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Disk amplitudes of closed strings in the pure spinor formalism

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We describe how to calculate closed string scattering on the disk in the pure spinor approach to string perturbation theory. We exemplify the methods with certain low point functions and compare to the known results of the RNS formalism. Moreover, we discuss some potential applications to the study of higher derivative corrections to D-brane actions. This talk summarises the results of [1].

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1. Motivation

Some of the predictions of string theory can be inferred from its perturbative quantum corrections. String perturbation theory is a double expansion with the independent expansion parameters $g_s \sim e^{\Phi}$ (weighing spacetime quantum effects) and $\sqrt{\alpha'}k$ with a momentum scale k (weighing quantum effects in the world-sheet theory). In theories with open strings, the leading g_s-correction relative to the sphere level comes from the disk (and the projective plane in unoriented theories like type I). This makes the disk contributions particularly relevant.

There are many applications that one might have in mind. For concreteness let us focus on a particular one – higher derivative corrections to the DBI action and in particular a term $\sim e^{-\Phi}\epsilon_{10}\epsilon_{10}R^4$. Here R^4 stands schematically for an expression involving four powers of the Riemann tensor (not the Ricci scalar) and ϵ_{10} is the ten dimensional epsilon tensor. All the indices of the Riemann and epsilon tensors are contracted in a way that will not play a role for us. The dilaton dependence (in string frame) characterises this term as arising from the disk (or projective plane). Such a term could have interesting phenomenological consequences as it would lead to a correction to the Einstein-Hilbert term in four dimensions when compactified on a Calabi-Yau threefold with non-vanishing Euler number [2]. For minimally supersymmetric four dimensional type IIB orientifolds with D9-branes, this would constitute the leading g_s-correction to the Einstein-Hilbert term in four dimensions, dominant compared with the one-loop corrections obtained in [3].[1] Such a correction in the string frame could then lead, via a Weyl rescaling, to a correction to the Kähler potential of the moduli and/or to a redefinition of the moduli fields, as discussed for instance in [5, 6].

An interesting indirect argument for the presence of such a term in the DBI action has been given by Green and Rudra [7], based on type I / heterotic duality. In order to understand their argument better it helps to remember that there are two different kinds of R^4-terms in ten dimensional string theory. In type II theory they arise at the sphere and torus level of the perturbative expansion, i.e.

$$\Big(\zeta(3)e^{-2\Phi} + \frac{\pi^2}{6}\Big)t_8t_8R^4 - \Big(\zeta(3)e^{-2\Phi} \pm \frac{\pi^2}{6}\Big)\frac{1}{8}\epsilon_{10}\epsilon_{10}R^4\ , \tag{1}$$

see for instance [2]. Again, t_8 is an eight-tensor and all its indices are contracted with the indices of the Riemann-tensors. While, as mentioned before, the $\epsilon_{10}\epsilon_{10}R^4$-terms lead to a correction of the four dimensional Einstein-Hilbert term in a compactification on a Calabi-Yau manifold with non-vanishing Euler number, the $t_8t_8R^4$-terms lead to a correction to the kinetic terms of the moduli scalars in four dimensions [2]. The lower sign in (1) gives the result in type IIA theory, while the upper sign corresponds to type IIB. These R^4-terms of type IIB are also inherited by type I after orientifolding and, thus, the $\epsilon_{10}\epsilon_{10}R^4$ term arises at least at two orders in the type I perturbation theory.

On the other hand in the heterotic theory the $\epsilon_{10}\epsilon_{10}R^4$ term only arises at sphere level, the torus correction is absent,

$$\Big(\zeta(3)e^{-2\Phi} + \frac{\pi^2}{6}\Big)t_8t_8R^4 - \zeta(3)e^{-2\Phi}\frac{1}{8}\epsilon_{10}\epsilon_{10}R^4\ , \tag{2}$$

[1] A similar disk level correction to the four dimensional Einstein-Hilbert term was discussed in type IIB orientifolds with D7-branes and O7-planes wrapping four-cycles with non-trivial first Chern form, cf. [4].

see [8–12]. Thus the question arises, how this can be consistent with heterotic / type I S-duality

$$g_{\mu\nu}^{(I)} = e^{-\Phi_{het}} g_{\mu\nu}^{(het)} \quad , \quad \Phi_I = -\Phi_{het} \ . \tag{3}$$

This question is not new and was asked for instance also in [13]. A possible answer was given a few years ago in [7]. The argument can be summarised as follows. The terms (2) in the ten dimensional heterotic action $S^{(het)}$ can be expressed in terms of the bosonic content of two superinvariants introduced in [14], i.e.

$$J_0 = t_8 t_8 R^4 - \frac{1}{8}\epsilon_{10}\epsilon_{10}R^4 \quad , \quad \mathcal{I}_2 = -\frac{1}{8}\epsilon_{10}\epsilon_{10}R^4 + \dots \ . \tag{4}$$

