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Mass hierarchies from residual modular symmetries

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We report on a mechanism to generate hierarchical fermion mass matrices in modular-invariant models of flavour, based on the proximity of the value of the modulus to a point of residual symmetry. No flavons are required. The mechanism is distinct from (and an improvement on) the known Froggatt-Nielsen method of obtaining hierarchies between the elements of fermion mass matrices. It crucially depends on the decomposition of field representations under the residual symmetry group. For each symmetric point and considering, in turn, the modular S_3, A_4, S_4, and A_5 groups and their double covers, we identify the pairs of representations which may explain charged-lepton and quark mass hierarchies. After formulating the conditions for obtaining a viable lepton mixing matrix in the symmetric limit, we construct a model in which both charged-lepton mass hierarchies and lepton mixing are free from fine-tuning.

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1. Introduction

The origin of the observed patterns of fermion masses and mixing and of CP violation is one of the most challenging unsolved problems in particle physics. The unsatisfactory status of the lepton (as well as quark) flavour problem and the remarkable progress in studies of neutrino oscillations have stimulated renewed attempts to seek solutions to the former. A step in this direction was made in Ref. [1], where the idea of using modular invariance as a flavour symmetry was put forward. The main feature of this approach is that the elements of the Yukawa coupling and fermion mass matrices are modular forms, functions of a single complex scalar field: the modulus τ. In the simplest class of such models, the VEV of τ is the only source of flavour symmetry breaking and no flavons are needed. Another appealing feature of the proposed framework is that the VEV of τ can be the only source of CP-symmetry breaking [2]. When the flavour symmetry is broken, a certain flavour structure arises and e.g. charged-lepton and neutrino masses, neutrino mixing and leptonic CPV phases are simultaneously determined in terms of a limited number of parameters.

In almost all phenomenologically-viable flavour models based on modular invariance constructed so far (see [3] for a rather complete list) the hierarchy of the charged-lepton and quark masses is obtained by fine-tuning some of the parameters, i.e. there is a high sensitivity of observables to model parameters or there are unjustified hierarchies between parameters introduced in the model on an equal footing.

The present contribution is based on the work of Ref. [3], wherein we develop a formalism that allows to construct models in which fermion (charged-lepton and quark) mass hierarchies follow solely from the properties of the modular forms, avoiding fine-tuning without the need to introduce extra fields. We also investigate the possibility of concurrently obtaining large mixing without fine-tuning in models of lepton flavour. As we will see below, residual modular symmetries play a crucial role in our analysis.

2. Modular symmetries as flavour symmetries

In the supersymmetric modular-invariance approach to flavour, one introduces the modulus chiral superfield τ transforming non-trivially under the modular group $\Gamma \equiv SL(2,\mathbb{Z})$. The latter is generated by the matrices

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad R = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{1}$$

obeying $S^2 = R$, $(ST)^3 = R^2 = \mathbb{1}$, and $RT = TR$. For $\gamma \in \Gamma$, one has

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : \quad \tau \to \gamma\tau = \frac{a\tau + b}{c\tau + d}, \tag{2}$$

while matter superfields transform as weighted multiplets [1, 4, 5],

$$\psi_i \to (c\tau + d)^{-k} \rho_{ij}(\gamma)\, \psi_j, \tag{3}$$

where ρ is a unitary representation of Γ. We restrict ourselves to integer modular weights k. To use modular symmetry as a flavour symmetry, one fixes a level $N \geq 2$ and assumes that $\rho(\gamma) = \mathbb{1}$ for elements of the principal congruence subgroup, $\Gamma(N) \equiv \{\gamma \in SL(2,\mathbb{Z}),\ \gamma \equiv \mathbb{1}(\text{mod } N)\}$, so that ρ is effectively a representation of the (finite) quotient $\Gamma'_N \equiv \Gamma \,/\, \Gamma(N) \simeq SL(2,\mathbb{Z}_N)$. In the case where matter fields transform trivially under R, ρ is effectively a representation of a smaller finite modular group $\Gamma_N \equiv \Gamma \,/\, \langle \Gamma(N) \cup \mathbb{Z}_2^R \rangle$. For small N, the groups Γ_N and Γ'_N are isomorphic to permutation groups and to their double covers (e.g. $\Gamma_2 \simeq S_3$, $\Gamma_3 \simeq A_4$, $\Gamma_4 \simeq S_4$, and $\Gamma_5 \simeq A_5$).

The VEV of τ is restricted to the upper half-plane and plays the role of a spurion, parameterising modular symmetry breaking. Modular symmetry may then constrain the Yukawa couplings and mass structures of a model in a predictive way. By requiring the invariance of the superpotential under modular transformations, one finds that couplings $Y_{I_1\ldots I_n}(\tau)$ appearing in terms of the type $\psi_{I_1}\ldots\psi_{I_n}$ must be special holomorphic functions of τ — they are modular forms of level N — obeying

$$Y_{I_1\ldots I_n}(\tau) \xrightarrow{\gamma} Y_{I_1\ldots I_n}(\gamma\tau) = (c\tau + d)^{k_Y} \rho_Y(\gamma)\, Y_{I_1\ldots I_n}(\tau)\,. \tag{4}$$

Modular forms carry weights $k_Y = k_{I_1} + \ldots + k_{I_n}$ and furnish unitary representations ρ_Y of the finite modular group such that $\rho_Y \otimes \rho_{I_1} \otimes \ldots \otimes \rho_{I_n} \supset \mathbf{1}$. Non-trivial modular forms span finite-dimensional linear spaces. These have relatively low dimensionalities for small values of k and N, leading to a predictive setup in which only a restricted number of τ-dependent Yukawa textures are allowed in the superpotential. For additional details, the reader is referred to Ref. [3]

2.1 Residual symmetries

While there is no value of the modulus VEV preserving the full symmetry group, at so-called symmetric points $\tau = \tau_{\text{sym}}$ the modular group is only partially broken, with unbroken generators giving rise to residual symmetries. Note that the R generator is unbroken for any value of τ, so that a $\mathbb{Z}_2^R$ symmetry is always preserved. The fundamental domain $\mathcal{D}$ and symmetric points of the modular group are shown in Figure 1. There are only three inequivalent such points [6],

- $\tau_{\text{sym}} = i\infty$, invariant under T, preserving $\mathbb{Z}_N^T \times \mathbb{Z}_2^R$;
- $\tau_{\text{sym}} = i$, invariant under S, preserving $\mathbb{Z}_4^S$ (note that $S^2 = R$); and
- $\tau_{\text{sym}} = \omega \equiv \exp(2\pi i/3)$, 'the left cusp', invariant under ST, preserving $\mathbb{Z}_3^{ST} \times \mathbb{Z}_2^R$.

