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Properties of Non-relativistic Neveu-Schwarz Gravity

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We show how the common low-energy effective action of the different non-relativistic string theories, called non-relativistic Neveu-Schwarz gravity, can be obtained by taking a particular limit of the relativistic low-energy effective action. We discuss some distinguishing features of this non-relativistic Neveu-Schwarz gravity theory.

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1. Introduction

To explain the idea behind this talk, it is useful to consider the so-called Bronstein cube of physical theories [1] given in in Figure 1. Starting from classical mechanics, one can extend in three different directions by either adding special relativity, gravity or quantum mechanics. Each extension introduces a constant of Nature that is absent in classical mechanics: (1) at large velocities with respect to the velocity of light c classical mechanics extends to special relativity; (2) a gravitational force can be introduced via Newton's constant G leading to Newtonian gravity and (3) at small distances certain physical quantities get quantized in units of the reduced Planck's constant $\hbar$ corresponding to quantum mechanics. Combining two of these extensions one can obtain general relativity or quantum field theory. To be precise, extending classical mechanics with high velocities and gravity leads to general relativity while extending classical mechanics to high velocities and small distances leads to quantum field theory. While general relativity is used to describe the gravitational force, quantum field theory can be applied to describe particle physics.

It is an outstanding problem to construct a theory of quantum gravity - the holy grail of theoretical physics - that combines all three extensions at the same time. Usually, this issue is addressed either by trying to quantize general relativity or by trying to formulate quantum field theory in a curved space. However, the cube shows that there is a third road to attack this problem, namely by trying to add special relativity to a combination of gravity and quantum mechanics, called *non-relativistic* (*NR*) *quantum gravity*. In order to follow this third road, one should first complete the cube and define NR quantum gravity as a theory that stands on its own. This leads to the fundamental question

To what extent does a consistent theory of quantum gravity rely on special relativity?

In case special relativity is required, this could lead to a first step on this third road to quantum gravity.

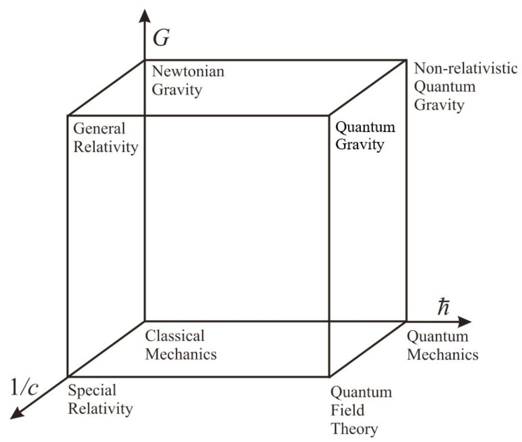

Figure 1: The so-called Bronstein cube shows the different physical theories discussed in the text.

Recently, we made considerable progress in defining a consistent theory of NR quantum gravity using the approach of NR string theory. In particular, we succeeded to derive the explicit expression of the part of the low-energy effective action of NR string theory, called *NR Neveu-Schwarz* (*NS*) *gravity*, that is common to all different NR string theories by making use of a subtle limit technique. In this talk we will discuss the construction of this NR NS gravity theory and discuss some of its particular properties.

2. Non-relativistic limits

To construct an expression for NR NS gravity, we will make use of a special NR limit. In this section we will discuss the peculiar features of this limit. Generically, to define a limit we will perform the following two steps:

- we make an invertible field redefinition writing all relativistic fields in terms of the would-be fields of the limiting theory and a dimensionless contraction parameter ω. The invertibility of the field redefinition implies that the number of fields before and after taking the limit remains the same. The would-be limiting fields only become the true limiting fields after taking the NR limit in the second step. Before this step we are just rewriting the relativistic theory.
- In a second step we take the limit that ω goes to ∞ either in the action or equations of motion. We do not allow divergent terms in the action. The limiting action is given by all terms of order ω^0. Taking the limit of the equations of motion the resulting non-relativistic equations of motion are given by the terms of leading order in ω. Independently of this one can also take the limit of the transformation rules.

