PROCEEDINGS OF SCIENCE

Uses of Killing and Killing-Yano Tensors

Ulf Lindström[a,b,*] **and Özgür Sarıoğlu**[b]

[a] *Theoretical Physics, Dept. of Physics and Astronomy, Uppsala University, SE-751 20 Uppsala, Sweden*

[b] *Department of Physics, Faculty of Arts and Sciences, Middle East Technical University, 06800, Ankara, Turkey*

E-mail: ulf.lindstrom@physics.uu.se, sarioglu@metu.edu.tr

In this contribution we have collected some facts about Killing and Killing-Yano tensors that we feel are of general interest for researchers working on problems that rely on differential geometry. We also include some of our recent studies pertaining to currents, charges and (super)invariants for particles and tensionless strings.

Corfu Summer Institute 2021 "School and Workshops on Elementary Particle Physics and Gravity"
29 August - 9 October 2021
Corfu, Greece

*Speaker

PoS(CORFU2021)149

Contents

1. Introduction

Killing tensors, Killing-Yano forms and Killing-Yano tensors have many uses: Killing tensors correspond to "hidden" symmetries of bosonic models [1–3]. They are also instrumental for separating variables in General Relativity [4, 5] and string equations [6], and are used in the study of G-structures [7, 8]. Killing tensors characterise the symmetries of Laplacians [9]. Killing-Yano tensors square to Killing tensors and characterise the symmetries of the Dirac equation [10] and are also related to novel supersymmetries in sigma models and strings [11, 12]. Supersymmetric Killing-Yano tensors [13] characterise the symmetries of super Laplacians [14–16]. Finally, KTs arise in the context of hyperKähler geometry [17]. Of particular interest in the present context is the relation of Killing-Yano tensors to asymptotic conserved charges [18, 19]. In this presentation we will give lightning reviews of a few of these topics and report on some new results [20, 21] and [22].

The paper is organised as follows: Section 2 contains the definition of Killing tensors and illustrations of their use in constructing invariants along geodesics, separating variables and their role in determining symmetries of the Laplacian. Section 3 contains the definition of Killing-Yano forms and illustrations of their use for constructing rank two Killing tensors, conserved currents, and asymptotic charges. In three dimensions we introduce a current based on the Cotton tensor and display its form in superspace as well as in ordinary space. Section 4 contains the definition of Killing-Yano tensors with both symmetric and antisymmetric sets of indices and an illustration of how they are used to find invariants of spinning particles and for spinning tensionless strings.

2. Killing tensors

In this section we introduce Killing tensors (KTs), their generalisation to conformal Killing tensors (CKTs) and illustrate their usefulness in a couple of examples.

An nth rank Killing tensor is a completely symmetric tensor $f^{\mu_1\cdots\mu_n}$ that satisfies the equation

$$\nabla_{(\mu_1} f_{\mu_2\ldots\mu_{n+1})} = 0 \,, \tag{1}$$

whereas a conformal Killing tensor is trace-free and satisfies

$$\nabla_{(\mu_1} f_{\mu_2\ldots\mu_{n+1})} = n g_{(\mu_1\mu_2} \bar{f}_{\mu_3\ldots\mu_{n+1})} \,, \tag{2}$$

where the rank $n-1$ tensor $\bar{f}$ is determined by tracing both sides:

$$\bar{f}_{\mu_1\ldots\mu_{n-1}} = \frac{1}{D+2(n-1)} \nabla_\nu f^\nu{}_{\mu_1\ldots\mu_{n-1}} \,. \tag{3}$$

2.1 Uses of KTs

In this subsection we indicate some of the most important uses of KTs.

Conservation along geodesics

Let $x^\mu(\tau)$ be a geodesic, an over dot denote τ-derivative and let $p^\mu := \dot{x}^\mu$ be the tangent vector to the geodesic. The covariant directional derivative along the geodesic is denoted

$$\frac{D}{d\tau} = p^\mu \nabla_\mu \tag{4}$$

and the geodesic equation is

$$\frac{Dp^\mu}{d\tau} = 0 \,. \tag{5}$$

If f is an nth rank Killing tensor, the quantity

$$Q = f_{\mu_1\ldots\mu_n} p^{\mu_1} \ldots p^{\mu_n} \tag{6}$$

is then conserved along the geodesic [23],

$$\begin{aligned} \frac{D}{d\tau} Q &= p^\mu \nabla_\mu Q = p^\mu \nabla_\mu f_{\mu_1\ldots\mu_n} p^{\mu_1} \ldots p^{\mu_n} \\ &= \nabla_{(\mu} f_{\mu_1\ldots\mu_n)} p^\mu p^{\mu_1} \ldots p^{\mu_n} = 0 \,, \end{aligned} \tag{7}$$

where we used (5) and (1). Note that this is a purely geometric construction. Since the equations of motion of point-particles are geodesics it can be used to construct invariants for particles moving in geometries that allow KTs. It can also be extended to provide invariants for spinning particles [13]

PoS(CORFU2021)149

and for tensionless strings [22].