In a CP- and modular-invariant theory [2, 7], an additional $\mathbb{Z}_2^{\text{CP}}$ symmetry is preserved for $\text{Re}\,\tau = 0$ or for τ on the border of $\mathcal{D}$, while is broken at generic values of τ. Note that all three symmetric values above preserve the CP symmetry.

3. Mass hierarchies without fine-tuning

3.1 Mass matrices close to symmetric points

At a symmetric point, flavour textures can be severely constrained by the residual symmetry group, which may enforce the presence of multiple zero entries in the mass matrices. As τ moves away from its symmetric value, these entries will generically become non-zero. In what follows it is shown that the magnitudes of such (residual-)symmetry-breaking entries are controlled by the size of the departure ϵ of τ from τ_{sym} and by the field transformation properties under the residual symmetry group, which may depend on modular weights.

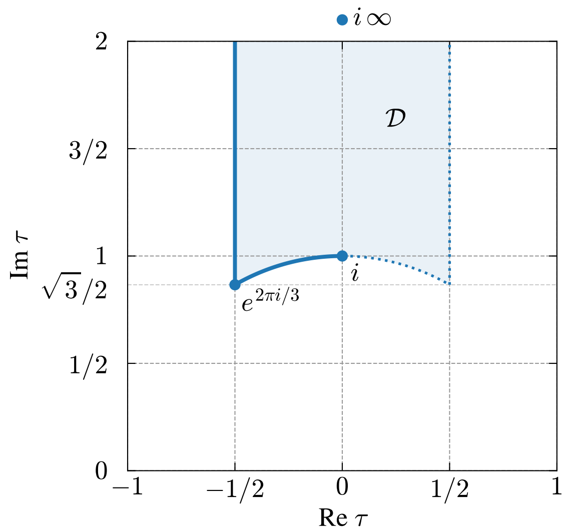

Figure 1: The fundamental domain $\mathcal{D}$ of the modular group Γ and its three symmetric points $\tau_{\text{sym}} = i\infty, i, \omega$. The value of τ can always be restricted to $\mathcal{D}$ by a suitable modular transformation. Figure from Ref. [7].

We start by considering a modular-invariant bilinear $\psi_i^c\, M(\tau)_{ij}\, \psi_j$, where the superfields ψ and ψ^c transform under the modular group as

$$\psi \xrightarrow{\gamma} (c\tau + d)^{-k} \rho(\gamma)\, \psi\,, \qquad \psi^c \xrightarrow{\gamma} (c\tau + d)^{-k^c} \rho^c(\gamma)\, \psi^c\,, \tag{5}$$

so that each $M(\tau)_{ij}$ is a modular form of level N and weight $K \equiv k + k^c$. Modular invariance requires $M(\tau)$ to transform as

$$M(\tau) \xrightarrow{\gamma} M(\gamma\tau) = (c\tau + d)^K \rho^c(\gamma)^* M(\tau) \rho(\gamma)^\dagger\,. \tag{6}$$

Taking τ to be close to the symmetric point, and setting γ to the residual symmetry generator, one can use this transformation rule to constrain the form of the mass matrix. Each of the three symmetric points is analysed in turn.

3.1.1 $\tau_{\text{sym}} = i\infty$

Consider the T-diagonal representation basis for group generators, in which $\rho^{(c)}(T) = \text{diag}(\rho_i^{(c)})$, and take τ 'close' to $\tau_{\text{sym}} = i\infty$, i.e. large enough $\text{Im}\,\tau$. By setting $\gamma = T$ in eq. (6), one finds

$$M_{ij}(T\tau) = \left(\rho_i^c \rho_j\right)^* M_{ij}(\tau)\,. \tag{7}$$

It is convenient to treat the M_{ij} as a function of $q \equiv \exp(2\pi i\tau/N)$, so that $\epsilon \equiv |q| = e^{-2\pi\,\text{Im}\,\tau/N}$ parameterises the deviation of τ from the symmetric point. Note that the entries $M_{ij}(q)$ depend analytically on q and that $q \xrightarrow{T} \zeta q$, with $\zeta \equiv \exp(2\pi i/N)$. Thus, in terms of q, eq. (7) reads

$$M_{ij}(\zeta q) = (\rho_i^c \rho_j)^* M_{ij}(q)\,. \tag{8}$$

Expanding both sides in powers of q, one finds

$$\zeta^n M_{ij}^{(n)}(0) = (\rho_i^c \rho_j)^* M_{ij}^{(n)}(0)\,, \tag{9}$$

where $M_{ij}^{(n)}$ denotes the n-th derivative of M_{ij} with respect to q. It follows that $M_{ij}^{(n)}(0)$ can only be non-zero for values of n such that $(\rho_i^c \rho_j)^* = \zeta^n$. It is clear that in the symmetric limit $q \to 0$ the entry $M_{ij} = M_{ij}^{(0)}$ is only allowed to be non-zero if $\rho_i^c \rho_j = 1$. More generally, if $(\rho_i^c \rho_j)^* = \zeta^l$ with $0 \leq l < N$,

$$M_{ij}(q) = a_0\, q^l + a_1\, q^{N+l} + a_2\, q^{2N+l} + \ldots \tag{10}$$

in the vicinity of the symmetric point. It crucially follows that the entry M_{ij} is expected to be $\mathcal{O}(\epsilon^l)$ whenever $\operatorname{Im}\tau$ is large. The power l only depends on how the representations of ψ and ψ^c decompose under the residual symmetry group $\mathbb{Z}_N^T$.