Before considering string effective actions, it is instructive to first discuss the case of particles and general relativity without the Kalb-Ramond field and dilaton. Using a second-order formulation of general relativity [1], we need to express the relativistic Vierbein field $E_\mu{}^{\hat{A}}$ into the would-be non-relativistic fields in an invertible way using a contraction parameter ω. Inspired by the standard Wigner-Inönu contraction of the Poincaré algebra we first write

$$E_\mu{}^0 = \omega\tau_\mu \,, \qquad\qquad E_\mu{}^{A'} = e_\mu{}^{A'} \,, \tag{1}$$

where we have decomposed $\hat{A} = (0, A')$, ω is a dimensionless parameter, τ_μ is the clock function and $e_\mu{}^{A'}$ are the rulers. [2] As it stands, this redefinition cannot give rise, after taking the limit that $\omega \to \infty$, to a Newtonian gravity theory because in the NR case energy is not the same as mass and hence we need two gauge fields τ_μ and m_μ for energy and mass, respectively. Indeed, the NR limit defined by eq. (1) gives rise to a NR gravity theory called Galilei gravity [2]. The additional mass operator gives rise to a central extension of the Galilei algebra called the Bargmann algebra.

[1] One could also use a first-order formulation. In the presence of matter, such as in supergravity, it is easiest to use a second-order formulation.

[2] Sometimes, one uses, instead of a dimensionless parameter ω, the dimensionfull velocity of light c. This leads to the same limit as can be seen by first redefining $c \to \omega c$ and, next, taking the limit $\omega \to \infty$. The reason that this way of taking the limit leads to the same answer is that the NR limit is such that after taking the limit the parameter c disappears from the NR theory.

In order to obtain this Bargmann algebra from a Wigner-Inönü contraction of a relativistic algebra, we must extend the Poncaré algebra with an additional U(1) generator. In terms of gauge fields this implies that we should extend general relativity with an additional gauge field M_μ before taking the limit. In order not to extend general relativity with extra degrees of freedom we impose by hand that the field equation of M_μ is given by the following zero flux condition [3]

$$[M]_{\mu\nu} = \partial_\mu M_\nu - \partial_\nu M_\mu = 0\,. \tag{2}$$

Note that, without extending general relativity any further, this field equation does not follow from a relativistic action and therefore the specific limit we are considering can only be taken at the level of the equations of motion, i.e. the Einstein equations.

Given the extended general relativity theory, we consider the following redefinitions [3]:

$$E_\mu{}^0 = \omega\tau_\mu + \frac{1}{\omega}m_\mu\,, \qquad E_\mu{}^{A'} = e_\mu{}^{A'}\,, \qquad M_\mu = \omega\tau_\mu\,. \tag{3}$$

Note that the relativistic inverse Vielbeine are redefined as follows

$$E_0^\mu = \frac{1}{\omega}\tau^\mu + \cdots\,, \qquad E^\mu{}_{A'} = e^\mu{}_{A'} + \cdots\,, \tag{4}$$

where the projective inverse Vielbeine τ^μ and $e^\mu{}_{A'}$ are defined by

$$\tau_\mu\tau^\mu = 1\,, \quad \tau_\mu e^\mu{}_{A'} = \tau^\mu e_\mu{}^{A'} = 0\,, \quad e_\mu{}^{A'}e^\nu{}_{A'} + \tau_\mu\tau^\nu = \delta_\mu{}^\nu\,. \tag{5}$$

We have only given here the leading order redefinitions. The lower order dotted terms in (4) do not contribute to the final answer when taking the NR limit. We note that in a Post-Newtonian expansion of general relativity there is no need to make use of the extra gauge field M_μ since the lowest order terms in such an expansion do not need to constitute an invertible field redefinition. In that case the NR central charge gauge field m_μ occurs as a next component in the expansion of the Vierbein field. The terms of lowest order in ω in such an expansion do not correspond to any limit.