Separation of variables

Here we briefly describe one of the most important uses of KTs, for separation of variables. The literature on this is vast, see, e.g., [1, 6]. Below we sketch the logic following the presentation in [6].

Let $S = W(X) + \frac{1}{2}t\mu^2$ be Hamilton's principal function and W his characteristic function, for a system with Hamiltonian $H = \frac{1}{2}pg^{-1}p$. The Hamilton-Jacobi equation

$$H(X, \frac{\partial S}{\partial q}, t) + \frac{\partial S}{\partial t} = 0 \tag{8}$$

then reads

$$g^{\mu\nu}\partial_\mu W \partial_\nu W + \mu^2 = 0 \,, \tag{9}$$

where μ is a constant. Assume that the coordinates (X) can be divided into two distinct groups denoted by (x) and (y). If the following conditions are satisfied,

$$\begin{aligned} &W = W_x(x_1, \ldots x_k) + W_y(y_{k+1}\ldots y_n) \\ &g^{\mu\nu} = \frac{X^{\mu\nu}(x) + Y^{\mu\nu}(y)}{f_x - f_y} \,, \quad X^{y\nu} = 0 \,, \quad Y^{x\nu} = 0 \,, \\ &\partial_y f_x = 0, \quad \partial_x f_y = 0 \,, \end{aligned} \tag{10}$$

then (9) separates as

$$X^{\mu\nu}\partial_\mu W \partial_\nu W + \mu^2 f_x = -Y^{\mu\nu}\partial_\mu W \partial_\nu W + \mu^2 f_y \tag{11}$$

with the left hand side a function of x only and the right hand side a function of only y. Applying the usual argument that the two sides must be separately constant, we find the following integral of the motion

$$I = X^{\mu\nu}\partial_\mu W_x \partial_\nu W_x + \mu^2 f_x = \left(X^{\mu\nu} - g^{\mu\nu} f_x\right)\partial_\mu W_y \partial_\nu W_y \,. \tag{12}$$

It can then be shown that

$$I = -\Big(\frac{f_y X^{\mu\nu} + f_x Y^{\mu\nu}}{f_x - f_y}\Big)\partial_\mu W \partial_\nu W =: K^{\mu\nu}\partial_\mu W \partial_\nu W \tag{13}$$

with $K^{\mu\nu}$ a second rank KT.

The key to the usefulness of this relation is that the converse is also true, as described in [6]: Given a second rank KT, one can use it to find the separation (11).

Symmetries of the Laplacian

A symmetry of the Laplacian Δ is a linear differential operator $\mathcal{D} \neq \Delta$ such that [9]

$$\Delta\mathcal{D} = \delta\Delta \tag{14}$$

for some linear differential operator δ. Any linear differential operator on a Riemannian manifold may be written in the form

$$D = V^{\mu\nu\ldots\rho}\nabla_\mu\nabla_\nu\ldots\nabla_\rho + \text{lower order terms}\;, \tag{15}$$

where $V^{\mu\nu\ldots\rho}$ is symmetric in its indices. This tensor is called *the symbol* of D.

It is shown in [9] that any symmetry $\mathcal{D}$ of the Laplacian on a Riemannian manifold is equivalent to one whose symbol is a CKT f:

$$\mathcal{D} = f^{\mu\nu\ldots\rho}\nabla_\mu\nabla_\nu\ldots\nabla_\rho + \text{lower order terms.} \tag{16}$$

Similar results hold for the Dirac operator [10] and in superspace for super-Laplacians [14].

Having exemplified the uses of KTs, we now turn to Killing-Yano and conformal Killing-Yano forms[1].

3. Killing-Yano forms

Killing-Yano forms (KYFs) are generalisations of KTs to antisymmetric covariant tensors.