3.1.2 $\tau_{\mathrm{sym}} = i$

For τ in the vicinity of $\tau_{\mathrm{sym}} = i$, it is convenient to work in the S-diagonal basis, where now $\rho^{(c)}(S) = \operatorname{diag}(\rho_i^{(c)})$. Define $\tilde{\rho}_i^{(c)} \equiv i^{k^{(c)}} \rho_i^{(c)}$. By setting $\gamma = S$ in eq. (6), one finds

$$M_{ij}(S\tau) = (-i\tau)^K \left(\tilde{\rho}_i^c \tilde{\rho}_j\right)^* M_{ij}(\tau)\,. \tag{11}$$

We now treat the M_{ij} as functions of $s \equiv (\tau - i)/(\tau + i)$ so that, in this context, $\epsilon \equiv |s|$ parameterises the deviation from the symmetric point. Note that the entries $M_{ij}(s)$ depend analytically on s and that $s \xrightarrow{S} -s$. In terms of s, eq. (11) reads

$$M_{ij}(-s) = \left(\frac{1+s}{1-s}\right)^K (\tilde{\rho}_i^c \tilde{\rho}_j)^* M_{ij}(s) \quad \Rightarrow \quad \tilde{M}_{ij}(-s) = (\tilde{\rho}_i^c \tilde{\rho}_j)^* \tilde{M}_{ij}(s)\,, \tag{12}$$

where $\tilde{M}_{ij}(s) \equiv (1-s)^{-K} M_{ij}(s)$. Expanding both sides in powers of s, one finds

$$(-1)^n \tilde{M}_{ij}^{(n)}(0) = (\tilde{\rho}_i^c \tilde{\rho}_j)^* \tilde{M}_{ij}^{(n)}(0)\,, \tag{13}$$

where $\tilde{M}_{ij}^{(n)}$ denotes the n-th derivative of $\tilde{M}_{ij}$ with respect to s. Thus, $M_{ij} \sim \tilde{M}_{ij}$ is only allowed to be $\mathcal{O}(1)$ when $\tilde{\rho}_i^c \tilde{\rho}_j = 1$. If instead $\tilde{\rho}_i^c \tilde{\rho}_j = -1$, the entry $M_{ij} \sim \tilde{M}_{ij}$ is expected to be $\mathcal{O}(\epsilon)$, with $\epsilon = |s|$. Note that, unlike in the previous case, the relevant factors $\tilde{\rho}_i^{(c)}$ depend (by definition) on the weights $k^{(c)}$.

3.1.3 $\tau_{\mathrm{sym}} = \omega$

Finally, for $\tau \simeq \tau_{\mathrm{sym}} = \omega$, it is useful to work in the basis where ST is diagonal and to define $\tilde{\rho}_i^{(c)} \equiv \omega^{k^{(c)}} \rho_i^{(c)}$. By setting $\gamma = ST$ in eq. (6), one finds

$$M_{ij}(ST\tau) = [-\omega(\tau+1)]^K \left(\tilde{\rho}_i^c \tilde{\rho}_j\right)^* M_{ij}(\tau)\,. \tag{14}$$

We now treat the M_{ij} as functions of $u \equiv (\tau - \omega)/(\tau - \omega^2)$ so that, in this context, $\epsilon \equiv |u|$ parameterises the deviation from the symmetric point. Note that the entries $M_{ij}(u)$ depend analytically on u and that $u \xrightarrow{ST} \omega^2 u$. In terms of u, eq. (14) reads

$$M_{ij}(\omega^2 u) = \left(\frac{1-\omega^2 u}{1-u}\right)^K (\tilde{\rho}_i^c \tilde{\rho}_j)^* M_{ij}(u) \quad \Rightarrow \quad \tilde{M}_{ij}(\omega^2 u) = (\tilde{\rho}_i^c \tilde{\rho}_j)^* \tilde{M}_{ij}(u)\,, \tag{15}$$

where $\tilde{M}_{ij}(u) \equiv (1-u)^{-K} M_{ij}(u)$. Expanding both sides in powers of u, one finds

$$\omega^{2n} \tilde{M}_{ij}^{(n)}(0) = (\tilde{\rho}_i^c \tilde{\rho}_j)^* \tilde{M}_{ij}^{(n)}(0)\,, \tag{16}$$

where $\tilde{M}_{ij}^{(n)}$ denotes the n-th derivative of $\tilde{M}_{ij}$ with respect to u. Thus, $M_{ij} \sim \tilde{M}_{ij}$ is only allowed to be $O(1)$ when $\tilde{\rho}_i^c \tilde{\rho}_j = 1$. More generally, if $\tilde{\rho}_i^c \tilde{\rho}_j = \omega^l$ with $l = 0, 1, 2$, then the entry $M_{ij} \sim \tilde{M}_{ij}$ is expected to be $O(\epsilon^l)$, with $\epsilon = |u|$. Like in the previous case, the factors $\tilde{\rho}_i^{(c)}$ depend on weights.

3.2 Decomposition under residual symmetries

We have just seen that as τ departs from a symmetric value τ_{sym} — with ϵ parameterising the deviation — the zero entries of fermion mass matrices become $O(\epsilon^l)$. We now show that the exponents l are extracted from products of factors which correspond to representations of the residual symmetry group.

Matter fields ψ furnish 'weighted' representations $(\mathbf{r}, k)$ of the finite modular group Γ'_N. Whenever a residual symmetry is preserved by τ, fields decompose into unitary representations of the residual symmetry group. Modulo a possible $\mathbb{Z}_2^R$ factor, these are the cyclic groups $\mathbb{Z}_N^T$, $\mathbb{Z}_4^S$, and $\mathbb{Z}_3^{ST}$ (cf. section 2.1). A cyclic group $\mathbb{Z}_n \equiv \langle a \mid a^n = 1 \rangle$ has n inequivalent 1-dimensional irreps,

$$\mathbf{1}_k \ : \quad \rho(a) = \exp\left(2\pi i \frac{k}{n}\right) , \tag{17}$$

where $k = 0, \ldots, n-1$. For odd n, the only real irrep of $\mathbb{Z}_n$ is the trivial one, 1_0. For even n, there is one more real irrep, $1_{n/2}$. All other irreps are complex, with $(\mathbf{1}_k)^* = \mathbf{1}_{n-k}$.

