2` [35].

Using the IBP relations and the fact that FI are functions of external momenta and internal masses, we can differentiate them with respect to the kinematic invariants, $S_k = \{s_{ij}, m_a^2\}$, and derive differential equations for the MI

$$\frac{\partial}{\partial S_k} G_i(\varepsilon, \{S_k\}) = \sum_{j=1}^{I} B_{ij}^k(\varepsilon, \{S_k\}) G_j(\varepsilon, \{S_k\}) \Rightarrow \partial^k \vec{G} = B^k \vec{G} \tag{12}$$

where I is the number of MI. One can solve this DE in a Laurent expansion around $\varepsilon = 0$ after first finding appropriate boundary conditions for the MI. From (12) we can see that by making a change of the basis $\vec{G} \to U\vec{G}$, B^k changes as $B^k \to UB^kU^{-1} + U\partial^k U^{-1}$.

One groundbreaking idea that has lead to numerous calculations of FI is that of the *Canonical DE* [36]. According to this idea, for a suitable choice of the basis of MI (which corresponds to a suitable choice of U) the DE can take the following form

$$\frac{\partial}{\partial S_k} \vec{G}'(\varepsilon, \{S_k\}) = \varepsilon \sum_i \frac{M_{ki}}{S_k - l_i} \vec{G}'(\varepsilon, \{S_k\}) \tag{13}$$

which is ε-factorized, *Fuchsian* and with M_{ki} being purely numerical. Thus it can be solved iteratively in ε. In order to obtain DE of canonical form, the basis of MI should be chosen such that it contains only functions with *uniform degree of transcendentality* (UT). The transcedentality of a function f, $\mathcal{T}(f)$, is defined by the number of iterated integrations needed to define the function f. Some Examples and properties of the transcedentality within DR, are [36]

- $\mathcal{T}(\log(x)) = 1$, $\mathcal{T}(Li_n) = n$, $\mathcal{T}(\pi) = 1$, $\mathcal{T}(\zeta(n)) = n$ and $\mathcal{T}(\text{Algebraic Factors}) = 0$.
- $\mathcal{T}(f_1 f_2) = \mathcal{T}(f_1) + \mathcal{T}(f_2)$ and $\mathcal{T}(\varepsilon) = -1$.

Although there doesn't exist a general method for obtaining a UT basis (or even proving its existence) for any given family of FI, there exist different methods [37–46] the combination of which can lead to a canonical DE.

3.1 The Simplified Differential Equations Approach

The SDE approach [9] is a variant of the standard DE method, where the external momenta are re-parametrized in terms of a dimensionless parameter, x, with respect to which the MI are differentiated in order to create a DE of the form

$$\frac{\partial}{\partial x} \mathbf{G} = \mathbf{M}\left(\{S_{ij}, m_a^2\}, x, \varepsilon \right) \mathbf{G} \tag{14}$$

where S_{ij} are the kinematic invariants defined by the external momenta in the SDE parametrization. As we will see in the next subsection, the SDE parametrization is not unique, but in general respects the rule that x is introduced in a way that captures the off-shellness of one external momentum. The rationalization of the square roots, with respect to x, that appear in the UT basis is also a fact that someone has to take into account in order to choose a viable parametrization.

In order to solve (14) one has to compute the boundary conditions of the basis elements at $x \to 0$, denoted as $\mathbf{g}_{\text{bound}}$. For this computation we follow the subsequent algorithmic approach [47, 48]

- First, we use as boundaries the already known integrals from other families or the ones known in closed form.

- Then, by comparing the asymptotic regions obtained for the MIs from expansion-by-regions method [49] (as implemented in the `asy` code) with the ones obtained by the DE, using the resummation matrix [50, 51] at $x = 0$

$$\mathbf{M}_0 = \mathbf{S}_0\mathbf{D}_0\mathbf{S}_0^{-1} \quad \longrightarrow \quad \mathbf{R}_0 = \mathbf{S}_0 e^{\varepsilon \mathbf{D}_0 \log(x)} \mathbf{S}_0^{-1} \quad \longrightarrow \quad \mathbf{TF}_{x\to 0} = \mathbf{R}_0 \mathbf{g}_{\text{bound}} \,, \tag{15}$$

  we obtain relations between different boundaries of the family.

- Finally, we are left with some asymptotic regions of MI or basis elements [52] to calculate, which we do so by obtaining Feynman-Parameter representation of the regions and integrating in the Feynman-parameters.

An extra feature of the SDE approach is that by taking the $x \to 1$ limit of the solution we can readily obtain the solution for the same family with one external massive momentum (the one the off-shellness of which we captured with the introduction of x) less. For more details and examples of taking the $x \to 1$ limit we refer to [48, 53–55].