An nth rank KYF is an n-form k that satisfies

$$\nabla_{(\mu_1}k_{\mu_2)\ldots\mu_{n+1}} = 0\;, \tag{17}$$

or, equivalently,

$$\nabla_{\mu_1}k_{\mu_2\ldots\mu_{n+1}} = \nabla_{[\mu_1}k_{\mu_2\ldots\mu_{n+1}]}\;. \tag{18}$$

A conformal Killing-Yano form (CKYF) satisfies

$$\nabla_{\mu_1}\mathscr{k}_{\mu_2\ldots\mu_{n+1}} = \nabla_{[\mu_1}\mathscr{k}_{\mu_2\ldots\mu_{n+1}]} + \frac{n}{D-n+1}g_{\mu_1[\mu_2}\bar{\mathscr{k}}_{\mu_3\ldots\mu_{n+1}]}\;, \tag{19}$$

where

$$\bar{\mathscr{k}}_{\mu_1\ldots\mu_{n-1}} = \nabla_\mu\mathscr{k}^\mu{}_{\mu_1\ldots\mu_{n-1}}\;. \tag{20}$$

Moreover, it is called *closed* if

$$\nabla_{[\mu_1}\mathscr{k}_{\mu_2\ldots\mu_{n+1}]} = 0\;. \tag{21}$$

[1]We refer to the totally antisymmetric versions as Killing-Yano forms and reserve the label Killing-Yano to tensors with mixed symmetries.

PoS(CORFU2021)149

3.1 Uses of KYFs

In this subsection we indicate some of the most important uses of KYFs.

The square of a KYF is a KT

In applications, it is often easier to find KYFs than KTs for a given geometry. It is then gratifying that KYFs square to second rank KTs:

Let $k_{\mu_1...\mu_n}$ be a KYF. Consider the second rank tensor

$$f_{\mu\nu} = k_{\mu\mu_2...\mu_n} k_\nu{}^{\mu_2...\mu_n} \; . \tag{22}$$

Then

$$\begin{aligned} \nabla_\sigma f_{\mu\nu} &= \nabla_{[\sigma} k_{\mu\mu_2...\mu_n]} k_\nu{}^{\mu_2...\mu_n} + k_\mu{}^{\mu_2...\mu_n} \nabla_{[\sigma} k_{\nu\mu_2...\mu_n]} \\ \Rightarrow \nabla_{(\sigma} f_{\mu\nu)} &= 0 \; . \end{aligned} \tag{23}$$

So $f_{\mu\nu}$ is a Killing tensor. This construction also produces rank 2 CKTs from two rank n CKYFs [24]. Note that when the CKYFs are different, we must explicitly symmetrise the free indices in (22).

Conserved currents

A covariantly conserved antisymmetric rank n tensor field J is equivalent to a co-closed n-form. By the Poincaré lemma, this means that it is equal to the co-derivative of an $(n+1)$-form ℓ in a simply-connected open set

$$\nabla^\mu J_{\mu...\mu_n} = 0 \Rightarrow J_{\mu_1...\mu_n} = \nabla^\mu \ell_{\mu\mu_1...\mu_n} \; . \tag{24}$$

This can be used to construct conserved charges for a given J. An interesting example is the Kastor-Traschen $\mathbb{KT}$ current [18]:

$$J^{\mu_1...\mu_n} = -\frac{(n-1)}{4} R^{[\mu_1\mu_2}{}_{\rho\sigma}\, k^{\mu_3...\mu_n]\rho\sigma} + (-1)^{n+1} R_\rho{}^{[\mu_1}\, k^{\mu_2...\mu_n]\rho} - \frac{1}{2n} R\, k^{\mu_1...\mu_n} \, , \tag{25}$$

where k is a KYF and the geometry is represented by the curvature tensor and its contractions. The covariant divergence of J vanishes due to Bianchi identities and the properties (18) of k.

In [20] we show how the $\mathbb{KT}$ current may be rewritten in terms of the Weyl and Schouten tensors and how it has several separately conserved constituents. Interestingly only the full current seems to allow for Abbott-Deser (AD) charges, to which we now turn.

AD charges

In the spirit of [25] one may construct asymptotic charges for the Kastor-Traschen $\mathbb{KT}$ current. If the metric has the asymptotic form

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} \tag{26}$$

and the background geometry defined by $\bar{g}_{\mu\nu}$ has a KYF $\bar{k}$, then the linearised $\mathbb{KT}$ current

$$J^{(L)\mu_1\ldots\mu_n} = \frac{(n-1)}{4} R_L^{[\mu_1\mu_2}{}_{\rho\sigma}\, \bar{k}^{\mu_3\ldots\mu_n]\rho\sigma} + (-1)^{n+1} R_{L\rho}{}^{[\mu_1}\, \bar{k}^{\mu_2\ldots\mu_n]\rho} - \frac{1}{2n} R_L\, \bar{k}^{\mu_1\ldots\mu_n} \tag{27}$$

will be background conserved for certain geometries such as asymptotically flat or asymptotically AdS ones. For $n = 2$, one finds

$$Q^{\mu\nu} \sim \int_\Sigma dS_i \sqrt{|\bar{\gamma}|}\, \bar{\ell}^{\mu\nu i} , \tag{28}$$

which then represents a conserved "charge". Note that this requires deriving the potential $\bar{\ell}^{\mu\nu\rho}$. This derivation prompted the development of some mathematical tools.

