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# Classical space-time geometry and the weak gravity regime in the IKKT matrix model

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We discuss the reconstruction of generic 3+1-dimensional space-time geometries from covariant quantum spaces as backgrounds in the IKKT matrix model. Generic classical geometries can be realized within the weak gravity regime, without inducing significant higher-spin contributions. In the strong gravity regime i.e. for strong curvature, the background acquires significant higher spin contributions, so that the classical geometry is no longer adequate. Assuming that the scale of noncommutativity is given by the Planck scale, the weak gravity regime is bounded by a curvature scale of the order $10^{-4}m$, and easily compatible with known gravitational physics. This justifies the framework for emergent gravity given by the semi-classical matrix model, supplemented by an induced Einstein-Hilbert action which arises in the presence of fuzzy extra dimensions.

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## 1. Introduction

The purpose of these notes is two-fold. The first and main purpose is to provide a justification for the geometric framework which is underlying the higher-spin gravity and gauge theory in the IKKT matrix model, as described in a series of recent papers [1–5]. We will show that generic 3+1-dimensional space-time geometries can indeed be realized as backgrounds within the IKKT matrix model, whose structure is that of covariant quantum spaces. This means that there is no explicit Poisson tensor or $B$ field on space-time which would manifestly break Lorentz invariance.

The second purpose of these notes is to summarize and discuss some further implications of emergent gravity in this framework, in particular the recent 1-loop computation leading to the Einstein-Hilbert action [5]. The underlying framework is now fully justified by the present reconstruction of generic geometries.

The main result of the paper is a recipe how to realize or reconstruct generic background geometries (with trivial topology) in the matrix model, starting from some metric $G_{\mu\nu}$ on space-time. Even though this was assumed in the above works, no full justification has been given, and the statement is in fact rather non-trivial and subtle. It turns out that the geometric reconstruction works *provided* we restrict ourselves to a certain regime dubbed *weak gravity regime*. This basically means that the gravitational curvature (length) scale should be sufficiently large, bounded by the geometric mean of a cosmic scale and a UV scale (of noncommutativity). For geometries with stronger curvature, their reconstruction as covariant quantum space contains significant higher-spin contributions (or "contaminations"), which goes beyond the classical framework of manifolds.

The matrix models under consideration have an extremely simple structure, given by

$$S_{YM} = \mathrm{Tr}[T^{\dot{a}}, T^{\dot{b}}][T^{\dot{a}'}, T^{\dot{b}'}]\eta_{\dot{a}\dot{a}'}\eta_{\dot{b}\dot{b}'} \; + \text{fermions}. \tag{1}$$

Here $T^{\dot{a}},\ \dot{a} = 0, ..., D-1$ are a set of hermitian matrices which transform under a global $SO(D-1,1)$ symmetry acting on the dotted Latin indices, and $\eta_{\dot{a}\dot{b}}$ can be interpreted as $SO(D-1,1)$- invariant metric on target space $\mathbb{R}^{D-1,1}$. The models are invariant under gauge transformations

$$T^{\dot{a}} \to U^{-1} T^{\dot{a}} U \ . \tag{2}$$

It is straightforward to include fermions, which is very important for the quantization; in fact we will require maximal supersymmetry, as realized in the IKKT model [6] with $D = 10$. There is no a priori notion of space-time or differential geometry; all geometrical structures relevant for the fluctuations on some given background solution emerge dynamically within the model. We will show how generic 3+1-dimensional space-time geometries as required for gravity can be realized as deformations of the covariant cosmic background $\bar{\mathcal{M}}^{3,1}$ introduced in [1].

A general framework which allows to make geometric sense of the matrix model is that of quantized symplectic spaces. We consider any given set of matrices $T^{\dot{a}}$ as a **matrix configuration**. Since the action is given by the square of commutators, only "almost-commutative" matrix configurations are expected to play a significant role at low energies, i.e. matrices whose commutators are much smaller in some sense than the matrices $T^{\dot{a}}$. One can then argue on rather general grounds [7, 8] that such matrix configurations can be interpreted in terms of a quantized symplectic space $(\mathcal{M}, \omega)$, where the algebra of functions $C(\mathcal{M})$ is replaced by the operator algebra $\mathrm{End}(\mathcal{H})$. More precisely, this is expected to hold for some subspace of IR functions and almost-local operators; more details can be found in [7]. Such functions

$$\Phi \in \mathrm{End}(\mathcal{H}) \sim \phi \in C(\mathcal{M}) \tag{3}$$

can be identified with their classical counterpart via some (de-) quantization map defined via quasi-coherent states. We will work mostly in the semi-classical regime indicated by ∼, where commutators can be replaced by Poisson brackets

$$[\Phi, \Psi] \sim i\{\phi, \psi\} \tag{4}$$

as familiar from quantum mechanics. In particular, the $T^{\dot{a}}$ can accordingly be viewed as quantized functions on $\mathcal{M}$, which thereby define an embedding of $\mathcal{M}$ into target space:

$$T^{\dot{a}} \sim t^{\dot{a}}: \quad \mathcal{M} \hookrightarrow \mathbb{R}^{9,1} . \tag{5}$$

This suggests to interpret $\mathcal{M}$ as a brane, very much like in string theory. However from the point of view of the physics on $\mathcal{M}$, the $T^{\dot{a}}$ and their commutators

$$\Theta^{\dot{a}\dot{b}} := i[T^{\dot{a}}, T^{\dot{b}}] \ \sim -\{T^{\dot{a}}, T^{\dot{b}}\} \tag{6}$$

play also another role, and can be related to geometric i.e. tensorial objects on $\mathcal{M}$.

The key to understand $T^{\dot{a}}$ and $\Theta^{\dot{a}\dot{b}}$ is to observe that they generate *Hamiltonian vector fields* on $\mathcal{M}$:

$$E^{\dot{a}}[\phi] := \{T^{\dot{a}}, \phi\} \tag{7}$$

$$\mathcal{T}^{\dot{a}\dot{b}}[\phi] := \{\Theta^{\dot{a}\dot{b}}, \phi\} \tag{8}$$

acting on some test-function $\phi \in C(\mathcal{M})$. These vector fields can be made more explicit by introducing local coordinates $y^\mu$ on the $n$-dimensional manifold $\mathcal{M}$. Define

$$E^{\dot{a}\mu} := \{T^{\dot{a}}, y^\mu\} , \tag{9}$$

$$\mathcal{T}^{\dot{a}\dot{b}\mu} := \{\Theta^{\dot{a}\dot{b}}, y^\mu\} ; \tag{10}$$