It is now a straightforward matter to substitute the field redefinitions (3) into the Einstein equations and to take the limit $\omega \to \infty$. This leads to the so-called Newton-Cartan (NC) equations of motion that we will not discuss further here. The important thing is that it gives a frame-independent reformulation of Newtonian gravity which can be obtained from NC gravity by gauge-fixing the gravitational fields leaving us with a Newton potential Φ only. It turns out that this Newton potential can be identified with the time component of the cental charge gauge field m_μ:

$$\Phi = \tau^\mu m_\mu\,. \tag{6}$$

Furthermore, due to the constraint (2), we end up with a NC gravity theory with zero intrinsic torsion [4], i.e.

$$\tau_{\mu\nu} = \frac{1}{2}\big(\partial_\mu\tau_\nu - \partial_\mu\tau_\nu\big) = 0\,. \tag{7}$$

The special thing about this torsion tensor, which has no analogue in the relativistic case, is that it defines a covariant tensor of rank 2. It indeed describes torsion as can be seen from the metric postulate for the clock function:

$$\partial_\mu\tau_\nu - \Gamma_{\mu\nu}{}^\rho\tau_\rho = 0 \quad\to\quad \Gamma_{[\mu\nu]}{}^\rho\tau_\rho = \partial_{[\mu}\tau_{\nu]}\,. \tag{8}$$

[3]Here and in the following we will indicate the equation of motion of a field with square brackets.

In general, one may distinguish between the following three different cases:[4]

$$\tau_{\mu\nu} = 0 \quad : \quad \text{zero torsion}\,, \tag{9}$$

$$\tau_{A'B'} = 0 \quad : \quad \text{twistless torsional}\,, \tag{10}$$

$$\tau_{\mu\nu} \neq 0 \quad : \quad \text{general torsion}\,. \tag{11}$$

We have used here the projective inverse Vierbein $e^\mu{}_{A'}$ to convert a curved index μ into a flat index A':

$$\tau_{A'B'} \equiv e^\mu{}_{A'} e^\nu{}_{B'} \tau_{\mu\nu}\,. \tag{12}$$

The zero torsion case defines a Newtonian spacetime with a co-dimension 1 foliation or, equivalently, a preferred time direction t given by $\tau_\mu = \partial_\mu t$. Any observer traveling along a curve C from a time slice Σ_{t_A} at $t = t_A$ to a time slice Σ_{t_B} at $t = t_B$ will measure a time difference ΔT given by

$$\Delta T = \int_{t_A}^{t_B} dx^\mu \tau_\mu = t_B - t_A \tag{13}$$

independent of the curve C. The twistless torsional case leads to a spacetime with a hypersurface orthogonality condition of the clock fucction τ_μ. Such spacetimes are encountered in Lifshitz holography [5]. Note that the twistless torsional condition is invariant under the following anisotropic local scale transformations

$$\delta\tau_\mu{}^A = \lambda_D \tau_\mu{}^A\,. \tag{14}$$

It is this property that explains the occurrence of the twistless torsional condition in Lifshitz holography.

Before we discuss the extension from particles to strings, we summarize a few notable features of the above limiting procedure that also will be relevant in the string case.

1. In general, when taking the limit $\omega \to \infty$, many objects, such as the redefined Vierbein field given in eq. (3) blow up. This is harmless, as long as the final result, in this case the NC equations of motion, do not blow up. The occurrence of these infinities are due to the fact that one goes from a regular geometry in the relativistic case to a degenerate geometry in the NR case.

2. The fact that there is no action could be avoided by deleting the zero flux condition (2) and adding a kinetic term for the extra gauge field M_μ to the Einstein-Hilbert action, possibly with other matter fields, such that there is a critical cancellation of divergences between the Einstein-Hilbert term and some of the matter fields. This is what will happen in the string case where the extra matter consists of the Kalb-Ramond 2-form and the dilaton.