Identities

The study of identities and integrability conditions for KTs, CKTs, KYFs and CKYFs has a long history: [26–29]. Here we relate some recent results along this line of investigations.

In deriving the explicit relation $J^{(L)}_{\mu_1\ldots\mu_n} = \bar{\nabla}^\mu \bar{\ell}_{\mu\mu_1\ldots\mu_n}$ needed for construction of the charge, we use

$$\nabla_\mu \nabla_\nu k_{\rho_1\ldots\rho_n} = (-1)^{n+1} \frac{(n+1)}{2} R^\sigma{}_{\mu[\nu\rho_1}\, k_{\rho_2\ldots\rho_n]\sigma} , \tag{29}$$

which generalises the Killing vector relation $\nabla_\mu \nabla_\nu f_\rho = R^\sigma{}_{\mu\nu\rho}\, f_\sigma$. From this, we derive a number of new relations such as

$$R_{\mu\nu}\, k^{\mu\sigma} + R^{\mu\sigma}\, k_{\mu\nu} = 0 \tag{30}$$

for a rank 2 KYF. The corresponding identity for a general rank KYF k reads

$$\begin{aligned} &R^\mu{}_\nu\, k_{[\sigma_1\ldots\sigma_{n-1}]\mu} + (-1)^n R^\mu{}_{[\sigma_1}\, k_{\sigma_2\ldots\sigma_{n-1}]\nu\mu} \\ &+(n-2)\left((-1)^n R_{\nu\mu\lambda[\sigma_1}\, k_{\sigma_2\ldots\sigma_{n-1}]}{}^{\lambda\mu} + \frac{1}{2} R_{\lambda\mu[\sigma_1\sigma_2}\, k_{\sigma_3\ldots\sigma_{n-1}]\nu}{}^{\lambda\mu} \right) = 0 . \end{aligned} \tag{31}$$

Several other new identities are to be found in [20]. The following one, involving the Einstein tensor

$$G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R, \tag{32}$$

reads

$$k^{\mu\nu} \nabla_\mu G_{\nu\rho} = 0 . \tag{33}$$

The corresponding identity for a general rank KYF k reads

$$(n-1)\left(\nabla^\nu R^\mu{}_{[\sigma_1}\right) k_{\sigma_2\ldots\sigma_{n-1}]\mu\nu} + \frac{1}{2}\left(\nabla^\mu R\right) k_{\mu[\sigma_1\ldots\sigma_{n-1}]} = 0 . \tag{34}$$

We use (33) to prove that

$$K^{\mu\nu} = 2\, G_\rho{}^{[\mu}\, k^{\nu]\rho} \tag{35}$$

is a conserved "current". Note that in a given geometry this leads to a relation between the KYF and the energy-momentum tensor on shell. More details may be found in [20].

The identity (29) may be generalized to CKYTs (19) (as well as to geometries with torsion, see [21]). For a second rank CKYT $\mathcal{k}$, it reads

$$\nabla_\mu \nabla_\nu \mathcal{k}_{\rho\sigma} = -\frac{3}{2} R^\tau{}_{\mu[\nu\rho} \mathcal{k}_{\sigma]\tau} - \frac{3}{D-1} g_{\mu[\nu} \nabla_\rho \bar{\mathcal{k}}_{\sigma]} + \frac{2}{D-1} \nabla_\mu \left(g_{\nu[\rho} \bar{\mathcal{k}}_{\sigma]} \right) . \tag{36}$$

PCCKYF

The closed conformal Killing-Yano forms defined in (21) play an important role when $n = 2$ and it is non-degenerate as a matrix. It is then called a *principal closed conformal Killing-Yano tensor* and is the starting point for constructing a hierarchy of KYTs and KTs that can be used to characterise the solution. E.g., for the Kerr-NUT-(A)dS family this principal tensor generates a hierarchy of Killing vectors that ensures complete integrability of geodesic motion and separability of the Hamilton-Jacobi, Klein-Gordon, and Dirac equations, see [3].

Cotton current comments

Above we have focused on the $\mathbb{KT}$ current. Another current, the Killing-Yano Cotton current, was recently constructed in [21]. Here we briefly describe this current.