Assuming that the coefficient of the J_0-invariant is given by the Eisenstein series

$$E_{3/2} = \zeta(3) e^{-3/2\Phi_{het}} + \frac{\pi^2}{6} e^{\Phi_{het}/2} + \mathrm{non-pert.} \ , \tag{5}$$

and [7] gives some indirect arguments for this assumption, based on heterotic M-theory, the heterotic R^4-terms are given by

$$S^{(het)} \sim \int d^{10}x \left(\sqrt{g^{(het)}} e^{-\Phi_{het}/2} J_0 E_{3/2}(e^{-\Phi_{het}}) - \sqrt{g^{(het)}} \frac{\pi^2}{6} \mathcal{I}_2 + \dots \right) . \tag{6}$$

The two perturbative terms in (5) would a priori lead to a sphere and a torus contribution to both kinds of R^4-terms, but the $\epsilon_{10}\epsilon_{10}R^4$-term at torus level is explicit subtracted again via the $\mathcal{I}_2$-term.

How does this now help with our original question concerning the fate of the heterotic / type I duality? The crucial observation is that the Eisenstein-series $E_{3/2}$ is S-duality invariant and, thus, the first term in (6) is left invariant under S-duality. On the other hand, the second term maps separately under (3), altogether leading to the following S-duality transformation:

$$S^{(het)} \longrightarrow \int d^{10}x \left(\sqrt{g^{(I)}} e^{-\Phi_I/2} J_0 E_{3/2}(e^{-\Phi_I}) - \sqrt{g^{(I)}} \frac{\pi^2}{6} e^{-\Phi_I} \mathcal{I}_2 + \dots \right) . \tag{7}$$

The first term in (7) encompasses the sphere and torus contributions of type I, inherited from type IIB, that we discussed above. We now see that the argument of [7] predicts the presence of an additional perturbative $\epsilon_{10}\epsilon_{10}R^4$-term in type I theory with a dilaton dependence appropriate for the disk and / or the projective plane. A direct check in ten dimensions would require the calculation of a 5-point graviton amplitude (given that the expansion of $\epsilon_{10}\epsilon_{10}R^4$ around flat space starts at order five in gravitons). As this would be a daunting task, one might hope to infer the presence of such a term indirectly, by checking for a disk or projective plane correction to the four dimensional Einstein-Hilbert term in type I compactified on a Calabi-Yau threefold Y_3. This would be proportional to the Euler number χ_{Y_3} of Y_3, i.e.

$$\sim e^{-\Phi} \chi_{Y_3} R \ . \tag{8}$$

However, this raises the interesting question of how a disk or projective plane amplitude of states with purely external polarisations could possibly depend on the Euler number χ_{Y_3} of the internal space Y_3.

2. Review of the pure spinor formalism

After motivating the interest in disk amplitudes in string theory, we would now like to address the second aspect of the present work and ask: Why the pure spinor formalism? Pure spinors were introduced into the field of string perturbation theory in [15]. They allow for a manifestly spacetime super-Poincaré covariant formalism. Hence, it allows to overcome simultaneously the shortcoming of the RNS formalism of not being manifestly spacetime supersymmetric and of the Green-Schwarz formalism of being quantisable only in light-cone gauge. The equivalence of the pure spinor and the RNS formalism has been shown in many concrete examples (mainly involving massless states). There also exists a prescription of how to calculate loop amplitudes even though this is more straightforward in the non-minimal formulation of the pure spinor formalism introduced in [16]. This will not be discussed any further in the following.

In view of the motivation we presented in the last section it should be noted that the pure spinor formalism can also lead to simplified calculations of string amplitudes. To underline this point let us mention three rather impressive examples of this. The complete quartic effective action of type II at sphere level (to all orders of α') was calculated in [17], including fermions and RR-fields. The massless closed string 4-point amplitude in type II was obtained at *3-loop* order in [18], even though only at low energy, i.e. the leading term in an expansion for small momenta. Finally, the pure spinor formalism allowed to find a rather compact expression for an arbitrary n-point amplitude of massless open strings on the disk [19, 20].

Let us next come to a short review of those technicalities of the pure spinor formalism which will be relevant for closed string scattering on the disk. We present in turn some details about the CFT, the massless vertex operators and the prescription to calculate tree level correlators. In collecting those details we profited mainly from the presentations in [21, 22] but there are many more reviews with additional aspects, like for instance [23–25].