To illustrate the decomposition of representations at symmetric points, take as an example a $(\mathbf{3}, k)$ triplet ψ of S'_4. It transforms under the unbroken $\gamma = ST$ at $\tau = \omega$ as

$$\psi_i \xrightarrow{ST} (-\omega - 1)^{-k} \rho_{\mathbf{3}}(ST)_{ij}\, \psi_j = \omega^k \rho_{\mathbf{3}}(ST)_{ij}\, \psi_j \,. \tag{18}$$

One can check that the eigenvalues of $\rho_{\mathbf{3}}(ST)$ are 1, ω and ω^2, and so in a suitable (ST-diagonal) basis the transformation rule explicitly reads

$$\psi \xrightarrow{ST} \omega^k \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} \psi = \begin{pmatrix} \omega^k & 0 & 0 \\ 0 & \omega^{k+1} & 0 \\ 0 & 0 & \omega^{k+2} \end{pmatrix} \psi \,, \tag{19}$$

which means that ψ decomposes as $\psi \rightsquigarrow \mathbf{1}_k \oplus \mathbf{1}_{k+1} \oplus \mathbf{1}_{k+2}$ under the residual $\mathbb{Z}_3^{ST}$.

One can similarly find the residual symmetry representations for any other 'weighted' multiplet:

- At $\tau = i\infty$, $\psi \sim (\mathbf{r}, k)$ transforms under the unbroken $\gamma = T$ as

  $$\psi_i \xrightarrow{T} \rho_{\mathbf{r}}(T)_{ij}\, \psi_j = \rho_i\, \psi_i \,, \tag{20}$$

  where we have assumed to be in a T-diagonal basis. The phase factors ρ_i correspond to the $\mathbb{Z}_N^T$ irreps into which ψ decomposes. It follows that each ρ_i is a power of $\zeta = \exp(2\pi i/N)$, depending on $\mathbf{r}$ but not on k.

- At $\tau = i$, $\psi \sim (\mathbf{r}, k)$ transforms under the unbroken $\gamma = S$ as

  $$\psi_i \xrightarrow{S} (-i)^{-k} \rho_{\mathbf{r}}(S)_{ij}\, \psi_j = i^k \rho_i\, \psi_i \,, \tag{21}$$

  in an S-diagonal basis. The phase factors $\tilde{\rho}_i = i^k \rho_i$ correspond to the $\mathbb{Z}_4^S$ irreps into which ψ decomposes. It follows that each $\tilde{\rho}_i$ is a power of i which depends both on $\mathbf{r}$ and on k (mod 4).

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- At $\tau = \omega$, $\psi \sim (\mathbf{r}, k)$ transforms under the unbroken $\gamma = ST$ as

$$\psi_i \xrightarrow{ST} (-\omega - 1)^{-k} \rho_{\mathbf{r}}(ST)_{ij}\, \psi_j = \omega^k \rho_i\, \psi_i\,, \tag{22}$$

in an ST-diagonal basis, as in the example of eq. (19). The phase factors $\tilde{\rho}_i = \omega^k \rho_i$ correspond to the $\mathbb{Z}_3^{ST}$ irreps into which ψ decomposes. It follows that each $\tilde{\rho}_i$ is a power of ω which depends both on $\mathbf{r}$ and on k (mod 3).

It follows from the above that, for $\tau \simeq i\infty$, the product $(\rho_i^c \rho_j)^*$ matches some power ζ^l with $0 \leq l < N$, while for $\tau \simeq \omega$ one has $\tilde{\rho}_i^c \tilde{\rho}_j = \omega^l$ with $l = 0, 1, 2$. These were tacitly taken as the most general possibilities in sections 3.1.1 and 3.1.3. For $\tau \simeq i$, it turns out that only two out of the four possibilities are viable, namely $l = 0, 2$ so that $\tilde{\rho}_i^c \tilde{\rho}_j = \pm 1$, as considered in section 3.1.2. This is due to the fact that $M(\tau)_{ij}$ is R-even and thus the fields ψ_i^c and ψ_j need to carry the same R-parity.

The decompositions of the weighted representations of Γ'_N $(N \leq 5)$ under the three residual symmetry groups have been collected in appendix A of Ref. [3].

3.3 Hierarchical structures

The results found so far allow us to construct hierarchical mass matrices in the vicinity of a symmetric point. Physical masses are the singular values of $M(\tau)$ and are also analytic functions of ϵ. To uncover the dependence of the physical spectrum on ϵ we make use of the following set of relations, valid for any $n \times n$ complex matrix M [8]:

$$\sum_{i_1 < \ldots < i_p} m_{i_1}^2 \ldots m_{i_p}^2 = \sum \left|\det M_{p\times p}\right|^2\,, \tag{23}$$

where $p = 1, \ldots, n$ is fixed, m_i are the singular values of M, and the sum on the right-hand side goes over all possible $p \times p$ submatrices $M_{p\times p}$ of M. For more details, see [3].