The Cotton tensor is defined in $D \geq 3$ dimensions as

$$C_{\mu\nu\rho} \equiv 2(D-2) \nabla_{[\rho} S_{\nu]\mu} = 2\nabla_{[\rho} R_{\nu]\mu} - \frac{1}{(D-1)} g_{\mu[\nu} \nabla_{\rho]} R \,. \tag{37}$$

It satisfies

$$\begin{aligned} C_{\mu\nu\rho} &= C_{\mu[\nu\rho]} \,, \\ C_{[\mu\nu\rho]} &= 0 \,, \\ \nabla^\mu C_{\mu\nu\rho} &= 0 \,. \end{aligned} \tag{38}$$

Recall that a second rank CKYF $\mathcal{k}$ satisfies

$$\nabla_\mu \mathcal{k}_{\nu\rho} = \nabla_{[\mu} \mathcal{k}_{\nu\rho]} + \frac{2}{D-1} g_{\mu[\nu} \bar{\mathcal{k}}_{\rho]} \,, \tag{39}$$

and consider

$$J^\mu \equiv C^{\mu\nu\rho} \mathcal{k}_{\nu\rho} \,. \tag{40}$$

That this is a conserved current follows from

$$\begin{aligned} \nabla_\mu J^\mu &= (\nabla_\mu C^{\mu\nu\rho}) \mathcal{k}_{\nu\rho} + C^{\mu\nu\rho} \nabla_\mu \mathcal{k}_{\nu\rho} \\ &= 0 + C^{\mu\nu\rho} (\nabla_{[\mu} \mathcal{k}_{\nu\rho]} + \frac{2}{D-1} g_{\mu[\nu} \bar{\mathcal{k}}_{\rho]}) = 0 \,, \end{aligned} \tag{41}$$

since C is conserved, symmetric and traceless. We define a charge for this current. Since

$$\nabla_\mu J^\mu = \frac{1}{\sqrt{|g|}} \partial_\mu \left(\sqrt{|g|}\, J^\mu \right) = 0 \,, \tag{42}$$

one can define a conserved charge Q as

$$Q \equiv \int_{\Sigma_\tau} d^{(D-1)} x \, J^\mu \, n_\mu \,, \tag{43}$$

with n^μ normal to the spacelike surface Σ. We apply this construction to the Plebański-Demiański metric where we can carry out the integration for certain values of the metric parameters in [21].

3D

In three dimensional topologically massive gravity, the Cotton tensor, or its descendant, the York tensor

$$C^{\mu\nu} = \frac{1}{\sqrt{|g|}}\epsilon^{\mu\sigma\rho}\nabla_\sigma S_\rho{}^\nu\,, \tag{44}$$

has a prominent role. In [30], it is used to define asymptotic charges using a current based on the energy-momentum tensor and an asymptotic Killing vector. In [21], we compare it to our Cotton current in 3D with some interesting results.

3D is also our starting point for lifting conserved currents to supergravity to which we now turn.

3D conformal supergravity

In this section we use Greek letters for spinor indices while vector indices are represented by Latin letters or, equivalently, by pairs of spinor indices.

In three dimensions, conformal supergravity may be defined by the following algebra of covariant derivatives [31, 32]

$$\begin{aligned}
&\{\nabla_\alpha, \nabla_\beta\} = 2i\nabla_{\alpha\beta}\,,\\
&[\nabla_a, \nabla_\alpha] = \tfrac{1}{4}(\gamma_a)_\alpha{}^\beta W_{\beta\gamma\delta}K^{\gamma\delta}\,,\\
&[\nabla_a, \nabla_b] = -\tfrac{i}{8}\epsilon_{abc}(\gamma^c)^{\alpha\beta}\nabla_\alpha W_{\beta\gamma\delta}K^{\gamma\delta} - \tfrac{1}{4}\epsilon_{abc}(\gamma^c)^{\alpha\beta}W_{\alpha\beta\gamma}S^\gamma\,.
\end{aligned} \tag{45}$$

The notation is that spinor indices are $\alpha, \beta, \ldots$, vector tangent space indices are $a, b, \ldots$. The usual convention that a vector index is represented by a symmetric pair of spinor indices also applies. Thus $K_{\alpha\beta}$ is the vector generator of special conformal transformations while S^γ generates S supersymmetry transformations.

The super Cotton tensor [32] W obeys

$$\begin{aligned}
&W_{\alpha\beta\gamma} = W_{(\alpha\beta\gamma)}\\
&\nabla^\alpha W_{\alpha\beta\gamma} = 0\\
&K_a W_{\alpha\beta\gamma} = 0\,,
\end{aligned} \tag{46}$$

where the last relation identifies W as a primary field.

The super Cotton current

Armed with these relations, we turn to the super Cotton current [21].