2.1 CFT

For concreteness we consider the world-sheet action for type IIB theory in ten dimensional flat spacetime. It is given by

$$S = \frac{1}{2\pi}\int d^2z \left(\tfrac{1}{2}\partial X^m \overline{\partial} X_m + p_\alpha \overline{\partial}\theta^\alpha + \overline{p}_\alpha \partial \overline{\theta}^\alpha + w_\alpha \overline{\partial}\lambda^\alpha + \overline{w}_\alpha \partial \overline{\lambda}^\alpha\right). \tag{9}$$

The fields θ^α and $\overline{\theta}^\alpha$ are the fermionic spacetime superpartners of X^m. They both have the same chirality as appropriate for the type IIB theory (spinors with upper and lower index transform in the two inequivalent chiral representations 16 and 16', respectively). The fields p_α and $\overline{p}_\alpha$ are their canonically conjugate momenta. The additional fields λ^α and $\overline{\lambda}^\alpha$ (with canonically conjugate momenta w_α and $\overline{w}_\alpha$) are the (bosonic, i.e. commuting) pure spinors which give the formalism its name. Hence, they satisfy the pure spinor constraint

$$(\lambda\gamma^m\lambda) = 0\,, \tag{10}$$

where the 16×16 matrices $\gamma^m_{\alpha\beta}$ are the ten dimensional Pauli matrices. Compared to the Green-Schwarz formalism, which also uses the spacetime fermions θ^α and $\bar{\theta}^\alpha$ as world-sheet fields, the action (9) is noticeably simpler. This simplification is made possible by treating the canonically conjugate momenta p_α and $\overline{p}_\alpha$ as independent variables, an idea that goes back to Siegel, cf. [26]. Another difference to the Green-Schwarz formalism is the presence of the pure spinors and their canonical momenta. They are necessary in order for the action (9) to have vanishing central charge $c = 0 = \bar{c}$. The spacetime spinors in (9) have first order kinetic terms and, thus, they behave as (anti-)commuting bc-systems like the (super)ghosts known from the quantisation of the RNS string. However, they have $\lambda = 1$ (in the notation of sec. 13.1 in [27]), in contrast to the ghost and superghost systems of the RNS string which have $\lambda = 2$ and $\lambda = 3/2$, respectively. This implies the following dimensions for the components of the spacetime spinors:

$$h(p_\alpha) = h(w_\alpha) = 1 \,, \quad h(\theta^\alpha) = h(\lambda^\alpha) = 0 \tag{11}$$

(and similarly for the right-movers). Moreover, given that p_α and θ^α are anti-commuting while w_α and λ^α are commuting, their contributions to the central charge of (9) would exactly cancel each other (leaving only the non-vanishing contribution of the X^m), if w_α and λ^α were unconstrained. It turns out that one can ensure a vanishing central charge of (9) by demanding the pure spinor condition (10) which only leaves 11 of the 16 components of λ^α as independent degrees of freedom.

The definition of the vertex operators in the pure spinor formalism makes use of the following supersymmetric composite fields

$$\Pi^m = \partial X^m + \tfrac{1}{2}(\theta\gamma^m\partial\theta) \,, \tag{12}$$

$$d_\alpha = p_\alpha - \tfrac{1}{2}\Big(\partial X^m + \tfrac{1}{4}(\theta\gamma^m\partial\theta)\Big)(\gamma_m\theta)_\alpha \,, \tag{13}$$

which have antiholomorphic analogs.

The basic fields appearing in (9) have the usual OPEs which will be needed to perform contractions:[2]

$$X^m(z,\overline{z})X^n(w,\overline{w}) \sim -\eta^{mn}\ln|z-w|^2 \,, \tag{14}$$

$$p_\alpha(z)\theta^\beta(w) \sim \frac{\delta_\alpha{}^\beta}{z-w} \,, \tag{15}$$

$$w_\alpha(z)\lambda^\beta(w) \sim -\frac{\delta_\alpha{}^\beta}{z-w} \,. \tag{16}$$

Note that we set $\alpha' = 2$ in (14), while (15) and (16) are the standard OPEs of (anti-)commuting bc-systems [27].

Another operator which plays a role for the definition of the vertex operators is the Lorentz current

$$J^{mn} = X^m\partial X^n - X^n\partial X^m + \tfrac{1}{2}(p\gamma^{mn}\theta) + \tfrac{1}{2}(w\gamma^{mn}\lambda) \tag{17}$$

and in particular its contribution from the pure spinor

$$N^{mn} = \tfrac{1}{2}(w\gamma^{mn}\lambda) \qquad \text{with} \qquad \gamma^{mn} = \frac{1}{2}(\gamma^m\gamma^n - \gamma^n\gamma^m) \,. \tag{18}$$

[2] Strictly speaking the case of the $w\lambda$-OPE is actually more complicated, cf. eq. (2.11) in [15], but this complication will not play any role in the following.