3.2 Recent Multi-loop Computations using the Simplified Differential Equations Approach

In this Subsection we quote some recent results for multi-loop MI using the SDE approach.

3.2.1 Planar Three-Loop Four-Point Massless Families with One External Massive Particle

We begin with the three-loop problem of the planar massless families for a process with four external particles, one of which is massive. These are the ladder-box (F1) and the two tennis-court (F2 and F3) families, which are depicted in Figure 3. The calculation of these families is important for the computation of N^3LO corrections to the cross section of particle processes like $e^+e^- \to \gamma^* \to 3j$, $pp \to Zj$ and $pp \to Hj$.

From the integral family point of view, each of these three families contains 15 propagators of which 5 are numerators produced by ISPs. F1 consists of 83 MI, while F2 and F3 consist of 117 and 166 MI, respectively. Regarding the kinematics, this scattering process has the following three independent invariants

$$s = (q_1 + q_2)^2, \quad t = (q_2 + q_3)^2 \quad \text{and} \quad m^2 = q_2^2. \tag{16}$$

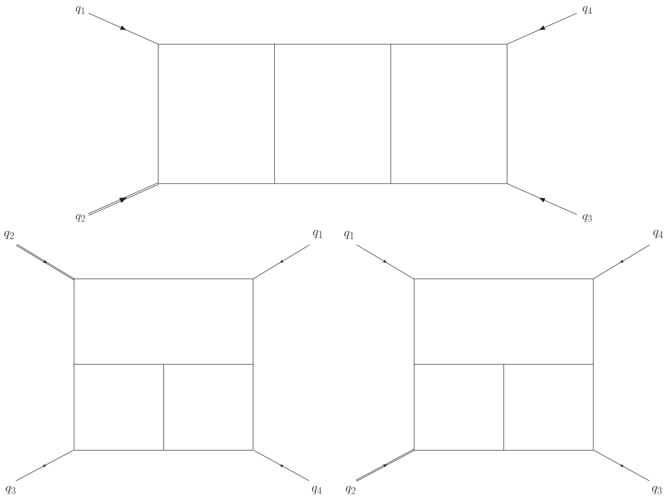

Figure 3: Diagrammatic representation of the F1 (top), F2 (bottom left) and F3 (bottom right) top-sector diagrams. The double line represents the massive particle and all external momenta are taken to be incoming.

In our computation [48, 56] we used the following parametrization[7]

$$q_1 \to x p_1, \quad q_2 \to p_{12} - x p_1, \quad \text{and} \quad q_3 \to p_3, \tag{17}$$

having as independent invariants the set $\{S_{12}, S_{23}, x\}$, with $S_{12} = (p_1 + p_2)^2$ and $S_{23} = (p_2 + p_3)^2$. Concerning the UT basis, for the ladder-box family we adopted the UT basis provided in [57], where this family was first studied, while for the two tennis-court families we derived it using different existing methods (see [58] for details). We solved the DE till weight 6 on ε, and in an Euclidean region of the invariants and, subsequently, we used fibration-basis techniques [59, 60] to analytically continue and express our results in the three physical regions of interest. Our solutions are analytic and expressed in terms of real-valued Goncharov polylogarithms (GPLs[8]) [61], thus being well-suited for phenomenological applications.

Interesting is the fact that, by studying the relation of the residue matrices of the standard DE method with the ones of SDE aprroach in [56], we found out that the adjacency conditions [63, 64] previously studied for the ladder-box family and the corresponding two-loop planar and non-planar families seem to apply also for the two tennis-court families.

3.2.2 Two-Loop Five-Point Massless Families with One External Massive Particle

We focus now on some families with one loop less but with three more kinematical invariants. These are the six two-loop massless families with four massless and one massive external particle, which are sketched in Figure 4 and contain 11 propagators (3 of which are numerators). The planar topologies (first row) are called penta-box families and we use the notation P_1 (74 MI), P_2 (75 MI) and P_3 (86 MI) for them, while the non-planar ones (second row) are called hexa-box families and we denote them N_1 (86 MI), N_2 (86 MI) and N_3 (135 MI). For the completion of the computation of all the FI families contributing to scattering processes involving 4 massless and 1 massive particle,

[7]For convenience we use the notation $p_{i\ldots j} = p_i + \cdots + p_j$.

[8]GPLs are numerically evaluated very fast when they are real valued using GinaC [62].