3. After taking the limit, the central charge gauge field m_μ has become part of the geometry in the sense that it transforms under Galilean boost transformations with parameters $\lambda_{A'}$ as follows:

$$\delta m_\mu = \lambda_{A'} e_\mu{}^{A'}\,. \tag{15}$$

[4]One cannot impose $\tau_{0A'} = 0$ since such a constraint is not invariant under Galilean boost transformations.

We now proceed from particles to strings. This leads to two changes with respect to the particle case. First of all, we need to replace the clock function τ_μ by two so-called longitudinal Vierbeine $\tau_\mu{}^A$ with $A = (0, 1)$, where we now decompose the relativistic Lorentz index $\hat{A}$ as $\hat{A} = (A, A')$. Secondly, we replace the relativistic vector M_μ that couples to a particle by a Kalb-Ramond 2-form field $B_{\mu\nu}$ which naturally couples to a string. Comparing with the particle case where we redefined the relativistic fields in terms of the would-be NR ones as follows:

$$\{E_\mu{}^{\hat{A}}, M_\mu\} \quad \rightarrow \quad \{\tau_\mu, e_\mu{}^{A'}, m_\mu\}\,,$$

we now redefine the following general relativity fields $E_\mu{}^{\hat{A}}$ plus Kalb-Ramond fields $B_{\mu\nu}$ and dilaton field Φ in terms of would be NR fields according to:

$$\{E_\mu{}^{\hat{A}}, B_{\mu\nu}, \Phi\} \quad \rightarrow \quad \{\tau_\mu{}^A, e_\mu{}^{A'}, b_{\mu\nu}, \phi\}\,.$$

Here, $b_{\mu\nu}$ and ϕ are the NR Kalb-Ramond and dilaton field, respectively.

Defining a limit, we redefine the 2-form field as follows:

$$B_{\mu\nu} = -\omega^2 \epsilon_{AB} \tau_\mu{}^A \tau_\nu{}^B \; + b_{\mu\nu}\,,$$

while the redefinitions of the Vierbein fields remain the same except for the index structure:

$$E_\mu{}^A = \omega \tau_\mu{}^A\,, \qquad\qquad E_\mu{}^{A'} = e_\mu{}^{A'}\,. \tag{16}$$

Together with this, we also redefine the relativistic dilaton Φ in terms of a would-be NR dilaton field ϕ as follows:

$$\Phi = \phi + \ln \omega\,. \tag{17}$$

After taking the limit that $\omega \rightarrow \infty$, the Newton potential Φ can now be identified as the longitudinal time-space component of the NR Kalb-Ramond field:

$$\Phi = \epsilon^{AB} \tau^\mu{}_A \tau^\nu{}_B b_{\mu\nu}\,. \tag{18}$$

Similar to the particle case, we find that, after taking the limit, the NR Kalb-Ramond field $b_{\mu\nu}$ becomes part of the geometry in the sense that it transforms under the stringy Galilean boost transformations with parameters $\lambda_{A'}{}^A$ in the following non-trivial way:

$$\delta b_{\mu\nu} = \partial_{[\mu} \lambda_{\nu]} + 2\epsilon_{AB} \lambda_{A'}{}^A \tau_{[\mu}{}^B e_{\nu]}{}^{A'}\,.$$

With these definitions, we are now ready to take the limit of the relativistic NS gravity action.

3. Non-relativistic Neveu-Schwarz gravity

Using a second-order formulation of general relativity, our starting point is the relativistic NS action

$$S_{\rm rel} = \frac{1}{2\kappa^2} \int d^{10}x\, E e^{-2\Phi} \Big(\mathcal{R}(\Omega(E)) - \frac{1}{12} \mathcal{H}_{\mu\nu\rho} \mathcal{H}^{\mu\nu\rho} + 4 \partial_\mu \Phi \partial^\mu \Phi \Big)\,, \tag{19}$$

where κ is the gravitational coupling constant, $E = \det E_\mu{}^{\hat{A}}$, Φ is the dilaton field, $\mathcal{R}(\Omega(E))$ is the Einstein-Hilbert scalar in terms of a dependent spin connection field $\Omega_\mu{}^{\hat{A}\hat{B}}$ and $\mathcal{H}_{\mu\nu\rho}$ is the curvature of the Kalb-Ramond field $B_{\mu\nu}$, i.e. $\mathcal{H}_{\mu\nu\rho} = 3\partial_{[\mu}B_{\nu\rho]}$.