Finally, an important ingredient in the CFT of the pure spinor formalism is the nilpotent BRST operator

$$Q = \oint \frac{\mathrm{d}z}{2\pi i} \lambda^\alpha(z) d_\alpha(z) \ . \tag{19}$$

Its nilpotency

$$Q^2 = 0 \tag{20}$$

follows straightforwardly from (10) and the OPE

$$d_\alpha(z) d_\beta(w) \sim -\frac{\gamma^m_{\alpha\beta}\Pi_m(w)}{z-w} \ . \tag{21}$$

Given that Q is nilpotent, one can consider its cohomology, which was shown to coincide with the superstring spectrum [28].

2.2 Massless vertex operators

Let us next come to the description of the massless vertex operators and begin with the open string. Similarly to the RNS formalism they come in two different versions, as integrated and unintegrated vertex operators. In the bosonic string the integrated and unintegrated vertex operators just differ by a factor of the c-ghost. In the pure spinor formalism the difference is more substantial. The unintegrated vertex operator is rather simple. It is given by

$$V^{(0)}(z) = [\lambda^\alpha A_\alpha(X,\theta)]\,(z) \ , \tag{22}$$

i.e. it contains a pure spinor factor and a superfield A_α whose explicit form we are going to present in a moment. The unintegrated vertex operator is BRST invariant, i.e.

$$QV^{(0)} = 0 \ . \tag{23}$$

As in the RNS formalism, the integrated vertex operator is defined via

$$QV^{(1)} = \partial V^{(0)} \ . \tag{24}$$

Consequently, it is BRST invariant only upon integration. The condition (24) can be solved, resulting in

$$V^{(1)}(z) = \left[\partial\theta^\alpha A_\alpha(X,\theta) + \Pi^m A_m(X,\theta) + d_\alpha W^\alpha(X,\theta) + \tfrac{1}{2} N^{mn} \mathcal{F}_{mn}(X,\theta)\right](z) \ . \tag{25}$$

The four contributions all have a similar structure, multiplying a dimension one field with one of the superfields A_α, A_m, W^α and $\mathcal{F}_{mn}$. These superfields are not independent and rather all contain the same degrees of freedom – the gluon and gluino of a massless ten dimensional $\mathcal{N} = 1$ vector multiplet. For example, neglecting the fermionic degrees of freedom, the superfields describing a gauge field with polarisation ξ take the following form (higher order terms in the θ-expansion are usually not needed for amplitude calculations):

$$A_\alpha(X,\theta) = e^{ik\cdot X}\left\{\frac{\xi_m}{2}(\gamma^m\theta)_\alpha - \frac{1}{16}(\gamma_p\theta)_\alpha(\theta\gamma^{mnp}\theta)ik_{[m}\xi_{n]} + O(\theta^5)\right\}, \quad (26)$$

$$A_m(X,\theta) = e^{ik\cdot X}\left\{\xi_m - \frac{1}{4}ik_p(\theta\gamma_m{}^{pq}\theta)\xi_q + O(\theta^4)\right\}, \quad (27)$$

$$W^\alpha(X,\theta) = e^{ik\cdot X}\left\{-\frac{1}{2}ik_{[m}\xi_{n]}(\gamma^{mn}\theta)^\alpha + O(\theta^3)\right\}, \quad (28)$$

$$\mathcal{F}_{mn}(X,\theta) = e^{ik\cdot X}\left\{2ik_{[m}\xi_{n]} - \frac{1}{2}ik_{[p}\xi_{q]}ik_{[m}(\theta\gamma_{n]}{}^{pq}\theta) + O(\theta^4)\right\}. \quad (29)$$

These formulas can easily be generalised to include the fermionic degrees of freedom. This makes pure spinor calculations of scattering amplitudes involving spacetime fermions very similar to those involving only bosons (and a similar statement holds for amplitudes involving RR-fields in the closed string theory). Note that in the pure spinor formalism there is no analog of the picture number of the RNS formalism (and there is also no need to sum over spin structures, which makes loop calculations conceptually simpler).

For the closed string we restrict ourselves to those massless fields which, in the RNS formalism, would reside in the NSNS sector (i.e. G_{mn}, B_{mn} and Φ). The polarisation tensors are tensor products of open string polarisation vectors,

$$\epsilon_{mn} = \xi_m \otimes \overline{\xi}_n \, , \quad (30)$$

and the closed string vertex operators are tensor products of open string vertex operators for the left- and the right-movers,

$$V^{(a,b)}(z,\overline{z}) = V^{(a)}(z) \otimes \overline{V}^{(b)}(\overline{z}) \, , \qquad a,b \in \{0,1\} \, . \quad (31)$$

We will see momentarily that for closed string calculations on the disk we have to allow for the mixed case that one of the open string factors in the closed string vertex operator takes the unintegrated form (22) while the other one is of the integrated form (25). $V^{(a,b)}$ contains a plane wave factor

$$e^{ik\cdot X(z)}e^{ik\cdot\overline{X}(\bar{z})} = e^{ik\cdot[X(z)+\overline{X}(\bar{z})]} = e^{ik\cdot X(z,\bar{z})} \, . \quad (32)$$

This involves the full embedding field $X(z,\bar{z})$ which, however, we want to view as the sum of a holomorphic and an antiholomorphic piece.