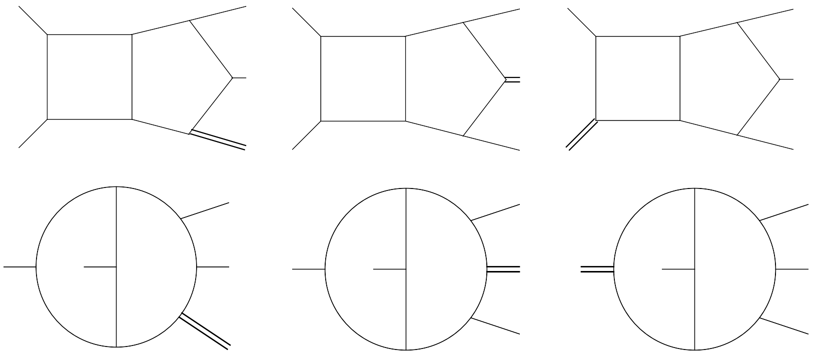

Figure 4: Diagrammatic representation of the planar penta-box and the non-planar hexa-box families with one external massive leg (double line). From left to right, in the first row we have the P_1, P_2 and P_3 penta-box families, while in the second row we have the N_1, N_2 and N_3 hexa-box ones.

such as W, Z and Higgs production in association with two jets, one needs to compute also the two, so-called, double-pentagon families.

Concerning the six independent kinematical invariants of this problem, in the standard approach these can be chosen to be $\{q_1^2, s_{12}, s_{23}, s_{34}, s_{45}, s_{15}\}$, with $s_{ij} = \left(q_i + q_j\right)^2$ and the external momenta satisfying $q_1^2 \neq 0$ and $q_i^2 = 0$, for $i = 2, \ldots, 5$. For our computation [47, 52], we adopted the UT basis from [46, 65], where these families were first studied and numerically solved using the method of generalized power series expansions [66], and we used the following parametrization

$$q_1 \to p_{123} - x p_{12}, \quad q_2 \to p_4, \quad q_3 \to -p_{1234}, \quad \text{and} \quad q_4 \to x p_1 \tag{18}$$

where the momenta p_i satisfy $p_i^2 = 0$, for $i = 1, \ldots, 5$. After the application of the above transformation the new set of independent invariants is $\{S_{12}, S_{23}, S_{34}, S_{45}, S_{51}, x\}$, with $S_{ij} = \left(p_i + p_j\right)^2$.

The reason for choosing the parametrization (18) is the fact that it rationalizes, with respect to x, some of the square roots that appear in the UT basis of the 6 families, making them in this way possible to be solved analyticaly in terms of GPLs. More specifically, using (18) we were able to rationalize the square roots appearing in P_1, P_2, P_3 and N_1, obtaining for them thus an analytic solution till weight 4 on ε in an Euclidean region. For the families N_2 and N_3, where the rationalization with respect to x was not possible, we acquired an analytic solution till weight 2 on ε and from thereon we established a one-dimensional integral representation in terms of GPLs for obtaining numerical results till weight 4. In order to analytically continue our results and make them suitable for phenomenological studies in the 5 physical regions of interest, we used the $+i\epsilon$ prescription [67] because an analytic continuation using [59] was not possible due to the appeareance of algebraic letters on the alphabet of the DE.

4. Conclusion

In conclusion, we presented the recent developments made on the development of `HELAC-2LOOP` and the computation of multi-loop FI using the SDE approach. Our next steps towards the automation of NNLO QCD computations using the `HELAC` framework consist of the completion of all upgrades needed by the HELAC code in order to be able to numerically compute the 4-dimensional part of a 2-loop numerator for any provided process, the creation of a general basis of master integrands (plus spurious terms) and the computation of the R_1^{2-loop} rational terms. On the same direction, progress is necessary to be made on the computation of 2-loop MI and thus we are currently working on the computation of the two massless double-pentagon families with one external massive particle. Being aware of the need for N^{3}LO computations for future comparisons with experimental data, we plan also to extend our work to the computation of the non-planar three-loop four-point MI with one massive leg.

Acknowledgments

The research work of D.C. was supported by the Hellenic Foundation for Research and Innovation (HFRI) under the HFRI Ph.D. Fellowship grant (Fellowship Number: 554). N.S. was supported by the Excellence Cluster ORIGINS funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy - EXC-2094 - 390783311. This research was supported in part by grant K 125105 of the National Research, Development and Innovation Fund in Hungary. A.K. is supported by the ÚNKP-21-Bolyai+ New National Excellence Program of the Ministry for Innovation and Technology from the source of the National Research, Development and Innovation Fund. A.K. kindly acknowledges further financial support from the Bolyai Fellowship programme of the Hungarian Academy of Sciences. The work of A.S. was supported by the Russian Science Foundation, agreement no. 21-71-30003.

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