We take a superconformal Killing supervector field ξ to be given by

$$\xi = \xi^a\nabla_a + \xi^\alpha\nabla_\alpha\,, \tag{47}$$

with ξ^a a primary field. It follows that

$$\begin{aligned}
&\nabla_{(a}\xi_{b)} = \tfrac{1}{3}\eta_{ab}\nabla_c\xi^c\\
&\nabla^{\beta\gamma}\xi^\alpha = -\tfrac{2}{3}\epsilon^{\alpha(\beta}\nabla^{\gamma)}{}_\sigma\xi^\sigma\,,
\end{aligned} \tag{48}$$

where the next to last relation defines a conformal Killing vector and the last one a conformal Killing spinor.

We may now construct a supergravity version of our Cotton current. To this end, we define

$$
\begin{aligned}
k_\alpha &= W_{\alpha\beta\gamma}\xi^{\beta\gamma} \\
k_{\alpha\beta} &= \nabla_\alpha k_\beta = (\nabla_{(\alpha}W_{\beta)\gamma\delta})\xi^{\gamma\delta} + 4iW_{\alpha\beta\gamma}\xi^\gamma \; .
\end{aligned}
\tag{49}
$$

These satisfy

$$
\begin{aligned}
\nabla^\alpha k_\alpha &= 0 \\
\nabla^{\alpha\beta} k_{\alpha\beta} &= 0 \; ,
\end{aligned}
\tag{50}
$$

and the lowest component of the first part of $k_{\alpha\beta}$ is the bosonic Cotton current. In a more covariant form we have

$$
(k^A) = (k^{\alpha\beta}, k^\alpha) \tag{51}
$$

with

$$
\nabla_A k^A = 0 \; . \tag{52}
$$

We note that this construction opens the novel field of super invariants from super Killing tensors.

4. Killing-Yano tensors

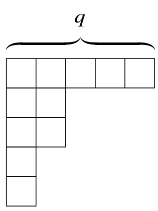
In this section we introduce the mixed KYTs and exemplify some of their uses. Here we only consider a Minkowski space background.

A product of a conformal Killing vector $\slashed{k}_a$ and a CKY 2-form $\slashed{k}_{\mu\nu}$ projected onto the highest weight representation gives an object

$$
A_{\mu,\nu\rho} := \slashed{k}_{\mu(\nu}\slashed{k}_{\rho)} + \frac{1}{(n-1)}\left(\eta_{\mu(\nu}(\slashed{k}\cdot\slashed{k})_{\rho)} - \eta_{\nu\rho}(\slashed{k}\cdot\slashed{k})_\mu\right) \; , \tag{53}
$$

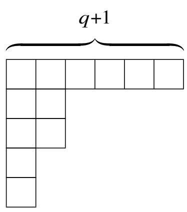
In [13] this construction is generalised[2] to tensors $A_{p_1,p_2,\ldots,q}$ with $p_1 \geq p_2 \geq \ldots$ being the number of boxes minus one in the columns starting from the left. For example,

$$
A_{p_1,p_2,q} \sim \quad , \tag{54}
$$



where there are $p_1 + 1$ boxes in the first column and $p_2 + 1$ in the second. The differential constraint satisfied by this tensor is

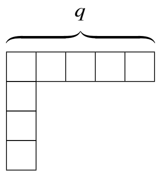
$$
\partial A_{p_1,p_2,q} \ni \quad = 0 \; . \tag{55}
$$



[2]There is a different generalisation of KYFs in [33].

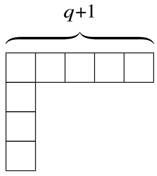
For the particular case $A_{p,q}$ the Young tableau is

$$A_{p,q} \sim \quad (56)$$



with $(p+1)$ boxes in the first column. The differential constraint satisfied by such a CKYT is that, when a derivative is applied to $A_{p,q}$, the traceless tensor corresponding to the Young tableau with one extra box on the first row has to vanish, i.e.,

$$\partial A_{p,q} \ni \quad = 0 \,. \quad (57)$$



This kind of tensors appears naturally in the context of the spinning particle [11, 34].

4.1 Uses of KYTs

These Killing-Yano tensors can, e.g., be used for finding invariants of spinning particles and for spinning tensionless strings.