2.3 Tree level correlators

Let us now come to the description of computing correlators in the pure spinor formalism. In the past the focus was put on sphere amplitudes with only closed strings and on disk amplitudes with only open strings. As in the RNS formalism the conformal Killing groups of the sphere and the disk ($PSL(2,\mathbb{C})$ and $PSL(2,\mathbb{R})$) allow to fix the positions of three closed and open strings, respectively, leading to

$$\mathcal{A}^{\text{closed}}_{S^2}(1,2,\ldots,n) = \left\langle V_1^{(0,0)}(z_1,\overline{z}_1)\prod_{i=2}^{n-2}\int \mathrm{d}^2 z_i V_i^{(1,1)}(z_i,\overline{z}_i)V_{n-1}^{(0,0)}(z_{n-1},\overline{z}_{n-1})V_n^{(0,0)}(z_n,\overline{z}_n)\right\rangle, \quad (33)$$

$$\mathcal{A}^{\text{open}}_{D_2}(1,2,\ldots,n) = \left\langle V^{(0)}_1(z_1)\prod_{i=2}^{n-2}\int_{z_{i-1}}^{z_{n-1}} \mathrm{d}z_i V^{(1)}_i(z_i) V^{(0)}_{n-1}(z_{n-1}) V^{(0)}_n(z_n)\right\rangle . \tag{34}$$

In the remainder of this section, for concreteness and for simplicity, we only discuss the open string disk amplitude. A generalisation to the closed string sphere amplitude is straightforward.

The first step is to integrate out all the non-zero modes via Wick's theorem, using the Green's functions

$$\begin{aligned} \langle X^m(z)X^n(w)\rangle &= -\eta^{mn}\ln(z-w)\ , & (35)\\ \langle p_\alpha(z)\theta^\beta(w)\rangle &= \frac{\delta_\alpha{}^\beta}{z-w}\ , & (36)\\ \langle w_\alpha(z)\lambda^\beta(w)\rangle &= -\frac{\delta_\alpha{}^\beta}{z-w}\ . & (37)\end{aligned}$$

Once this is done, one is left solely with the zero modes. At tree level only the $h=0$ fields $X^m, \theta^\alpha, \lambda^\alpha$ have zero modes, not the $h=1$ fields $\partial\theta^\alpha, \Pi^m, d_\alpha, N^{mn}$, which appear in the integrated vertex operators. This is a big simplification arising at sphere and disk level. Thus, after eliminating all the non-zero modes via Wick contractions and after integrating out the X^m zero modes (leading to the usual momentum delta function), one ends up with

$$\left\langle V^{(0)}_1(z_1)\prod_{i=2}^{n-2} V^{(1)}_i(z_i) V^{(0)}_{n-1}(z_{n-1}) V^{(0)}_n(z_n)\right\rangle = \delta(\Sigma_{i=1}^n k_i)\ \left\langle \lambda^\alpha\lambda^\beta\lambda^\gamma f_{\alpha\beta\gamma}(\theta;z_i)\right\rangle_0 , \tag{38}$$

where on the right hand side λ and θ stand for the zero modes of these fields and the subscript 0 at the angular bracket denotes the integration over these zero modes. The three factors of the pure spinor λ originate from the three unintegrated vertex operators and $f_{\alpha\beta\gamma}(\theta;z_i)$ is some expression depending on the components of (the zero modes of) θ and on the positions of the vertex operators. It can be expanded in powers of θ, using (26)-(29).

The integration over the θ and λ zero modes now proceeds via the following prescription. Out of all the terms in the θ-expansion of $f_{\alpha\beta\gamma}(\theta;z_i)$, only the terms with five powers of θ contribute according to

$$\langle(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta)\rangle_0 = 1\ , \tag{39}$$

all other terms are set to zero. This prescription can be justified by noticing that the expression on the left hand side of (39) is the unique scalar element of the Q-cohomology with three factors of λ. The terms that contribute to the amplitude in the end should indeed lie in the Q-cohomology. They should be Q-closed because the expression we started with was the product of Q-closed vertex operators. Moreover, Q-exact pieces should not contribute in order to ensure gauge invariance. Effectively, the prescription (39) projects out the coefficients of the $\lambda^3\theta^5$-terms of

$$\lambda^\alpha\lambda^\beta\lambda^\gamma f_{\alpha\beta\gamma}(\theta;z_i)\ . \tag{40}$$

Finally, we still need to integrate over the positions of the integrated vertex operators, z_i for $i=2,\ldots,n-2$.