their significance will be clarified shortly. We must carefully distinguish the different types of indices: Greek indices $\mu, \nu = 1, ..., n$ will denote local coordinate indices on $\mathcal{M}$, which play the role of tensor indices. Dotted Latin indices $\dot{a}, \dot{b} = 0, ..., 9$ indicate frame-like indices which are unaffected by a change of coordinates $y^\mu$, but transform under the global $SO(1,9)$ symmetry of the matrix model. These frame-like indices will be raised and lowered with $\eta_{\dot{a}\dot{b}}$. In particular, the $E^{\dot{a}\mu}$ define vector fields

$$E^{\dot{a}} = E^{\dot{a}\mu}\partial_\mu \tag{11}$$

on $\mathcal{M}$, which play a role of a (generalized) *frame* on $\mathcal{M}$. This will allow to understand the effective geometry and the gauge theory which arises on $\mathcal{M}$ through the matrix model. In particular, we can recognize the infinitesimal gauge transformations in the matrix model

$$\delta_\Lambda T^{\dot{a}} = [T^{\dot{a}}, \Lambda] \sim i\{T^{\dot{a}}, \Lambda\} = iE^{\dot{a}\mu}\partial_\mu\Lambda \tag{12}$$

as generators of a sub-sector of diffeomorphisms on $\mathcal{M}$, namely of the symplectomorphisms. Finally, the tensor $\mathcal{T}^{\dot{a}\dot{b}\mu}$ can be recognized as torsion of the Weitzenböck connection associated to the frame $E^{\dot{a}}$, which is very useful to describe the non-linear regime of the matrix model in the semi-classical regime [3, 9].

**Covariant quantum space-time.** In the following we will focus on branes $\mathcal{M}$ which are embedded in target space along the $\dot{a}, \dot{b} = 0, ..., 3$ directions. Then the extra dotted indices will mostly be ignored, but they play a role once fuzzy extra dimensions are included. However, this assumption does not mean that $\mathcal{M}$ is a 4-dimensional manifold; if $\mathcal{M}$ is 4-dimensional, then the Poisson tensor $\theta^{\mu\nu}$ on $\mathcal{M}$ plays the role of some background tensor on space-time, which is problematic since it breaks Lorentz invariance. To avoid this we will consider a different class of **covariant quantum spaces**, which have the structure of a $S^2$ bundle over space or space-time

$$\mathcal{M} \cong S^2 \times \mathcal{M}^{3,1} \qquad \text{locally} . \tag{13}$$

The prototype $\bar{\mathcal{M}}$ of such a structure [1] is obtained as a certain projection of the fuzzy hyperboloid $H^4_n$ [10, 11], and gives rise to a quantum space-time $\bar{\mathcal{M}}^{3,1}_n$ with FLRW geometry and Minkowski signature. For other examples and approaches to covariant quantum spaces[1] see e.g. [11–18].

Let us describe the structure of the covariant quantum space-time $\bar{\mathcal{M}}$ in some detail. In the semi-classical limit $n \to \infty$, $\bar{\mathcal{M}}$ reduces to an $SO(3,1)$- equivariant $S^2$ bundle over $\bar{\mathcal{M}}^{3,1}$. The functions on the 6-dimensional $\bar{\mathcal{M}}$ are generated by generators $x^\mu$ which describe $\bar{\mathcal{M}}^{3,1}$, and $t_\mu$ which generate the internal sphere $S^2$. Both sets of generators transform covariantly under $SO(3,1)$, and satisfy the constraints

$$x_\mu x^\mu = -R^2 - x_4^2 = -R^2 \cosh^2(\eta) , \qquad R \sim \frac{r}{2} n \tag{14}$$

$$t_\mu t^\mu = r^{-2} \cosh^2(\eta) \tag{15}$$

$$t_\mu x^\mu = 0 \tag{16}$$

where indices are contracted with $\eta^{\mu\nu}$. Here $\eta \in (-\infty, \infty)$ plays the role of a FLRW time parameter, featuring a big bounce at $\eta = 0$. The space of functions decomposes into a direct sum $\mathrm{End}(\mathcal{H}_n) = \oplus \mathcal{C}^s$ of higher spin ($\mathfrak{hs}$) modes on $\mathcal{M}^{3,1}$, which in the semi-classical regime can be organized in terms of totally symmetric traceless tensors

$$\begin{aligned} \phi^{(s)} &= \phi_{\mu_1 \ldots \mu_s}(x) t^{\mu_1} \ldots t^{\mu_s} \\ \phi_{\mu_1 \ldots \mu_s} x^{\mu_i} &= 0 = \phi_{\mu_1 \ldots \mu_s} \eta^{\mu_i \mu_j} . \end{aligned} \tag{17}$$

$\bar{\mathcal{M}}$ is a symplectic manifold (which is quantized in the matrix model), and the Poisson tensor $\theta^{\mu\nu} = \{x^\mu, x^\nu\}$ vanishes upon projection to space-time $\mathcal{M}^{3,1}$. This projection or averaging over $S^2$ will be denoted by $[.]_0$:

$$[\theta^{\mu\nu}]_0 \equiv \int\limits_{S^2} \theta^{\mu\nu} = 0 . \tag{18}$$

The more generic covariant quantum spaces under consideration here are by definition the *same symplectic bundle* $\mathcal{M} \cong \bar{\mathcal{M}}$ , realized as a background of the model through a different, perturbed embedding map $T^{\dot{a}} \sim t^{\dot{a}}$. More explicitly,

$$T^{\dot{a}} = \bar{T}^{\dot{a}} + \mathcal{A}^{\dot{a}} \sim t^{\dot{a}} + \mathcal{A}^{\dot{a}} \tag{19}$$

[1] The framework of [19] is also somewhat similar to ours, but the bundles under consideration there are vastly bigger.

where $\mathcal{A}^{\dot{a}}$ are functions on $\mathcal{M}$ or equivalently $\mathfrak{hs}$-valued functions on $\mathcal{M}^{3,1}$ which can be expanded in the form (17). In particular, all these backgrounds are equivalent as symplectic spaces, and we will always use the standard coordinate functions $x^\mu$ and $t^\mu$ as for the undeformed background $\bar{\mathcal{M}}$, with the same the symplectic form or Poisson structure. This is very natural since symplectic manifolds are rigid, so that any deformation is equivalent (locally, at least) to the undeformed space by some diffeomorphism.