Making the redefinitions explained in the previous section, we end up with a Lagrangian that contains pieces $\overset{(n)}{S}$ proportional to ω^n for $n = 2, 0, -2, -4$, i.e.

$$S = \omega^2 \overset{(2)}{S} + \overset{(0)}{S} + \omega^{-2} \overset{(-2)}{S} + \omega^{-4} \overset{(-4)}{S} \,. \tag{20}$$

Note that at this point we have not imposed any constraint on the geometry.

Now something special happens. Both the Einstein-Hilbert term and the kinetic term of the Kalb-Ramond field contribute to $\overset{(2)}{S}$ with equal terms but of opposite sign involving specific projections of the torsion tensor

$$\tau_{\mu\nu}{}^A = \frac{1}{2}\left(\partial_\mu \tau_\nu{}^A - \partial_\nu \tau_\mu{}^A\right) \tag{21}$$

such that a cancellation of divergences takes place and we find that $\overset{(2)}{S} = 0$.

Substituting all redefinitions given in the previous section into the relativistic action (19)and taking the limit $\omega \to \infty$ we find the following NR action [6]:

$$S_{\rm NR} = \frac{1}{2\,\kappa^2}\int {\rm d}^{10}x\, e\Big({\rm R}(J) - \frac{1}{12}\, h_{A'B'C'}h^{A'B'C'} \\ - 4\,\mathcal{D}_{A'}b^{A'} - 4\, b_{A'}b^{A'} - 4\,\tau_{A'\{AB\}}\tau^{A'\{AB\}}\Big)\,. \tag{22}$$

Here $e = \det(\tau_\mu{}^A, e_\nu{}^{A'})$ and

$$R(J) = e^\mu{}_{A'} e^\nu{}_{B'} R_{\mu\nu}{}^{A'B'}(J)\,, \tag{23}$$

where $R_{\mu\nu}{}^{A'B'}(J)$ is the curvature of spatial rotations whose precise definition can be found in [6]. Furthermore, $h_{A'B'C'}$ and $\tau_{A'\{AB\}}$ are intrinsic torsion components defined by

$$\tau_{A'\{AB\}} = e_{A'}{}^\mu \tau_{(A}{}^\nu \tau_{\mu\nu B)}\,, \qquad h_{A'B'C'} = e_{A'}{}^\mu e_{B'}{}^\nu e_{C'}{}^\rho h_{\mu\nu\rho}\,, \tag{24}$$

where $h_{\mu\nu\rho}$ is the curvature of the NR Kalb-Ramond field $b_{\mu\nu}$, i.e. $h_{\mu\nu\rho} = 3\partial_{[\mu}b_{\nu\rho]}$. The gauge field b_μ with a$b_{A'} = e_{A'}{}^\mu b_\mu$ is a dependent gauge field for scale transformations whose definition can be found in [6] to which we refer for more details.

It turns out that the NR NS action (22) has the following special features:

(i) the action (22) is invariant under an emergent dilatation symmetry with parameter λ_D given by

$$\delta\tau_\mu{}^A = \lambda_D \tau_\mu{}^A\,, \qquad \delta\phi = \lambda_D\,. \tag{25}$$

This means that the number of NR background fields that are present in the NR action (22) is one less than the number of relativistic background fields that are present in the relativistic action (19). This implies that there is one ‘missing equation of motion’ as compared to the number of relativistic equations of motion.