3. Adaption to disk amplitudes of closed strings

We are now ready to adapt the prescription to the case of disk amplitudes with only closed strings [1]. There are earlier results on mixed disk amplitudes of closed and open strings, in particular on 3-point functions of one closed and two open strings [29, 30]. However, these avoid a small subtlety with fixing the conformal Killing group that arises on disk amplitudes with only closed strings. We will come back to this in a moment.

Concretely, we consider type IIB theory with Dp-branes along the first p spatial dimensions $X^1, \dots, X^p$. The closed string vertex operators $V^{(a,b)}(z,\overline{z}) = V^{(a)}(z)\overline{V}^{(b)}(\overline{z})$ now 'live" on the upper half plane, i.e. $z \in \mathbb{H}_+$. Similarly to the RNS theory, one can deal with the right-moving part of the vertex operators by employing the doubling trick. This amounts to extending the X^m fields to the full complex plane. The extension to the lower half plane (where $\overline{z}$ lives) proceeds via the value of $\overline{X}^m$. Concretely one defines

$$\overline{X}^m(\overline{z}) = D^m{}_n X^n(\overline{z}) \,, \tag{41}$$

where the matrix

$$D^{mn} = \begin{cases} \eta^{mn} & m,n \in \{0,1,\dots,p\} \\ -\eta^{mn} & m,n \in \{p+1,\dots,9\} \\ 0 & \text{otherwise} \end{cases} \tag{42}$$

ensures the correct boundary conditions, i.e. Neuman boundary conditions parallel to the brane and Dirichlet boundary conditions transversal to it. The doubling trick can be extended to spinors, which is also reminiscent from the RNS formalism where the relevant spinors are the spin fields [31–33]. In our case the doubling trick for the spinors of the two different chiralities (denoted collectively as Ψ^β and Ψ_β) requires the introduction of two spinorial analogs of the D-matrix (42),

$$\overline{\Psi}^\alpha(\overline{z}) = M^\alpha{}_\beta \Psi^\beta(\overline{z}) \quad , \quad \overline{\Psi}_\alpha(\overline{z}) = N_\alpha{}^\beta \Psi_\beta(\overline{z}) \,. \tag{43}$$

In order to proceed, we do not need to know the concrete form of the matrices M and N. It suffices to derive some relations between D, M and N (e.g. $N = (M^T)^{-1}$) by demanding consistency of the OPEs. These relations allow the following rewriting of the right-moving parts of the closed string vertex operators [1]

$$\overline{V}^{(0)}(\overline{z}) = \Big(\overline{\lambda}^\alpha \overline{A}_\alpha[\overline{\xi},k](\overline{X},\overline{\theta})\Big)(\overline{z}) = \Big(\lambda^\alpha A_\alpha[D\cdot\overline{\xi}, D\cdot k](X,\theta)\Big)(\overline{z}) \,, \tag{44}$$

$$\begin{aligned} \overline{V}^{(1)}(\overline{z}) \;=\; & \Big(\overline{\partial}\theta^\alpha A_\alpha[D\cdot\overline{\xi}, D\cdot k](X,\theta) + \Pi^m A_m[D\cdot\overline{\xi}, D\cdot k](X,\theta) \\ & + d_\alpha W^\alpha[D\cdot\overline{\xi}, D\cdot k](X,\theta) + \frac{1}{2} N^{mn}\mathcal{F}_{mn}[D\cdot\overline{\xi}, D\cdot k](X,\theta)\Big)(\overline{z}) \,. \end{aligned} \tag{45}$$

Thus, the doubling trick allows to express the right-moving parts of the vertex operators in terms of the left-moving fields X and θ, albeit evaluated at $\overline{z}$ and the only remaining effect is that the polarisation vector and the momentum is multiplied by the D-matrix. The upshot of all this is that we can use the same contractions as introduced above (cf. (35)-(37)), but we allow both z- and $\overline{z}$-dependence.

We can now come back to the small subtlety with fixing the conformal Killing group mentioned above. The conformal Killing group $PSL(2,\mathbb{R})$ of the disk only allows to fix one and a half closed string vertex operators. For a disk amplitude with some closed and at least one open string, one can always treat the conformal Killing group by fixing the position of one closed and one open string. However, in the case with only closed strings this is not possible. Following [34] when treating the conformal Killing group, we find [1]

$$\mathcal{A}_{D_2}^{\text{closed}}(1,\dots,n) = 2ig_s^n\tau_p \int_0^1 \mathrm{d}y \left\langle V_1^{(0)}(iy)\overline{V}_1^{(1)}(-iy) \prod_{j=2}^{n-1} \int_{\mathbb{H}_+} \mathrm{d}^2 z_j V_j^{(1,1)}(z_j,\bar{z}_j) V_n^{(0)}(i)\overline{V}_n^{(0)}(-i) \right\rangle , \quad (46)$$

where τ_p is the tension of the Dp-brane. Hence, one closed string vertex operator appears in the mixed version with one factor taking the form of the unintegrated open string vertex operator and the other factor taking the form of the integrated open string vertex operator. Note however, that both of these factors do depend on a real variable y which is still integrated.