The purpose of this short paper is to clarify if and under what conditions the higher-spin gauge theory on $\mathcal{M}^{3,1}$ can be reduced to (or is dominated by) the classical geometry i.e. the lowest spin sector on $\mathcal{M}^{3,1}$, which is supposed to play the role of physical space-time. More explicitly, we want to understand if it is consistent to restrict to fluctuations of the form

$$\mathcal{A}^{\dot{a}} = \mathcal{A}^{\dot{a}\mu}(x)\, t_\mu \; , \tag{20}$$

dropping or neglecting higher-spin $\mathfrak{hs}$ contributions $\mathcal{A}^{\dot{a}\mu_1\ldots\mu_s}\, t_{\mu_1}\ldots t_{\mu_s}$. We will indeed establish that backgrounds of the structure

$$T^{\dot{a}} = T^{\dot{a}\mu}(x)\, t_\mu \tag{21}$$

are sufficiently rich to describe generic 3+1-dimensional space-time geometries, and provide a self-consistent class of configurations in the matrix model where higher-spin corrections are negligible in the **weak gravity regime**, to be discussed below.

## 2. Effective metric and frame on covariant quantum space-time

Now we establish the interpretation of $E^{\dot{a}\mu}$ as frame on $\mathcal{M}^{3,1}$. As in any field theory, the effective metric governing some field or fluctuation mode is encoded in the kinetic term of the action. Consider a matrix background corresponding to some $2n$-dimensional brane $\mathcal{M} \hookrightarrow \mathbb{R}^{3,1} \subset \mathbb{R}^{9,1}$. Then the kinetic (=quadratic) term for transversal fluctuations[2] in Yang-Mills matrix models has the structure

$$\begin{aligned}
S[\phi] = \mathrm{Tr}([T^{\dot{a}},\phi][T_{\dot{a}},\phi]) &\sim -\frac{1}{(2\pi)^n}\int\limits_{\mathcal{M}} \Omega\, \{T^{\dot{a}},\phi\}\{T_{\dot{a}},\phi\} \\
&= -\frac{1}{(2\pi)^n}\int\limits_{\mathcal{M}} \Omega\, \eta_{\dot{a}\dot{b}} E^{\dot{a}\mu} E^{\dot{b}\nu} \partial_\mu\phi\partial_\nu\phi \\
&= -\frac{1}{(2\pi)^n}\int\limits_{\mathcal{M}} \Omega\, \gamma^{\mu\nu} \partial_\mu\phi\partial_\nu\phi
\end{aligned} \tag{22}$$

in the semi-classical regime, recognizing (9). Here $\Omega$ is the symplectic volume form on $\mathcal{M}$, and

$$\gamma^{\mu\nu} := \eta_{\dot{a}\dot{b}} E^{\dot{a}\mu} E^{\dot{b}\nu} \; . \tag{23}$$

This is clearly the metric determined by the frame $E^{\dot{a}\mu}$; however the effective metric acquires an extra conformal factor, which arises as follows. In the case of covariant quantum spaces under

[2]The case of tangential fluctuations can be analyzed similar and leads to the same metric.

consideration, we can assume that $\mathcal{M} = \bar{\mathcal{M}} = S^2 \times \mathcal{M}^{3,1}$, with a global $SO(3)$ symmetry acting on $S^2$ and $\mathcal{M}^{3,1}$ simultaneously. Then $\Omega$ factorizes into the volume of the $S^2$ fiber times the effective density $\rho_M$ on space-time $\mathcal{M}^{3,1}$ [1]:

$$\Omega = \rho_M d^4x\, \Omega_2 , \qquad \rho_M = \frac{1}{r^2 R^2 \sinh(\eta)} \sim L_{\rm NC}^{-4} . \tag{24}$$

Here $S^2$ is normalized with volume $4\pi$, and $x^\mu$ are the Cartesian coordinates (14) on $\mathcal{M}^{3,1}$ or[3] $\bar{\mathcal{M}}^{3,1}$. $L_{\rm NC}$ characterizes the scale of noncommutativity. Then (22) can be written in a more familiar form

$$S[\phi] \sim -\frac{1}{2\pi^2} \int\limits_{\mathcal{M}} \rho_M\, \gamma^{\mu\nu} \partial_\mu\phi \partial_\nu\phi = -\frac{1}{2\pi^2} \int\limits_{\mathcal{M}^{3,1}} d^4x\, \sqrt{|G_{\mu\nu}|} G^{\mu\nu} \partial_\mu\phi\partial_\nu\phi . \tag{25}$$

We can now read off the **effective metric** on $\mathcal{M}^{3,1}$:

$$G^{\mu\nu} = \rho^{-2}\, \gamma^{\mu\nu} \tag{26}$$

where $\rho$ is the **dilaton**, which relates the symplectic density $\rho_M$ to the Riemannian density via

$$\rho^{-2}\sqrt{|G_{\mu\nu}|} = \rho_M \ = \rho^2 \sqrt{|\gamma_{\mu\nu}|} \tag{27}$$

using $\sqrt{|G_{\mu\nu}|} = \rho^4\sqrt{|\gamma_{\mu\nu}|}$. From the string theory point of view, the metric $G_{\mu\nu}$ can be interpreted as open-string metric on $\mathcal{M}^{3,1}$. Noting that

$$\sqrt{|\gamma^{\mu\nu}|} = |\det E^{\dot\alpha\mu}| , \tag{28}$$

the dilaton is determined by the frame as

$$\rho^2 = \rho_M\, |\det E^{\dot\alpha\mu}| . \tag{29}$$

It is important that the frame $E^{\dot\alpha\mu}$ in the present context does *not* admit local $SO(3,1)$ gauge transformations acting on $\dot\alpha$, only global $SO(3,1)$ transformations are allowed. The frame is a physical object here which is subject to certain constraints (68), and determines not only the metric but also additional physical information, such as the dilaton $\rho$ and also an axion $\tilde\rho$ (104).

### 2.1 Cosmological FLRW solution

A special case of the above class of backgrounds is given by

$$T^\mu = \frac{1}{R} M^{\mu 4} \ \sim t^\mu \tag{30}$$

where $M^{ab}$ are generators of the doubleton representation $\mathcal{H}_n$ of $\mathfrak{so}(4,1) \subset \mathfrak{so}(4,2)$. It is easy to see that $T^\mu$ is a solution of the matrix model in the presence of a suitable mass term; we shall simply discuss some of its properties here. $T^\mu$ defines a matrix configuration with manifest $SO(3,1)$ symmetry, which in the semi-classical regime reduces to a 6-dimensional background $\bar{\mathcal{M}}$ which is an $S^2$ bundle over $\bar{\mathcal{M}}^{3,1}$. The Cartesian coordinate functions on the base manifold $\bar{\mathcal{M}}^{3,1}$ arise as

$$X^\mu = r\, M^{\mu 5} \ \sim x^\mu . \tag{31}$$

[3] Recall that $\mathcal{M} = \bar{\mathcal{M}}$ as a manifold, only the embedding and the frame are deformed.