Spinning Particle

In [21] invariants for the spinning particle are constructed using $A_{p,q}$ CKYTs[3]. It is shown that invariants take the form

$$F = \lambda^p A_{p,q} p^q + \alpha(p,q)\lambda^{p+2} dA_{p+2,q-1} p^{q-1} := A + dA \,, \quad (58)$$

where $A_{p,q}$ is in the representation (56), satisfies the constraint (57) and

$$\alpha(p,q) := i\frac{(-1)^{(p+1)}q}{(1+p+q)} \,, \quad (59)$$

$$\left(dA_{p+2,q-1}\right)_{\nu_1\ldots\nu_{p+2},\mu_1\ldots\mu_{q-1}} := \partial_{[\nu_1} A_{\nu_2\ldots\nu_{p+1},\nu_{p+2}]\mu_1\ldots\mu_{q-1}} \,. \quad (60)$$

Spinning tensionless string

In [22] we take advantage of the close relation between the spinning tensionless string [36] and the spinning particle [37]: In a particular gauge the spinning tensionless string is a collection of spinning particles obeying certain constraints. We are then able to generalise the construction of invariants for the spinning particle to the spinning tensionless string.

5. Summary

We have given the definitions of KTs, CKTs, KYFs, CKYFs and KYTs and listed a number of applications for each of these. In addition, we have presented some recent results on conserved currents, (asymptotic)

[3]Integrability for the spinning particle is discussed in a different context in [35].

charges, nontrivial identities and their applications to spinning particles and spinning tensionless strings. In particular, the extension of conserved currents to $3D$ supergravity represents a novel and promising line of research.

Acknowledgments
We thank Sergei Kuzenko for commenting on the supergravity part and David Kubiznak for comments and references. The research of U.L. is supported in part by the 2236 Co-Funded Scheme2 (CoCirculation2) of TÜBİTAK (Project No:120C067)[4].

References

[1] O. P. Santillan, "Hidden symmetries and supergravity solutions," *J. Math. Phys.* **53** (2012) 043509; arXiv:1108.0149 [hep-th].

[2] V. P. Frolov and D. Kubiznak, "Higher-Dimensional Black Holes: Hidden Symmetries and Separation of Variables," *Class. Quant. Grav.* **25** 154005 (2008); arXiv:0802.0322 [hep-th].

[3] V. Frolov, P. Krtous and D. Kubiznak, "Black holes, hidden symmetries, and complete integrability," *Living Rev. Rel.* **20**, no.1, 6 (2017); arXiv:1705.05482 [gr-qc].

[4] B. Carter, "Global Structure of the Kerr Family of Gravitational Fields," *Phys. Rev.* **174** (1968) 1559.

[5] M. Walker and R. Penrose, "On quadratic first integrals of the geodesic equations for type {22} spacetimes," *Commun. Math. Phys.* **18** (1970) 265-274.

[6] Y. Chervonyi and O. Lunin, "Killing(-Yano) Tensors in String Theory," *JHEP* **09** (2015) 182; arXiv:1505.06154 [hep-th].

[7] G. Papadopoulos, "Killing-Yano equations and G-structures," *Class. Quant. Grav.* **25** (2008) 105016; arXiv:0712.0542 [hep-th].

[8] G. Papadopoulos, "Killing-Yano equations with torsion, worldline actions and G-structures," *Class. Quant. Grav.* **29** (2012) 115008; arXiv:1111.6744 [hep-th].

[9] M. G. Eastwood, "Higher symmetries of the Laplacian," *Annals Math.* **161** (2005), 1645-1665; arXiv:hep-th/0206233 [hep-th].

[10] M. Cariglia, "Hidden Symmetries of the Dirac Equation in Curved Space-Time," *In: Bičák J., Ledvinka T. (eds) Relativity and Gravitation. Springer Proceedings in Physics, vol 157. Springer, Cham.* **157** (2014) 25; arXiv:1209.6406 [gr-qc].

[11] G. W. Gibbons, R. H. Rietdijk and J. W. van Holten, "Susy in the sky," *Nucl. Phys. B* **404** (1993) 42-64; arXiv:hep-th/9303112 [hep-th].

[12] F. De Jonghe, K. Peeters and K. Sfetsos, "Killing-Yano supersymmetry in string theory," *Class. Quant. Grav.* **14** (1997) 35-46; arXiv:hep-th/9607203 [hep-th].

[13] P. S. Howe and U. Lindström, "Some remarks on (super)-conformal Killing-Yano tensors," *JHEP* **11** (2018) 049; arXiv:1808.00583 [hep-th].

[4]However the entire responsibility for the publication is ours. The financial support received from TÜBİTAK does not mean that the content of the publication is approved in a scientific sense by TÜBİTAK.

[14] P. S. Howe and U. Lindström, "Super-Laplacians and their symmetries," *JHEP* **05** (2017), 119; arXiv:1612.06787 [hep-th].

[15] S. M. Kuzenko, U. Lindström, E. S. N. Raptakis and G. Tartaglino-Mazzucchelli, "Symmetries of $\mathcal{N}$ = (1, 0) supergravity backgrounds in six dimensions," *JHEP* **03** (2021), 157; arXiv:2012.08159 [hep-th].