4. Application to 2- and 1-point functions

Next we would like to be more concrete and calculate the closed string 2- and 1-point functions on the disk.

4.1 2-point function

Plugging the form of the unintegrated (22) and integrated vertex operators (25) into (46) and performing the Wick contractions results in

$$\begin{aligned}
\mathcal{A}_{D_2}^{\text{closed}}(1,2) &= 2ig_s^2\tau_p \int_0^1 \mathrm{d}y \left\langle V_1^{(0)}(iy)\overline{V}_1^{(1)}(-iy) V_2^{(0)}(i)\overline{V}_2^{(0)}(-i) \right\rangle & (47)\\
&= 2ig_s^2\tau_p \int_0^1 \mathrm{d}y \Big\langle (\lambda A_1[\xi_1,k_1])(iy)\Big(\bar{\partial}\theta^\alpha A_{1\alpha}[D\cdot\bar{\xi}_1, D\cdot k_1] + \Pi^m A_{1m}[D\cdot\bar{\xi}_1, D\cdot k_1] \\
&\qquad + d_\alpha W_1^\alpha[D\cdot\bar{\xi}_1, D\cdot k_1] + \frac{1}{2} N^{mn}\mathcal{F}_{1mn}[D\cdot\bar{\xi}_1, D\cdot k_1]\Big)(-iy) \\
&\qquad (\lambda A_2[\xi_2,k_2])(i)(\lambda A_2[D\cdot\bar{\xi}_2, D\cdot k_2])(-i)\Big\rangle & (48)\\
&\sim g_s^2\tau_p \int_0^1 \mathrm{d}y \left(\frac{4y}{(1+y)^2}\right)^{k_1\cdot D\cdot k_1} \left(\frac{(1-y)^2}{(1+y)^2}\right)^{k_1\cdot k_2} \left(\frac{d_1}{2y} + \frac{d_2}{1+y} + \frac{d_3}{1-y}\right), & (49)
\end{aligned}$$

where the first two factors in the integrand of (49) correspond to the Koba-Nielsen factor and d_1, d_2 and d_3 are kinematic factors which still contain an implicit zero mode integration. For instance

$$\begin{aligned}
d_1 = \Big\langle & i(\lambda A_1[\xi_1,k_1])k_1\cdot A_1[D\cdot\bar{\xi}_1, D\cdot k_1](\lambda A_2[\xi_2,k_2])(\lambda A_2[D\cdot\bar{\xi}_2, D\cdot k_2]) \\
&+ A_{1m}[\xi_1,k_1](\lambda\gamma^m W_1[D\cdot\bar{\xi}_1, D\cdot k_1])(\lambda A_2[\xi_2,k_2])(\lambda A_2[D\cdot\bar{\xi}_2, D\cdot k_2])\Big\rangle_0 . \qquad (50)
\end{aligned}$$

The analogous expressions for d_2 and d_3 can be found in [1]. As mentioned before, the zero mode prescription instructs one to expand all the superfields appearing in (50) in powers of θ and then extract the coefficient of the $\lambda^3\theta^5$-terms. This step constitutes the main calculational task of the amplitude computation and we performed it with the help of the computer program Cadabra [35]. When the dust settles, the final result can be expressed with the help of the gamma function Γ as

$$\mathcal{A}_{D_2}^{\text{closed}}(1,2) \sim g_s^2 \tau_p \frac{\Gamma(-t/2)\Gamma(2q^2)}{\Gamma(1-t/2+2q^2)} \left(2q^2 a_1 + \frac{t}{2} a_2\right), \tag{51}$$

with

$$q = \tfrac{1}{2} k_1 \cdot D \cdot k_1 \ , \quad t = -2 k_1 \cdot k_2 \tag{52}$$

and the kinematic factors

$$\begin{aligned} a_1 &= \text{Tr}(\epsilon_1 \cdot D) k_1 \cdot \epsilon_2 \cdot k_1 - k_1 \cdot \epsilon_2 \cdot D \epsilon_1 \cdot k_2 - k_1 \cdot \epsilon_2 \cdot \epsilon_1^T \cdot D \cdot k_1 \\ &\quad - k_1 \cdot \epsilon_2^T \cdot \epsilon_1 \cdot D \cdot k_1 - k_1 \cdot \epsilon_2 \cdot \epsilon_1^T \cdot k_2 + q^2 \text{Tr}(\epsilon_1 \cdot \epsilon_2^T) + \{1 \leftrightarrow 2\} \end{aligned} \tag{53}$$