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We will focus on the semi-classical (Poisson) limit $n \to \infty$, working with commutative functions of $x^\mu$ and $t^\mu$, but keeping the Poisson structure $[.,.] \sim i\{.,.\}$. Then End$(\mathcal{H}_n) \sim \mathcal{C}$ reduces to the algebra of functions on the bundle space $\mathcal{M} \cong \mathbb{C}P^{2,1}$, dropping the bar for now. The sub-algebra $\mathcal{C}^0 \subset \mathcal{C}$ of functions on the base space $\mathcal{M}^{3,1}$ is generated by the

$$x^\mu : \mathcal{M}^{3,1} \hookrightarrow \mathbb{R}^{3,1} \tag{32}$$

for $\mu = 0, ..., 3$, which are interpreted as Cartesian coordinate functions. The generators $x^\mu$ and $t^\mu$ satisfy the constraints (16), which arise from the special properties of $\mathcal{H}_n$. The $t^\mu$ generators describe the $S^2$ fiber over $\mathcal{M}^{3,1}$, which is space-like due to (16). Here $\eta$ plays the role of a time parameter, defined via

$$x^4 = R \sinh(\eta) \ . \tag{33}$$

Hence $\eta = const$ defines a foliation of $\mathcal{M}^{3,1}$ into space-like surfaces $H^3$; this can be related to the scale parameter of a FLRW cosmology with $k = -1$. Note that $\eta$ runs from $-\infty$ to $\infty$, and the sign of $\eta$ distinguishes the two degenerate sheets of $\mathcal{M}^{3,1}$ linked by a Big Bounce, cf. [20]. The Poisson brackets on $\bar{\mathcal{M}}$ are given explicitly by

$$\begin{aligned} \{x^\mu, x^\nu\} &= \theta^{\mu\nu} = -r^2 R^2 \{t^\mu, t^\nu\} \ , \\ \{t^\mu, x^\nu\} &= \frac{x^4}{R}\eta^{\mu\nu} \ , \end{aligned} \tag{34}$$

where the Poisson tensor $\theta^{\mu\nu}$ satisfies the constraints

$$t_\mu \theta^{\mu\alpha} = -\sinh(\eta) x^\alpha, \tag{35a}$$

$$x_\mu \theta^{\mu\alpha} = -r^2 R^2 \sinh(\eta) t^\alpha, \tag{35b}$$

$$\eta_{\mu\nu}\theta^{\mu\alpha}\theta^{\nu\beta} = R^2 r^2 \eta^{\alpha\beta} - R^2 r^4 t^\alpha t^\beta + r^2 x^\alpha x^\beta \ . \tag{35c}$$

$\theta^{\mu\nu}$ can be expressed in terms of $t^\mu$ as

$$\theta^{\mu\nu} = \frac{r^2}{\cosh^2(\eta)}\Big(\sinh(\eta)(x^\mu t^\nu - x^\nu t^\mu) + \epsilon^{\mu\nu\alpha\beta} x_\alpha t_\beta\Big) \ , \tag{36}$$

and can therefore be viewed as spin 1 valued "function" on $\mathcal{M}^{3,1}$. More generally, the space of functions $\mathcal{C}$ on $\mathcal{M}$ decomposes into a tower of higher-spin ($\mathfrak{hs}$) valued functions

$$\mathcal{C} = \bigoplus_{s \geq 0} \mathcal{C}^s \tag{37}$$

on $\mathcal{M}^{3,1}$, where $\mathcal{C}^s$ is spanned by irreducible polynomials (17) of degree $s$ in $t^\mu$. The Poisson brackets do not respect the decomposition into $\mathcal{C}^s$, but the following holds

$$\{\mathcal{C}^s, x^\mu\} \in \mathcal{C}^{s+1} \oplus \mathcal{C}^{s-1} \tag{38}$$

noting that $\theta^{\mu\nu} \in \mathcal{C}^1$.

**Frame, metric and torsion on $\bar{\mathcal{M}}^{3,1}$.** Following the general strategy discussed above, we can extract the effective metric on $\bar{\mathcal{M}}^{3,1}$. Frame and metric are obtained in Cartesian coordinates from (34) as

$$\begin{aligned} E^{\dot{\alpha}} &= \{t^{\dot{\alpha}}, \cdot\} = E^{\dot{\alpha}\mu}\partial_\mu, \qquad E^{\dot{\alpha}\mu} = \eta^{\dot{\alpha}\mu}\sinh(\eta)\,, \\ \gamma^{\mu\nu} &= \eta_{\dot{\alpha}\dot{\beta}}E^{\dot{\alpha}\mu}E^{\dot{\beta}\nu} = \sinh^2(\eta)\eta^{\mu\nu}\,. \end{aligned} \tag{39}$$

Recalling that $\rho_M \sim \sinh(\eta)^{-1}$, the effective metric on $\bar{\mathcal{M}}^{3,1}$ and the dilaton are obtained as

$$\begin{aligned} G_{\mu\nu} &= \sinh^3(\eta)\gamma_{\mu\nu} = \sinh(\eta)\eta_{\mu\nu}\,, \\ \rho^2 &= \sinh^3(\eta)\,. \end{aligned} \tag{40}$$

This metric is $SO(3,1)$-invariant with signature $(-+++)$ and conformal to the induced ("closed-string") metric $\eta_{\mu\nu}$. It can be written in standard FLRW form as follows [1]

$$ds^2_G = G_{\mu\nu}dx^\mu dx^\nu = -dt^2 + a^2(t)d\Sigma^2 \tag{41}$$

where $d\Sigma^2$ is the metric on $H^3$, and the FLRW time $t$ is related to the time parameter $\eta$ via

$$a(t) \sim R\sinh^{3/2}(\eta) =: L_{\rm cosm}, \qquad t\to\infty\,. \tag{42}$$

One finds $a(t) \sim \frac{3}{2}t$ for late times, and $a(t) \sim t^{1/5}$ near the Big Bounce. The torsion tensor (10) is also easily computed using $\Theta^{\dot{\alpha}\dot{\beta}} = \frac{1}{R^2}\mathcal{M}^{\dot{\alpha}\dot{\beta}}$, which gives

$$\mathcal{T}^{\dot{\alpha}\dot{\beta}\mu} = \{\Theta^{\dot{\alpha}\dot{\beta}}, x^\mu\} = \frac{1}{R^2}\big(\eta^{\dot{\alpha}\mu}x^{\dot{\beta}} - \eta^{\dot{\beta}\mu}x^{\dot{\alpha}}\big) \tag{43}$$

in Cartesian coordinates $x^\mu$. This can be recast as a rank 3 tensor on $\mathcal{M}^{3,1}$ using the frame $E^{\dot{\alpha}\mu}$,

$$\mathcal{T}_{\nu\sigma}{}^{\mu} = \frac{1}{R^2\rho^2}\big(\delta^\mu_\nu\tau_\sigma - \delta^\mu_\sigma\tau_\nu\big) \tag{44}$$

where

$$\tau_\mu = G_{\mu\nu}\tau^\nu = G_{\mu\nu}x^\nu = \sinh(\eta)\eta_{\mu\nu}x^\nu \tag{45}$$

is a global time-like $SO(3,1)$-invariant vector field on the FLRW background.