(ii) It turns out that the missing equation of motion is precisely the one that contains the Poisson equation of the Newton potential. This missing equation can be obtained by taking the NR limit of the relativistic equations of motion. In other words, taking the limit of the action and then going on-shell is not the same as first going on-shell and then taking the NR limit. We find that there is a relation though between the Poisson equation and all the other equations that follow from varying the NR action (22) in the sense that together they form a *reducible but undecomposable representation* under the Galilean boost transformations. In practice this means that under Galilean boosts the Poisson equation transforms to the equations of motion that follow from varying the NR action (22) but not the other way around: none of these equations of motion transform under Galilean boosts back to the Poisson equation.

(iii) The equations of motion of the Newton potential itself gives the following non-linear geometric constraint:

$$\tau_{B'C'A}\tau^{B'C'A} = 0 \qquad \text{with} \qquad \tau_{A'B'}{}^{C} = e_{A'}{}^{\mu} e_{B'}{}^{\nu} \tau_{\mu\nu}{}^{A}\,. \tag{26}$$

This prevents an over-determined set of equations of motion since the total number of equations of motion (before and after taking the limit of the relativistic equations of motion) remains the same but, due to the emerging dilatations, there is one missing field in the NR equations of motion. It has been pointed out that this non-linear constraint prohibits that the quantum effective sigma model describing the non-relativistic string theory develops a term quadratic in the Lagrange multipliers that are present in the classical sigma model [7]. The presence of such a term would imply that the Lagranger multipliers become auxiliary fields that can be integrated out and that the non-relativistic string sigma model does not stay NR but, instead, flows towards a relativistic string sigma model.

It is instructive to compare the different ways to derive the equations of motion of the background fields. We have already seen that it makes a difference whether one first takes the limit of the relativistic action and next determines the equations of motion or whether one first determines the relativistic equations of motion by varying the relativistic action and then takes the limit of these relativistic equations of motion. A third and independent way to determine the non-relativistic equations of motion is by calculating the beta functions of the non-relativistic sigma model [8, 10, 12]. All methods lead to the same ('common') equations of motion except for the Poisson equation of the Newton potential and the non-linear geometric constraint (26). Due to the emerging dilatation symmetry there are the following differences:

$$\text{Varying non-relativistic NS action} \;\rightarrow\; \text{common equations} \;+\text{Non-linear}\,, \tag{27}$$

$$\text{Calculating non-relativistic } \beta\text{-functions} \;\rightarrow\; \text{common equations} \;+\text{Poisson}\,, \tag{28}$$

$$\text{Taking limit of relativistic e.o.m.} \;\rightarrow\; \text{common equations} \;+\text{Poisson}+\text{Non-linear}\,. \tag{29}$$

In some sense one can view the non-linear geometric constraint (26) as an additional beta function for the Lagrange multipliers. From now on we will assume that the dynamics of the background fields of NR string theory is determined by the total set of equations, i.e. the sum of the common equations plus the Poisson equation and the non-linear geometric contraint (26).

4. Torsional String Newton-Cartan Geometry

The low-energy effective NS action (22) is based upon a non-Lorentzian so-called Torsional String Newton-Cartan (TSNC) geometry that replaces the Riemannian geometry of general relativity, see also [6, 11]. The basic variables are given by

$$\{\tau_\mu{}^A, e_\mu{}^{A'}, b_{\mu\nu}, \phi\}\,, \tag{30}$$

where $\tau_\mu{}^A$ $(A = 0, 1)$ are the two longitudinal Vierbeine, $e_\mu{}^{A'}$ are the remaining transverse Vierbeine, $b_{\mu\nu}$ is the non-relativistic Kalb-Ramond field and ϕ is the non-relativistic dilaton. There are also a number of dependent gauge fields

$$\{\omega_\mu\,, \omega_\mu{}^{AA'}\,, \omega_\mu{}^{A'B'}\,, b_\mu\} \tag{31}$$

for the longitudinal Lorentz transformations, stringy Galilean boost transformations, spatial rotations and an-isotropic dilatations, respectively.