As an example, consider a model at level $N = 5$ with large $\operatorname{Im}\tau$ and matter fields $\psi \sim (\mathbf{3}, k)$ and $\psi^c \sim (\mathbf{3}', k^c)$. One has the decompositions $\psi \rightsquigarrow 1_0 \oplus \mathbf{1}_1 \oplus \mathbf{1}_4$ and $\psi^c \rightsquigarrow 1_0 \oplus \mathbf{1}_2 \oplus \mathbf{1}_3$ under the residual group at the symmetric point $\tau_{\text{sym}} = i\infty$. One can then identify $\rho_i = \operatorname{diag}(1, \zeta, \zeta^4)$ and $\rho_i^c = \operatorname{diag}(1, \zeta^2, \zeta^3)$, with $\zeta = \exp(2\pi i/5)$, and derive the power structure

$$M(\tau(\epsilon)) \sim \begin{pmatrix} 1 & \epsilon^4 & \epsilon \\ \epsilon^3 & \epsilon^2 & \epsilon^4 \\ \epsilon^2 & \epsilon & \epsilon^3 \end{pmatrix}, \quad \text{with } \epsilon = e^{-2\pi \operatorname{Im}\tau/5}\,, \tag{24}$$

which corresponds to a hierarchical $(1, \epsilon, \epsilon^4)$ spectrum. .

Note that $K = k + k^c$ must be large enough that sufficient modular forms contribute to $M(\tau)$. For instance, for $K = 2$ the superpotential may turn out to include a unique contribution:

$$W \supset \sum_s \alpha_s \left(Y_{\mathbf{5}}^{(5,2)}(\tau)\psi^c\psi\right)_{\mathbf{1},s} \quad \Rightarrow \quad M(\tau) = \alpha \begin{pmatrix} \sqrt{3}Y_1 & Y_5 & Y_2 \\ Y_4 & -\sqrt{2}Y_3 & -\sqrt{2}Y_5 \\ Y_3 & -\sqrt{2}Y_2 & -\sqrt{2}Y_4 \end{pmatrix}_{Y_{\mathbf{5}}^{(5,2)}}\,, \tag{25}$$

where the α_s are coupling constants, the sum is taken over all possible singlets s and $Y_{\mathbf{r}(,\mu)}^{(N,K)}$ denotes the modular form multiplet of level N, weight K and irrep $\mathbf{r}$, with μ possibly labelling

N	Γ'_N	Pattern	Sym. point	Viable $\mathbf{r}\otimes\mathbf{r}^c$
2	S_3	$(1,\epsilon,\epsilon^2)$	$\tau\simeq\omega$	$[\mathbf{2}\oplus\mathbf{1}^{(\prime)}]\otimes[\mathbf{1}\oplus\mathbf{1}^{(\prime)}\oplus\mathbf{1}']$
3	A'_4	$(1,\epsilon,\epsilon^2)$	$\tau\simeq\omega$	$[\mathbf{1}_a\oplus\mathbf{1}_a\oplus\mathbf{1}'_a]\otimes[\mathbf{1}_b\oplus\mathbf{1}_b\oplus\mathbf{1}''_b]$
			$\tau\simeq i\infty$	$[\mathbf{1}_a\oplus\mathbf{1}_a\oplus\mathbf{1}'_a]\otimes[\mathbf{1}_b\oplus\mathbf{1}_b\oplus\mathbf{1}''_b]$ with $\mathbf{1}_a\neq(\mathbf{1}_b)^*$
4	S'_4	$(1,\epsilon,\epsilon^2)$	$\tau\simeq\omega$	$[\mathbf{3}_a$, or $\mathbf{2}\oplus\mathbf{1}^{(\prime)}$, or $\hat{\mathbf{2}}\oplus\hat{\mathbf{1}}^{(\prime)}]\otimes[\mathbf{1}_b\oplus\mathbf{1}_b\oplus\mathbf{1}'_b]$
		$(1,\epsilon,\epsilon^3)$	$\tau\simeq i\infty$	$\mathbf{3}\ \otimes[\mathbf{2}\oplus\mathbf{1}$, or $\mathbf{1}\oplus\mathbf{1}\oplus\mathbf{1}']$, $\mathbf{3}'\otimes[\mathbf{2}\oplus\mathbf{1}'$, or $\mathbf{1}\oplus\mathbf{1}'\oplus\mathbf{1}']$, $\hat{\mathbf{3}}'\otimes[\hat{\mathbf{2}}\oplus\hat{\mathbf{1}}$, or $\hat{\mathbf{1}}\oplus\hat{\mathbf{1}}\oplus\hat{\mathbf{1}}']$, $\hat{\mathbf{3}}\ \otimes[\hat{\mathbf{2}}\oplus\hat{\mathbf{1}}'$, or $\hat{\mathbf{1}}\oplus\hat{\mathbf{1}}'\oplus\hat{\mathbf{1}}']$
5	A'_5	$(1,\epsilon,\epsilon^4)$	$\tau\simeq i\infty$	$\mathbf{3}\otimes\mathbf{3}'$

Table 1: Hierarchical mass patterns which can be realised in the vicinity of symmetric points. Subscripts run over irreps of a certain dimension, and $\mathbf{1}'''_a=\mathbf{1}_a$ for $N=3$, while $\mathbf{1}''_a=\mathbf{1}_a$ for $N=4$.

linearly independent multiplets of the same type (Y_i are the corresponding components). At leading order in $\epsilon=|q|$, one has $(Y_1,Y_2,Y_3,Y_4,Y_5)\simeq\left(-1/\sqrt{6},\,q,\,3q^2,\,4q^3,\,7q^4\right)$ up to normalisation and the power structure indeed matches that of eq. (24). However, one can check that the determinant of M vanishes identically for any τ and the spectrum is $\sim(1,\epsilon,0)$, with one massless fermion. This issue is solved at weight $K=4$. Then, the multiplets $Y^{(5,4)}_{\mathbf{4}}$, $Y^{(5,4)}_{\mathbf{5},1}$, and $Y^{(5,4)}_{\mathbf{5},2}$ are available and the spectrum is indeed of the type $(1,\epsilon,\epsilon^4)$ without a massless fermion. While the ϵ power-counting in eq. (25) may resemble that of a Froggatt-Nielsen mechanism [9], our framework is unrelated and can be regarded as an improvement. Instead of having an unknown $O(1)$ coefficient for each mass matrix entry, entries depend only on τ and a limited number of superpotential parameters.