$$\begin{aligned} a_2 &= \text{Tr}(\epsilon_1 \cdot D)(k_2 \cdot D \cdot \epsilon_2 \cdot D \cdot k_2 + k_1 \cdot \epsilon_2 \cdot D \cdot k_2 + k_2 \cdot D \cdot \epsilon_2 \cdot k_1) \\ &\quad + k_1 \cdot D \cdot \epsilon_1 \cdot D \cdot \epsilon_2 \cdot D \cdot k_2 - k_2 \cdot D \cdot \epsilon_2 \cdot \epsilon_1^T \cdot D \cdot k_1 + q^2 \text{Tr}(\epsilon_1 \cdot D \cdot \epsilon_2 \cdot D) \\ &\quad - q^2 \text{Tr}(\epsilon_1 \cdot \epsilon_2^T) - (q^2 - \tfrac{t}{4}) \text{Tr}(\epsilon_1 \cdot D) \text{Tr}(\epsilon_2 \cdot D) + \{1 \leftrightarrow 2\} \ . \end{aligned} \tag{54}$$

The details are less relevant for us than the fact that the formula (51) exactly agrees with the result obtained with the RNS formalism, cf. [32, 33].

4.2 1-point function

The 1-point function has been considered earlier in the bosonic theory and the RNS theory in [36–38]. For the pure spinor formalism there is an obvious puzzle. Even when one chooses the form of an unintegrated closed string vertex operator $V^{(0,0)}$, there are at most two factors of λ^α present, while the zero mode prescription (39) requires three factors.

The resolution of this puzzle lies in an alternative zero mode prescription (which is equivalent for higher-point tree amplitudes) [39]. It is given by

$$\langle 1 \rangle_0 = 1 \ , \tag{55}$$

where 1 corresponds to the only alternative scalar element of the Q-cohomology in addition to the scalar on the left hand side of (39).

Given that for the alternative zero mode prescription you do not want to have any factors of λ in your expression, you are instructed to work with the integrated vertex operator. Thus, one is led to calculate

$$\mathcal{A}_{D_2}^{\text{closed}}(1) = g_s \tau_p \int_{\mathbb{H}_+} \frac{\mathrm{d}^2 z}{V_{\text{CKG}}} \left\langle V^{(1)}(z,\bar{z}) \right\rangle = g_s \tau_p \int_{\mathbb{H}_+} \frac{\mathrm{d}^2 z}{V_{\text{CKG}}} \left\langle V^{(1)}(z) \overline{V}^{(1)}(\bar{z}) \right\rangle \ . \tag{56}$$

The infinite volume V_{CKG} of the conformal Killing group can be treated in the usual way by fixing the position of $V^{(1)}(z,\bar{z})$ for example to $z = i$ and subsequently dividing only by the volume of the subgroup $K \subset PSL(2,\mathbb{R})$ which leaves $z = i$ invariant, i.e.

$$K = \left\{ \left(\begin{array}{cc} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{array} \right) \middle| \ \theta \in [0, 2\pi] \right\} . \tag{57}$$

One can easily check that this subgroup is indeed the stabiliser group of $z = i$:

$$\frac{\cos\theta\, i + \sin\theta}{-\sin\theta\, i + \cos\theta} = i\ . \tag{58}$$

Now K only has the finite volume 2π. Performing the Wick contractions and using the zero mode prescription (55) leads to

$$\mathcal{A}^{\text{closed}}_{D_2}(1) \sim g_s \tau_p \text{Tr}(\epsilon \cdot D)\ , \tag{59}$$

which is again the same as for the RNS formalism. For a graviton this corresponds indeed to the linearisation of the DBI action

$$\tau_p \int d^p x \sqrt{-G} \tag{60}$$

(note that τ_p contains a dilaton factor, cf. below (8.7.20) in [40]).

Let us finally note that an alternative proposal to deal with the zero mode puzzle has been put forward in [41] and it would be interesting to better understand the relation between the two methods.

5. Outlook

Let us end by mentioning a few possible directions for future work. As alluded to before, the pure spinor formalism is particularly powerful when dealing with RR-fields and fermions. Thus, generalising our calculations to include those fields would be an obvious next step. Moreover, we only discussed the disk amplitude, even though the projective plane contributes at the same order in perturbation theory. An extension of our analysis to the projective plane would therefore be natural. The analogous calculation of the closed string 2-point function on the projective plane in the RNS formalism can be found in [42]. Finally, in view of our motivation presented in section 1, we would like to generalise our explicit calculation of the 2-point function in sec. 4.1 to higher n-point functions with the aim of inferring higher derivative corrections to the DBI action. We hope to be able to report on this soon [43].

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