The main feature that distinguishes the TSNC structures from other geometric structures is the fact that the Kalb-Ramond field strength $h_{\mu\nu\rho} = 3\,\partial_{[\mu}b_{\nu\rho]}$ and that of the dilaton $\partial_\mu\phi$ do transform with a derivative of the boost and an-isotropic dilatation parameter, respectively. In fact, they are part of the dependent gauge fields and are needed to give these dependent gauge fields the correct transformation rules of a gauge field:

$$b_\mu = e_\mu{}^{A'}\,\tau_{A'A}{}^A + \tau_\mu{}^A\partial_A\phi\,, \tag{32a}$$

$$\omega_\mu = \big(\tau_\mu{}^{AB} - \frac{1}{2}\,\tau_\mu{}^C\tau^{AB}{}_C\big)\epsilon_{AB} - \tau_\mu{}^A\,\epsilon_{AB}\partial^B\phi\,, \tag{32b}$$

$$\omega_\mu{}^{AA'} = -e_\mu{}^{AA'} + e_{\mu B'}e^{AA'B'} + \frac{1}{2}\,\epsilon^A{}_B\,h_\mu{}^{BA'} + \tau_{\mu B}W^{BAA'}\,, \tag{32c}$$

$$\omega_\mu{}^{A'B'} = -2\,e_\mu{}^{[A'B']} + e_{\mu C'}e^{A'B'C'} - \frac{1}{2}\,\tau_\mu{}^A\,\epsilon_{AB}\,h^{BA'B'}\,, \tag{32d}$$

where $\tau_{\mu\nu}{}^A = \partial_{[\mu}\tau_{\nu]}{}^A$ and $e_{\mu\nu}{}^{A'} = \partial_{[\mu}e_{\nu]}{}^{A'}$. It turns out that not all components of the above gauge fields can be solved for. This is reflected by the undetermined quantity $W^{ABA'}$ which is traceless symmetric in the (A, B) indices, but otherwise arbitrary. Since all the relevant expressions—such as the action, equations of motion and symmetry transformation rules—follow from a limit of the corresponding relativistic expressions it is clear that nothing depends on W. For more details about this TSNC geometry, we refer to [6] and the overview talk by Jan Rosseel at this workshop.

5. Generalizations and New developments

The work presented here can be generalized in several ways.

(i) The limit technique we applied here to obtain the non-relativistic NS action (22) was based upon a cancellation of divergences coming from the Einstein-Hilbert term and a matter term (the kinetic term of the NR Kalb-Ramond field). We expect that a similar cancellation of divergences occurs for limits with a co-dimension 1 foliation and other matter content. An explicit example would be to start from a spatial reduction of the relativistic action (19) but other examples should exist as well.

(ii) A natural generalization of our work is to add a Yang-Mills matter action and obtain the action of non-relativistic heterotic gravity with its characteristic Yang-Mills Chern-Simons term. At the level of the sigma model description this case is more subtle due to the occurrence of so-called sigma model anomalies.

(iii) In this talk we only discussed the case of closed strings. The generalization to open strings was discussed in [12, 13].

(iv) One can also include supersymmetry. A supersymmetric version of NS gravity, called ten-dimensional minimal supergravity, was given in [14], see also the talk by Lahnsteiner at this workshop. It turns out that the bosonic theory can only be embedded into a supersymmetric one provided that an additional geometric constraint is imposed which specializes the general TSNC geometry to a so-called DSNC$^-$ geometry. We refer to [14] for more details. Further aspects of (the bosonic sector of) NR M-theory and IIA/IIB supergravity have been discussed in [15, 16].

After this talk was presented a few interesting developments took place. In particular, we derived a target space description of the so-called T-duality rules and applied the so-called longitudinal T-duality rules to construct the basic (half-supersymmetric) string solutions corresponding to the NR NS action (22). It turns out that there are two basic string solutions [17]. One is the so-called *winding string* solution that occurs as a physical state in the string spectrum. The other is a so-called *unwound* string solution that does not correspond to a physical state. Instead, it describes an off-shell state needed to describe the instantaneous Newtonian force between two winding strings [18–20].

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