[16] S. M. Kuzenko and E. S. N. Raptakis, "Symmetries of supergravity backgrounds and supersymmetric field theory," *JHEP* **04** (2020), 133; arXiv:1912.08552 [hep-th].

[17] U. Lindström and M. Roček, "Properties of hyperkähler manifolds and their twistor spaces," *Commun. Math. Phys.* **293** (2010) 257; arXiv:0807.1366 [hep-th].

[18] D. Kastor and J. Traschen, "Conserved gravitational charges from Yano tensors," *JHEP* **08** (2004) 045; arXiv:hep-th/0406052 [hep-th].

[19] S. Ölmez, Ö. Sarıoğlu, and B. Tekin, "Mass and angular momentum of asymptotically ads or flat solutions in the topologically massive gravity," *Class. Quant. Grav.* **22**, 4355-4362 (2005); arXiv:gr-qc/0507003 [gr-qc].

[20] U. Lindström and Ö. Sarıoğlu, "New currents with Killing-Yano tensors," *Class. Quant. Grav.* **38** no.19, 195011 (2021); arXiv:2104.12451 [hep-th].

[21] U. Lindström and Ö. Sarıoğlu, "Killing-Yano Cotton Currents," arXiv:2110.03470 [hep-th]. Accepted for publication in JHEP.

[22] U. Lindström and Ö. Sarıoğlu, "Tensionless Strings and Killing(-Yano) Tensors," arXiv:2202.06542 [hep-th].

[23] R. Penrose and W. Rindler, SPINORS AND SPACE-TIME, VOL. 2: SPINOR AND TWISTOR METHODS IN SPACE-TIME GEOMETRY, Cambridge Univ. Press (1986).

[24] T. Houri, D. Kubiznak, C. M. Warnick and Y. Yasui, "Generalized hidden symmetries and the Kerr-Sen black hole," *JHEP* **07** (2010), 055; arXiv:1004.1032 [hep-th].

[25] L. F. Abbott and S. Deser, "Stability of Gravity with a Cosmological Constant," *Nucl. Phys. B* **195** (1982) 76-96.

[26] T. Kashiwada, "On conformal Killing tensor," *Nat. Sci. Rep. Ochanomizu Univ.* **19** (1968) no.2, 67-74.

[27] S. Tachibana and T. Kashiwada, "On the integrability of Killing-Yano's equation," *J. Math. Soc. Japan* **21** (1969) No.2, 259-265.

[28] S. Tachibana, "On conformal Killing tensor in a Riemannian space", *Tôhoku Math. Journ.* **21** (1969), 56-64.

[29] C. Batista, "Integrability Conditions for Killing-Yano Tensors and Conformal Killing-Yano Tensors," *Phys. Rev. D***91**, 024013 (2015); arXiv:1406.3069 [gr-qc].

[30] S. Deser and B. Tekin, "Energy in topologically massive gravity," *Class. Quant. Grav.* **20**, L259 (2003); arXiv:gr-qc/0307073 [gr-qc].

[31] D. Butter, S. M. Kuzenko, J. Novak and G. Tartaglino-Mazzucchelli, "Conformal supergravity in three dimensions: New off-shell formulation," *JHEP* **09** (2013), 072; arXiv:1305.3132 [hep-th].

[32] S. M. Kuzenko and G. Tartaglino-Mazzucchelli, "Conformal supergravities as Chern-Simons theories revisited," *JHEP* **03** (2013), 113; `arXiv:1212.6852 [hep-th]`.

[33] J. Kress, "Generalised Conformal Killing-Yano Tensors: Applications to Electrodynamics," `Doctoral Thesis`, University of Newcastle, Dept. of Math. (1997).

[34] J. W. van Holten and R. H. Rietdijk, "Symmetries and motions in manifolds," *J. Geom. Phys.* **11** (1993) 559; `arxiv:hep-th/9205074`.

[35] D. Kubiznak and M. Cariglia, "Integrability of spinning particle motion in higher-dimensional black hole spacetimes," *Phys. Rev. Lett.* **108**, 051104 (2012); `arXiv:1110.0495 [hep-th]`.

[36] U. Lindström, B. Sundborg and G. Theodoridis, "The zero tension limit of the spinning string," *Phys. Lett. B* **258** (1991) 331-334.

[37] P. S. Howe, S. Penati, M. Pernici and P. K. Townsend, "Wave Equations for Arbitrary Spin From Quantization of the Extended Supersymmetric Spinning Particle," *Phys. Lett. B* **215** (1988) 555-558.

PoS(CORFU2021)149