We are interested in identifying all possible 3×3 hierarchical mass matrices arising from the described mechanism for $N\leq5$. We scan over representations $\mathbf{r}$ and $\mathbf{r}^c$, rejecting spectra with massless fermions. In the reducible case, the same weight and the same $\rho(R)$ is shared across the decomposition. For $\tau\simeq i$ the hierarchical pattern cannot be produced solely as a consequence of the smallness of ϵ, since mass matrix entries are either $O(1)$ or $O(\epsilon)$. The full results of the scan are given in appendix B of Ref. [3]. It is only possible to obtain *hierarchical* spectra for a small list of representation pairs, the most promising of which are collected in Table 1. We have excluded from this summary table reducible representations made up of three copies of the same singlet, as in those cases the number of superpotential parameters is unappealingly high.

4. Charged-lepton masses and large lepton mixing without fine-tuning

4.1 Viable PMNS matrix in the symmetric limit

Inspired by the above results, we have searched and built viable and predictive S'_4 and A'_5 lepton flavour models, see section 3.4 of [3]. In these models, the slightly-broken residual symmetry allows to successfully produce hierarchical charged-lepton masses without tuning the corresponding couplings. However, tuning is still present in the neutrino sector, as residual symmetries constrain the PMNS matrix, forcing some of its entries to be zero. It is known that only a limited number of

N	Γ'_N	Pattern	Sym. point	Viable $\mathbf{r}_{E^c} \otimes \mathbf{r}_L$	Property
2	S_3	$(1, \epsilon, \epsilon^2)$	$\tau \simeq \omega$	$[\mathbf{2} \oplus \mathbf{1}^{(\prime)}] \otimes [\mathbf{1} \oplus \mathbf{1}^{(\prime)} \oplus \mathbf{1}']$	1 or 4
			$\tau \simeq \omega$	$[\mathbf{1}_a \oplus \mathbf{1}_a \oplus \mathbf{1}'_a] \otimes [\mathbf{1}_b \oplus \mathbf{1}_b \oplus \mathbf{1}''_b]$	2
3	A'_4	$(1, \epsilon, \epsilon^2)$	$\tau \simeq i\infty$	$[\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}'] \otimes [\mathbf{1}'' \oplus \mathbf{1}'' \oplus \mathbf{1}']$, $[\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}''] \otimes [\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}'']$	2
4	S'_4	$(1, \epsilon, \epsilon^2)$	$\tau \simeq \omega$	$[\mathbf{3}_a$, or $\mathbf{2} \oplus \mathbf{1}^{(\prime)}$, or $\hat{\mathbf{2}} \oplus \hat{\mathbf{1}}^{(\prime)}] \otimes [\mathbf{1}_b \oplus \mathbf{1}_b \oplus \mathbf{1}'_b]$	1 or 4
5	A'_5	–	–	–	–

Table 2: Hierarchical charged-lepton mass patterns which may be realised in the vicinity of symmetric points without fine-tuned mixing (PMNS close to the observed one in the symmetric limit).

flavour symmetry representation choices for lepton fields L and E^c may give rise to a viable PMNS matrix in the symmetric limit [10]. Viability in our case means that either none of its entries vanish, or only the (13) entry vanishes as $\epsilon \to 0$. A modular-symmetric model of lepton flavour with hierarchical charged-lepton masses may be free of fine-tuning if it satisfies any of the properties [3]:

1. $L \rightsquigarrow 1 \oplus 1 \oplus 1$, $E^c \rightsquigarrow 1 \oplus r$, where 1 is some real singlet and r is some (possibly reducible) representation such that $r \not\supset 1$;

2. $L \rightsquigarrow \mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}^*$, $E^c \rightsquigarrow \mathbf{1}^* \oplus r$, where $\mathbf{1}$ is some complex singlet, $\mathbf{1}^*$ is its conjugate, and r is some (possibly reducible) representation such that $r \not\supset \mathbf{1}, \mathbf{1}^*$.

3. all charged-lepton masses vanish in the symmetric limit, i.e. the corresponding hierarchical pattern involves only positive powers of ϵ, e.g. $(\epsilon, \epsilon^2, \epsilon^3)$;

4. all light neutrino masses vanish in the symmetric limit, i.e. L decomposes into three (possibly identical) complex singlets none of which are conjugated to each other.

Applying this filter to the promising hierarchical cases of Table 1, one is left with the representation pairs of Table 2. Note there is no surviving possibility for $A_5^{(\prime)}$.

4.2 Scan of predictive S'_4 models with $\tau \simeq \omega$

Finally, we consider the most structured surviving cases within Table 2. These arise for S'_4, $\tau \simeq \omega$ and E^c and L being a triplet and the direct sum of three singlets, respectively. The expected charged-lepton spectrum is $(1, \epsilon, \epsilon^2)$. We have performed a systematic scan restricting ourselves to promising models involving the minimal number of effective parameters (9, including $\operatorname{Re}\tau$ and $\operatorname{Im}\tau$). Right-handed neutrino fields N^c are present since Weinberg dimension-5 operator models require more parameters. Aiming at minimal and predictive models, we impose a generalised CP symmetry enforcing the reality of coupling constants [2]. Out of 48 models, we have identified the only one which is viable and not fine-tuned, and is consistent with the 2σ range for the Dirac CPV phase, predicting $\delta \simeq \pi$. For this model, $L = L_1 \oplus L_2 \oplus L_3$ with $L_1, L_2 \sim (\hat{\mathbf{1}}, 2)$, $L_3 \sim (\hat{\mathbf{1}}', 2)$,