**Late-time regime and noncommutativity scale.** Consider the regime of late time or large $\eta$, so that $\sinh(\eta) \gg 1$. Then the Poisson tensor $\theta^{\mu\nu}$ (36) reduces to

$$\theta^{\mu\nu} \sim \frac{r^2}{\cosh(\eta)}(x^\mu t^\nu - x^\nu t^\mu)\,, \qquad \eta\to\infty. \tag{46}$$

More specifically, consider some given reference point $\xi = (x^0, 0, 0, 0)$ on $\mathcal{M}$. Then this reduces to

$$\begin{aligned} \theta^{0i} &\overset{\xi}{=} \frac{r^2}{\cosh^2(\eta)}\sinh(\eta)x^0 t^i \sim r^2 R t^i \quad = O(L^2_{\rm NC}) \\ \theta^{ij} &\overset{\xi}{=} \frac{r^2}{\cosh^2(\eta)}x^0\epsilon^{0ijk}t_k \sim \frac{1}{\sinh(\eta)}r^2 R\epsilon^{ijk}t^k \quad = O(rR)\,, \end{aligned} \tag{47}$$

where

$$L_{\rm NC}^2 = Rr\cosh(\eta) \tag{48}$$

is the effective scale of noncommutativity on $\mathcal{M}^{3,1}$ (cf. (24)), using $|t| \sim r^{-1}\cosh(\eta)$ (15). Even though this grows with $\eta$, it is much shorter than the cosmic curvature scale (42):

$$\frac{L_{\rm cosm}^2}{L_{\rm NC}^2} \sim \frac{R}{r}\cosh^2(\eta) \sim n\cosh^2(\eta)\ . \tag{49}$$

Therefore there is plenty of space for interesting physics in between. In particular, $\theta^{0i} \sim r^2 R t^i \gg \theta^{ij}$ at late times $\eta \gg 1$. The space-like generators $t^i$ describe the internal fuzzy sphere $S^2_n$ with

$$\{t^i, t^j\} \stackrel{\xi}{=} -\frac{1}{r^2R^2}\theta^{ij} = -\frac{1}{R\sinh(\eta)}\epsilon^{ijk}t^k \tag{50}$$

and generate the higher-spin algebra $\mathfrak{hs}$. Even though $t^0 \stackrel{\xi}{=} 0$ vanishes as function at $\xi$, it is a non-trivial generator which induces local time translations via $\{t^0, .\}$.

### 2.2 Derivations

**Fuzzy hyperboloid $H^4_n$.** The above space-time $\mathcal{M}^{3,1}$ can be understood as a projection of the fuzzy hyperboloid $H^4_n$ [10], which can be viewed as a submanifold of $\mathbb{R}^{4,1}$ defined in terms of the 5 generators

$$X^a = r\, M^{a5} \sim x^a\ , \qquad a = 0, \ldots, 4 \tag{51}$$

(cf. (31)) which transform as vectors of $SO(4,1)$. The underlying symplectic space is the same as for $\mathcal{M}^{3,1}$, given by the non-compact projective space $\mathbb{C}P^{2,1}$ which is nothing but (projective) twistor space, cf. [21]. The Poisson structure on the bundle space allows to define derivations as follows

$$\eth^a\phi := -\frac{1}{r^2R^2}\theta^{ab}\{x_b,\phi\} = \frac{1}{r^2R^2}x_b\{\theta^{ab},\phi\}, \qquad \phi \in \mathcal{C}\ . \tag{52}$$

They satisfy the useful identities

$$\begin{aligned} x^a\eth_a\phi &= 0\ ,\\ \eth^a x^c &= \eta^{ab} + \frac{1}{R^2}x^a x^b\ ,\\ \eth^a(\{x_a,\phi\}) &= 0 \end{aligned} \tag{53}$$

for any $\phi \in \mathcal{C}$. Furthermore, we note that all (even $\mathfrak{hs}$-valued) Hamiltonian vector fields on $H^4_n$ are tangential to $H^4 \subset \mathbb{R}^{4,1}$, due to the identity

$$x^a\{x_a, \Lambda\} = 0\ . \tag{54}$$

**Derivatives on $\mathcal{M}^{3,1}$.** Since the algebra of functions $\mathcal{C}$ for $\mathcal{M}^{3,1}$ and $H^4_n$ is the same, we can use the above derivative operators to define the following derivations on $\mathcal{M}^{3,1}$

$$\partial_\mu := \eth_\mu - x_\mu \frac{1}{x_4}\eth_4 \qquad \text{on} \quad \mathcal{C} \,. \tag{55}$$

Using the identities (53), it is easy to show

$$\begin{aligned} \partial_\mu x^\nu &= \delta^\nu_\mu \\ \partial_\mu(\rho_M \theta^{\mu\nu}) &= 0 \,. \end{aligned} \tag{56}$$

This will imply that all Hamiltonian vector fields on $\mathcal{M}^{3,1}$, in particular the frame, are conserved.

## 3. Divergence-free vector fields on $H^4$ and $\mathcal{M}^{3,1}$

Divergence-free vector fields will play an important role in the following. Clearly any vector field $V^a$ on $H^4$ can be mapped to a vector field $V^\mu$ on $\mathcal{M}^{3,1}$, by simply dropping the $V^4$ component (in Cartesian coordinates). This can be understood as push-forward via a projection [1]. For example, a Hamiltonian vector field $V^a = \{T, x^a\}$ is mapped to $V^\mu = \{T, x^\mu\}$ in Cartesian coordinates. Conversely, any vector field $V^\mu$ on $\mathcal{M}^{3,1}$ can be lifted to $H^4$ by defining

$$V^4 := -\frac{1}{x_4} x_\mu V^\mu \,, \tag{57}$$

which defines a tangential vector field $V^a x_a = 0$ on $H^4$. We claim that this correspondence maps divergence-free vector fields $\eth_a V^a = 0$ on $H^4$ to divergence-free vector fields on $\mathcal{M}^{3,1}$, in the sense that

$$\partial_\mu(\rho_M V^\mu) = 0 \,. \tag{58}$$

Here $\rho_M$ is the symplectic density (24) on $\mathcal{M}^{3,1}$, which in Cartesian coordinates is given by $\rho_M = \sinh(\eta)^{-1}$. In fact the following more general result holds:

**Lemma 3.1.** *Let $V^a$ be a (tangential) vector field on $H^4$, i.e. $V^a x_a = 0$. Then its reduction (or push-forward) $V^\mu$ to $\mathcal{M}^{3,1}$ satisfies*

$$\eth_a V^a = \sinh(\eta)\partial_\mu(\rho_M V^\mu) \tag{59}$$

*Conversely, the lift of $V^\mu$ to $H^4$ defined by (57) satisfies (59). If $V^a$ is divergence-free on $H^4$ i.e. $\eth_a V^a = 0$, then its reduction to $\mathcal{M}^{3,1}$ satisfies*

$$\partial_\mu(\rho_M V^\mu) = 0 \,. \tag{60}$$

*In particular, all Hamiltonian vector fields on fuzzy $H^4$ and $\mathcal{M}^{3,1}$ are conserved, in the sense*

$$\eth_a\{x^a, T\} = 0, \qquad \partial_\mu(\rho_M\{x^\mu, T\}) \tag{61}$$

*Proof.* Using the definition of $\partial_\mu$ (55) on $C$, we compute

$$\begin{aligned}
ð_a V^a &= ð_\mu V^\mu + ð_4 V^4 \\
&= \big(\partial_\mu + \frac{1}{x_4} x_\mu ð_4\big) V^\mu + ð_4 V^4 \\
&= \partial_\mu V^\mu + \frac{1}{x_4} ð_4 (x_\mu V^\mu) - \frac{1}{x_4} V^\mu ð_4 x_\mu + ð_4 V^4 \\
&= \partial_\mu V^\mu - \frac{1}{x_4} ð_4 (x_4 V^4) - \frac{1}{R^2} x_\mu V^\mu + ð_4 V^4 \\
&= \partial_\mu V^\mu - \frac{1}{x_4} V^4 ð_4 x_4 + \frac{1}{R^2} x_4 V^4 \\
&= \partial_\mu V^\mu - \frac{1}{x_4} V^4 \\
&= \sinh(\eta) \partial_\mu \big(\frac{1}{\sinh(\eta)} V^\mu\big) \ .
\end{aligned} \tag{62}$$

(61) now follows using (53).

□

In particular, the identity (56) can now be understood by noting that $V^a = \{x^\nu, x^a\}$ is conserved on $H^4$. We also note that the divergence constraint (60) for vector fields on $\mathcal{M}^{3,1}$ can be written using (27) in covariant form in terms of the effective metric $G^{\mu\nu}$ on $\mathcal{M}^{3,1}$:

$$0 = \nabla_\mu (\rho^{-2} V^\mu) = \frac{1}{\sqrt{|G|}} \partial_\mu \big(\rho_M V^\mu\big) \tag{63}$$

where $\nabla$ is the Levi-Civita connection corresponding to $G$.

## 4. Generic backgrounds from deformed $\mathcal{M}^{3,1}$

Starting from the above FLRW background, we can obtain more generic geometries as deformations, by simply adding fluctuations of the background:

$$T^a = \bar{T}^a + \mathcal{A}^a \tag{64}$$

The fluctuations $\mathcal{A}^a$ are any $\mathfrak{hs}$ valued gauge fields, which are governed by a Yang-Mills gauge theory. We want to focus in the following on purely geometric deformations, leaving aside the higher spin modes. We therefore focus on fluctuations of the form

$$\mathcal{A}^a = \mathcal{A}^{a\mu}(x) t_\mu \tag{65}$$

Since we don't want to restrict ourselves to the linearized perturbations, we simply consider generic backgrounds of the form

$$T^a = T^{a\mu}(x) t_\mu \tag{66}$$

which include the cosmic background for $T^{\dot{\alpha}\mu}(x) = \eta^{\dot{\alpha}\mu}$. As discussed in section 2, such a background defines a frame (66)

$$E^{\dot{\alpha}\mu} = \{T^{\dot{\alpha}}, x^\mu\} \tag{67}$$

Taking into account the above results, we conclude that any such frame satisfies the divergence constraint [2]

$$\boxed{\partial_\mu(\rho_M E^{\dot\alpha\mu}) = \nabla_\mu(\rho^{-2}E^{\dot\alpha\mu})\ .} \tag{68}$$

In the following we will establish the converse statement: any frame given by divergence-free vector fields can indeed be implemented as above, for a suitable background of the form (66). Moreover, the $T^{\dot\alpha}$ can be computed explicitly. This entails in general some extra $\mathfrak{hs}$ valued contribution to the frame, which will be shown to be insignificant in section 5 in the weak gravity regime.

### 4.1 Reconstruction of divergence-free vector fields

We start by recalling two results given in [4], starting with the Euclidean case:

**Lemma 4.1.** *Given any divergence-free tangential vector field $\eth_a V^a = 0$ on $H^4$ with $V^a \in \mathcal{C}^0$, there is a unique generator $T \in \mathcal{C}^1$ such that*

$$V^a = \{T, x^a\}_0\ . \tag{69}$$

*This $T$ is given explicitly by*

$$\boxed{T := -3(\Box_H - 4r^2)^{-1}\{V^a, x_a\} \qquad \in \mathcal{C}^1} \tag{70}$$

*where $\Box_H = \{x^a, \{x_a, .\}\}$.*

However, the Hamiltonian vector field $\{T, x^a\}$ generated by the above $T \in \mathcal{C}^1$ contains in general also a spin 2 component

$$V^{(2)a} := \{T, x^a\}_2 \qquad \in \mathcal{C}^2 \tag{71}$$

due to (38). Since $\eth$ respects $\mathcal{C}^n$, this is also divergence-free $\eth_a V^{(2)a} = 0$. One might hope that is can be canceled by adding higher corrections to $T$, but this is not possible in general. Therefore the above reconstruction of vector fields on $H^4$ generically leads to extra $\mathfrak{hs}$ components $V^{(2)a} \in \mathcal{C}^2$ (71), which however encode the same information as $V^a$. It remains an open question if these can be cancelled by allowing higher-spin corrections to the coordinate generators $x^a$.

We can use this to obtain an analogous "reconstruction" statement on $\mathcal{M}^{3,1}$ [4]:

**Lemma 4.2.** *Given any $\mathcal{C}^0$-valued divergence-free vector field $V^\mu$ on $\mathcal{M}^{3,1}$,*

$$\partial_\mu(\rho_M V^\mu) = 0 \tag{72}$$

*there is a generating function $T \in \mathcal{C}^1$ such that*

$$V^\mu = \{T, x^\mu\}_0\ . \tag{73}$$

*Explicitly, $T$ is given by*

$$\boxed{T = -3(\Box_H - 4r^2)^{-1}\big(\{V^\mu, x_\mu\} + \{V^4, x_4\}\big)} \tag{74}$$

*where*

$$V^4 = -\frac{1}{x_4}x_\mu V^\mu\ . \tag{75}$$

This is simply obtained by lifting $V^\mu$ to a divergence-free vector field $V^a$ on $H^4$ as in Lemma 3.1. Then the result (69) on $H^4$ states that $V^a = \{T, x^a\}_0$ for some $T \in \mathcal{C}^1$, which implies $V^\mu = \{T, x^\mu\}_0$. Moreover, $T$ is uniquely determined by (73). One can show that this spin 2 component vanishes only for $T \in \mathfrak{so}(4,1)$.