$E^c \sim (\hat{\mathbf{3}}, 4)$ and $N^c \sim (\mathbf{3}', 1)$. The corresponding superpotential reads:

$$\begin{aligned} W = &\Big[\alpha_1 \left(Y^{(4,6)}_{\mathbf{3}',1} E^c L_1\right)_{\mathbf{1}} + \alpha_2 \left(Y^{(4,6)}_{\mathbf{3}',2} E^c L_1\right)_{\mathbf{1}} + \alpha_3 \left(Y^{(4,6)}_{\mathbf{3}',1} E^c L_2\right)_{\mathbf{1}} + \alpha_4 \left(Y^{(4,6)}_{\mathbf{3}',2} E^c L_2\right)_{\mathbf{1}} + \alpha_5 \left(Y^{(4,6)}_{\mathbf{3}} E^c L_3\right)_{\mathbf{1}}\Big] H_d \\ &+ \Big[g_1 \left(Y^{(4,3)}_{\hat{\mathbf{3}}} N^c L_1\right)_{\mathbf{1}} + g_2 \left(Y^{(4,3)}_{\hat{\mathbf{3}}} N^c L_2\right)_{\mathbf{1}} + g_3 \left(Y^{(4,3)}_{\hat{\mathbf{3}}'} N^c L_3\right)_{\mathbf{1}}\Big] H_u + \Lambda \left(Y^{(4,2)}_{\mathbf{2}} (N^c)^2\right)_{\mathbf{1}} . \end{aligned} \tag{26}$$

Since L_1 and L_2 are indistinguishable, one can set $\alpha_2 = 0$ without loss of generality.

At leading order in a small parameter $|\epsilon|$, with $\epsilon \equiv 1 - \frac{1+\sqrt{3}}{1-i}\frac{\varepsilon}{\theta}$ and $|\epsilon| \simeq 2.8 \left|\frac{\tau-\omega}{\tau-\omega^2}\right|$ in the context of this section,[1] the charged-lepton mass matrix reads

$$M_e^\dagger \simeq -\frac{3(\sqrt{3}-1)^6}{\sqrt{13}} v_d \alpha_1 \theta^{12} \begin{pmatrix} 1 & \tilde\alpha_3 + \frac{\sqrt{13}}{2}\tilde\alpha_4 & \frac{i\sqrt{39}}{2}\tilde\alpha_5 \\ \sqrt{3}\,\epsilon & \sqrt{3}\left(\tilde\alpha_3 - \frac{\sqrt{13}}{2}\tilde\alpha_4\right)\epsilon & \frac{i\sqrt{13}}{2}\tilde\alpha_5\,\epsilon \\ \frac{5}{2}\,\epsilon^2 & \frac{1}{4}\left(10\tilde\alpha_3 + \sqrt{13}\tilde\alpha_4\right)\epsilon^2 & -\frac{5i\sqrt{13}}{4\sqrt{3}}\tilde\alpha_5\,\epsilon^2 \end{pmatrix}, \tag{27}$$

while the charged-lepton mass ratios follow the expected ϵ-pattern and are given by

$$\frac{m_e}{m_\mu} \simeq 2\frac{|\tilde\alpha_4\tilde\alpha_5|\sqrt{4+\left(2\tilde\alpha_3+\sqrt{13}\tilde\alpha_4\right)^2+39\tilde\alpha_5^2}}{3\tilde\alpha_4^2+\left[1+\left(\tilde\alpha_3-\sqrt{13}\tilde\alpha_4\right)^2\right]\tilde\alpha_5^2}\,|\epsilon|\,, \qquad \frac{m_\mu}{m_\tau} \simeq 4\sqrt{13}\frac{\sqrt{3\tilde\alpha_4^2+\left[1+\left(\tilde\alpha_3-\sqrt{13}\tilde\alpha_4\right)^2\right]\tilde\alpha_5^2}}{4+\left(2\tilde\alpha_3+\sqrt{13}\tilde\alpha_4\right)^2+39\tilde\alpha_5^2}\,|\epsilon|\,, \tag{28}$$

with $\tilde\alpha_i \equiv \alpha_i/\alpha_1$. Up to an overall normalisation $\mathcal{K}$, the light neutrino mass matrix is given by

$$M_\nu \simeq \mathcal{K}\,\epsilon \begin{pmatrix} 0 & 0 & \tilde g_3 \\ 0 & 0 & \tilde g_2 \tilde g_3 \\ \tilde g_3 & \tilde g_2 \tilde g_3 & 2i\sqrt{\frac{2}{3}}\tilde g_3^2 \end{pmatrix} \tag{29}$$

at leading order in $|\epsilon|$, where $\tilde g_i \equiv g_i/g_1$. The smallness of $|\epsilon|$ does not constrain the M_ν contribution to mixing, which depends only on the g_i, and large mixing angles are allowed. Note that there is a massless neutrino even though N^c is a triplet. The fit of the model yields ($N\sigma \simeq 0.563$):

$$\begin{aligned} &\frac{m_e}{m_\mu} = 0.00475^{+0.00061}_{-0.00052}\,, \quad \frac{m_\mu}{m_\tau} = 0.0556^{+0.0136}_{-0.0116}\,, \quad \Sigma m_\nu = 0.0588^{+0.0002}_{-0.0002}\ \text{eV}\,, \\ &\delta m^2 = 7.38^{+0.35}_{-0.44} \times 10^{-5}\ \text{eV}^2\,, \quad |\Delta m^2| = 2.48^{+0.05}_{-0.04} \times 10^{-3}\ \text{eV}^2\,, \quad r = 0.0298^{+0.00196}_{-0.0023}\,, \\ &\sin^2\theta_{12} = 0.304^{+0.039}_{-0.036}\,, \quad \sin^2\theta_{13} = 0.0221^{+0.0019}_{-0.002}\,, \quad \sin^2\theta_{23} = 0.539^{+0.0522}_{-0.099}\,, \\ &m_{\beta\beta} = 0.00144^{+0.00035}_{-0.00033}\ \text{eV}\,, \quad \frac{\delta}{\pi} = 1 \pm \mathcal{O}(10^{-6})\,, \quad \frac{\alpha}{\pi} = 1 \pm \mathcal{O}(10^{-5})\,. \end{aligned} \tag{30}$$