To summarize, we have shown that every divergence-free vector field on $\mathcal{M}^{3,1}$ can be realized or reconstructed as Hamiltonian vector field, i.e. $V^\mu = \{T, x^\mu\}_0$. However, this entails the presence of a spin two sibling $V^{(2)\mu} = \{T, x^\mu\}_2 \in \mathcal{C}^2$. In other words, the Hamiltonian vector field generated by $T \in \mathcal{C}^1$ acts on a function $\phi = \phi(x) \in \mathcal{C}^0$ via

$$\{T, \phi\} = \{T, x^\mu\}\partial_\mu\phi = (V^\mu + V^{(2)\mu})\partial_\mu\phi \ . \tag{76}$$

Both components of $V^\mu + V^{(2)\mu} \in \mathcal{C}^0 \oplus \mathcal{C}^2$ are isomorphic as $\mathfrak{so}(4,1)$ modes. This applies in particular to the frame in the effective field theory on $\mathcal{M}^{3,1}$ arising from matrix models.

### 4.2 Reconstruction of classical geometry

Now we apply the results of the previous section to reconstruct a classical frame $e^{\dot\alpha\mu}$ within the present framework. This is the basis for describing gravity through the effective metric on a suitable covariant quantum spaces. It is clear from (68) that only divergence-free frames can be realized here, but this does not restrict the possible metrics as explained in section 6. Hence for any divergence-free classical frame $e^{\dot\alpha\mu}$, there is a unique $T^{\dot\alpha} \in \mathcal{C}^1$ given by

$$\begin{aligned} T^{\dot\alpha} &= -3(\Box_H - 4r^2)^{-1}\big(\{e^{\dot\alpha\mu}, x_\mu\} + \{e^{\dot\alpha 4}, x_4\}\big) \\ &= -3(\Box_H - 4r^2)^{-1}\big(\{e^{\dot\alpha\mu}, x_\mu\} - \frac{1}{x_4}\{e^{\dot\alpha\mu}x_\mu, x_4\}\big) \end{aligned} \tag{77}$$

such that

$$e^{\dot\alpha\mu} = \{T^{\dot\alpha}, x^\mu\}_0 \ . \tag{78}$$

E.g. for the cosmic frame $e^{\dot\alpha\mu} = \sinh(\eta)\eta^{\dot\alpha\mu}$ on $\mathcal{M}^{3,1}$, this gives

$$e^{\dot\alpha 4} = -\frac{x_\mu}{x_4}e^{\dot\alpha\mu} = -\frac{1}{r}x^{\dot\alpha} \tag{79}$$

and we recover the background (30)

$$\begin{aligned} T^{\dot\alpha} &= -3(\Box_H - 4r^2)^{-1}\big(\{e^{\dot\alpha\mu}, x_\mu\} - \frac{1}{r}\{x^{\dot\alpha}, x_4\}\big) \\ &= 6(\Box_H - 4r^2)^{-1}\{\sinh(\eta), x^{\dot\alpha}\} \\ &= t^{\dot\alpha} \end{aligned} \tag{80}$$

using $\Box_H t^\mu = -2r^2 t^\mu$. The generator $T^{\dot\alpha} \in \mathcal{C}^1$ is uniquely determined by (67). However, the reconstructed frame will in general contain higher spin $\mathfrak{hs}$ components $\{T^{\dot\alpha}, x^\mu\}_+ \in \mathcal{C}^2$ due to (38). Even though these drop out in the linearized theory upon averaging over $S^2_n$, this is no longer true in the non-linear regime, and we must clarify the importance of these contributions.

## 5. Weak gravity regime and classical geometry

Finally we address the crucial question if and under what conditions the reconstructed frame can be approximated by its classical component in $\mathcal{C}^0$:

$$\boxed{E^{\dot\alpha\mu} = \{T^{\dot\alpha}, x^\mu\} \stackrel{?}{\approx} \{T^{\dot\alpha}, x^\mu\}_0 = e^{\dot\alpha\mu} \ .} \tag{81}$$

Here $\{.,.\}_0$ denotes the projection of $\{T^{\dot\alpha}, x^\mu\} \in \mathcal{C}^0 \oplus \mathcal{C}^2$ to $\mathcal{C}^0$. That approximation was used or assumed in the description of the non-linear regime of gravity in [2, 3], where the (Weitzenböck) torsion and the Riemannian curvature are given by expressions of the form

$$\begin{aligned} \mathcal{T}_{\mu\nu}{}^\sigma &\sim e^{-1}\partial e \\ \mathcal{R}_{\mu\nu\sigma\kappa} &\sim e^{-1}\partial e e^{-1}\partial e \ . \end{aligned} \tag{82}$$

We will show that this holds in the "weak gravity regime", i.e. as long as the curvature of space-time is not too large. This is the regime where the frame $E^{\dot\alpha\mu} = \{T^{\dot\alpha}, x^\mu\}$ can be considered as a function of $x$ only, while the $\mathfrak{hs}$ contributions are negligible. If the curvature becomes too large, then the $\mathfrak{hs}$ components arising from the internal $S^2$ can no longer be neglected, and the theory becomes more complicated.

To understand the validity of the approximation (81), consider some generic background

$$T^{\dot\alpha} = T^{\dot\alpha\sigma}(x) t_\sigma \qquad \in \ \mathcal{C}^1 \tag{83}$$

which defines a frame

$$\begin{aligned} E^{\dot\alpha\mu} = \{T^{\dot\alpha}, x^\mu\} &= T^{\dot\alpha\sigma}\{t_\sigma, s^\mu\} + \{T^{\dot\alpha\kappa}, x^\mu\} t_\sigma \\ &= \sinh(\eta) T^{\dot\alpha\mu} + \frac{\partial T^{\dot\alpha\sigma}}{\partial x^\nu}\theta^{\nu\mu} t_\sigma \\ &=: e^{\dot\alpha\mu} \ + \delta E^{\dot\alpha\mu} \ . \end{aligned} \tag{84}$$

This decomposes into a local $\mathcal{C}^0$-valued background

$$e^{\dot\alpha\mu} := \sinh(\eta) T^{\alpha\mu} \qquad \in \ \mathcal{C}^0 \tag{85}$$

and a perturbation

$$\delta E^{\dot\alpha\mu} := \frac{\partial T^{\dot\alpha\sigma}}{\partial y^\nu}\theta^{\nu\mu} t_\sigma \qquad \in \ \mathcal{C}^0 \oplus \mathcal{C}^2 \ . \tag{86}$$