The viable region in the τ plane corresponds to a neutrino spectrum with NO and is located very close to $\tau_{\text{sym}} = \omega$, as can be seen from Figure 2. The annular form of the region is explained by the fact that the phase of $(\tau - \omega)$ has no effect on the observables, as it enters only through ϵ and its effects are suppressed by the smallness of $|\epsilon|$. Therefore, in the regime $\tau \simeq \omega$ this model is effectively described by 8 rather than 9 parameters:

$$\begin{aligned} &|\epsilon(\tau)| = 0.0186^{+0.0028}_{-0.0023}\,, \quad \tilde\alpha_3 = 2.45^{+0.44}_{-0.42}\,, \quad \tilde\alpha_4 = -2.37^{+0.36}_{-0.30}\,, \quad \tilde\alpha_5 = 1.01^{+0.06}_{-0.06}\,, \\ &\tilde g_2 = 1.5^{+0.15}_{-0.14}\,, \quad \tilde g_3 = 2.22^{+0.17}_{-0.15}\,, \quad v_d\,\alpha_1 = 4.61^{+1.32}_{-1.33}\ \text{GeV}\,, \quad \frac{v_u^2\, g_1}{\Lambda} = 0.268^{+0.057}_{-0.063}\ \text{eV}\,. \end{aligned} \tag{31}$$

[1] This local definition is motivated by the fact that $\varepsilon/\theta = (1-i)/(1+\sqrt{3})$ at $\tau = \omega$, with ε, θ defined in Ref. [7].

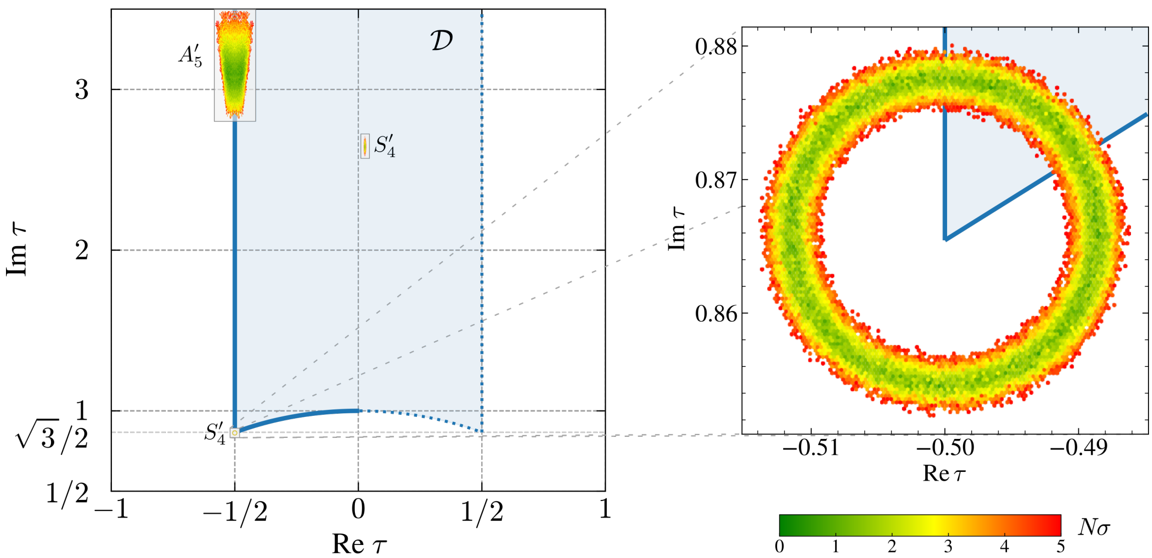

Figure 2: Allowed regions in the τ plane for the viable S'_4 and A'_5 lepton flavour models of section 3.4 of [3] and for the S'_4 model discussed here (left). The region corresponding to the latter is magnified (right).

5. Summary and Conclusions

In modular-invariant theories of flavour, hierarchical fermion masses may arise solely due to the proximity of the modulus to a point of residual symmetry $\tau_{\rm sym} = i, \omega$ or $i\infty$. In particular, if ϵ parameterises the deviation of τ from $\tau_{\rm sym}$ with $|\epsilon| \ll 1$, the degree of suppression of mass matrix elements is given by $|\epsilon|^l$ where l can take the values $l = 0, 1, \ldots, N-1$ if $\operatorname{Im}\tau$ is large; $l = 0, 1, 2$ if $\tau \simeq \tau_{\rm sym} = \omega$; or $l = 0, 1$ if $\tau \simeq \tau_{\rm sym} = i$. Here, N is the level of the finite modular group $\Gamma_N^{(\prime)}$. As shown, the specific value of l depends only on how the representations of the fermion fields entering the mass term bilinear decompose under the corresponding residual symmetry group.

Furthermore, we have found that it is only possible to obtain *hierarchical* spectra for a small list of representation pairs, the most promising of which correspond to the patterns $(1, \epsilon, \epsilon^2)$, $(1, \epsilon, \epsilon^3)$ and $(1, \epsilon, \epsilon^4)$, see Table 1. Having scanned these models, we found two viable ones based on S'_4 and A'_5, both in the 'vicinity' of $\tau_{\rm sym} = i\infty$, in which charged-lepton mass hierarchies arise naturally as a consequence of the described mechanism. However, a certain degree of fine-tuning is still required due to the need for large corrections to the symmetric-limit PMNS matrix. One may avoid it only if the model satisfies at least one of four conditions (see section 4.1). Accordingly, we have constructed and presented a viable model based on S'_4 modular symmetry with $\tau \simeq \omega$, which is free of fine-tuning in both the charged-lepton and neutrino sectors (see section 4.2). The charged-lepton mass pattern is predicted to be $(m_\tau, m_\mu, m_e) \sim (1, \epsilon, \epsilon^2)$ with $\epsilon \simeq 0.02$.[2]

These results demonstrate that the requirement of no fine-tuning in models based on modular invariance is remarkably restrictive. One hopes that such constraints may allow to identify not more than a few — if not just one — modular-invariant models providing a simultaneous, viable and appealing solution to the joint lepton and quark flavour puzzle.

[2]It was recently shown that values of τ corresponding to a deviation of $\epsilon \simeq 0.02$ from $\tau_{\rm sym} = \omega$ as required by the fit, cf. eq. (31), naturally arise in simple SUGRA-motivated potentials for the modulus [11].

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