We would like to understand if and when the latter can be neglected, so that the first term provides a good approximation:

$$\{T^{\dot\alpha\sigma}(x) t_\sigma, x^\mu\} \stackrel{!}{\approx} T^{\dot\alpha\sigma}\{t_\sigma, x^\mu\} = e^{\dot\alpha\mu} \tag{87}$$

so that

$$\boxed{E^{\dot\alpha\mu} = \{T^{\dot\alpha}, x^\mu\} \approx e^{\dot\alpha\mu} \qquad \in \mathcal{C}^0 \ .} \tag{88}$$

This is justified if the derivatives of $T^{\dot{\alpha}\kappa}(x)$ is sufficiently small, more precisely

$$\delta E^{\dot{\alpha}\mu}|_p = \frac{\partial T^{\dot{\alpha}\sigma}}{\partial x^\nu}\theta^{\nu\mu}t_\sigma \stackrel{!}{\ll} \sinh(\eta)T^{\dot{\alpha}\mu} = e^{\dot{\alpha}\mu} \; . \tag{89}$$

Recalling that $\theta^{\nu\mu} = O(L_{\rm NC}^2) = Rr\cosh(\eta)$ (47) and $t = O(r^{-1}\cosh(\eta))$ (15), this holds provided

$$R\cosh^2(\eta)\frac{\partial T^{\dot{\alpha}\sigma}}{\partial x^\nu} \ll \sinh(\eta)T^{\dot{\alpha}\mu} = e^{\dot{\alpha}\mu} \; . \tag{90}$$

The lhs can be rewritten by the same token in terms of $e$, and this condition becomes

$$\frac{\partial e^{\dot{\alpha}\sigma}}{\partial x^\nu} \ll \frac{e^{\dot{\alpha}\mu}}{R\cosh(\eta)} \tag{91}$$

for all components. This means that the torsion and curvature tensors (82) are bounded in Cartesian coordinates by

$$\mathcal{T} \sim e^{-1}\partial e \;\ll\; \frac{1}{R\cosh(\eta)} \tag{92}$$

and

$$\mathcal{R} = e^{-1}\partial e e^{-1}\partial e \ll \frac{1}{R^2\cosh^2(\eta)} \; . \tag{93}$$

All of these conditions boil down to the requirement that the characteristic length scale $\lambda$ of the geometry (as encoded in the torsion and curvature) should satisfy

$$\boxed{\lambda \;\gg\; L_{\rm grav} \;:=\sqrt{L_{\rm cosm}L_{\rm NC}} = R\cosh(\eta) \; .} \tag{94}$$

This defines the **weak gravity regime** within the present framework, where the approximation (88) holds. Then the frame $E^{\dot{\alpha}\mu} \approx e^{\dot{\alpha}\mu}$ can be treated as $C^0$-valued, and

$$\{T^{\dot{\alpha}}, \phi(x)\} = E^{\dot{\alpha}\mu}\partial_\mu\phi \approx e^{\dot{\alpha}\mu}\partial_\mu\phi \qquad \in C^0 \tag{95}$$

is justified. Therefore all the considerations on volume-preserving frames, torsion, and curvature in [2–5] go through. If we tentatively identify $L_{\rm NC} \approx L_{\rm Pl} = 10^{-35}m$ with the Planck scale and set $L_{\rm cosm} \approx 10^{28}m$ which is 10 times the size of the visible universe, then $L_{\rm grav} \approx 10^{-4}m$. This regime leaves plenty of space for interesting gravitational physics, and e.g. event horizons of macroscopic black holes are easily within this regime by a large margin. Note also that (49)

$$\frac{L_{\rm cosm}}{L_{\rm grav}} \sim \cosh^{1/2}(\eta) \;\; \to \infty \tag{96}$$

grows with the cosmic expansion. Moreover, the lower bound $L_{\rm grav}$ for the weak gravity regime should be regarded as conservative, since the higher-spin components of the frame at any point $p$ can always be locally absorbed in terms of adapted local coordinates[4]; however it is not known if this can also be achieved in a finite neighborhood of $p$.

[4]This is not hard to see, and will be published elsewhere.

To summarize, the characteristic property of the weak gravity regime is that the full frame $E^{\dot{\alpha}\mu}$ reconstructed from a classical frame $e^{\dot{\alpha}\mu}$ according to section 4.2 approximately reduces to the classical frame,

$$E^{\dot{\alpha}\mu} = \{T^{\dot{\alpha}}, x^\mu\} \in \mathcal{C} \;\approx\; e^{\dot{\alpha}\mu} \;\in \mathcal{C}^0 \,. \tag{97}$$

This means that such geometry can indeed be faithfully realized in the matrix model, and all the standard geometric considerations go through, including the discussion in [2–5] on volume-preserving frames, torsion, and curvature. In particular, the gauge transformations (12) on covariant quantum spaces include all volume-preserving diffeos generated by divergence-free vector fields $\xi^\mu \in \mathcal{C}^0$, which can be reconstructed from their classical counterparts as discussed above. For shorter wavelengths, these gauge transformations are typically accompanied by $\mathfrak{hs}$-valued components.

Some further remarks are in order. First, it is worth pointing out that the above steps can be easily adapted to obtain similarly

$$\{t^\mu, T^{\dot{\alpha}}\} = \{t^\mu, T^{\dot{\alpha}\sigma}(x) t_\sigma\} \approx T^{\dot{\alpha}\sigma}\{t^\mu, t_\sigma\} \tag{98}$$

in analogy to (88), as the derivatives of $T^{\dot{\alpha}\sigma}(x)$ are again negligible in the weak gravity regime. It should also be noted that the above discussion is based on the rescaled frame $e$ corresponding to the metric $\gamma_{\mu\nu}$ (23) rather than the effective frame for the effective metric $G_{\mu\nu}$, which differ by a factor of the dilaton $\rho$. However since the dilaton $\rho^2$ is determined by the metric $G_{\mu\nu}$ via (27), the weak gravity condition applies equally to both metrics. Finally, the above considerations do not seem to require that the frame $e^{\dot{\alpha}\mu}$ is divergence free. However, this is taken care of by the $\mathcal{C}^0$ component of $\delta E^{\dot{\alpha}\mu}$ (86), which is sub-leading in the weak gravity regime.

## 6. Realization of generic $3+1$-dimensional geometries in matrix models

Finally, we address the question if any given metric $G_{\mu\nu}$ can be realized in terms of a divergence-free frame. The first step is to determine the dilaton, which is obtained from (27) as

$$\rho^2 = \rho_M^{-1}\sqrt{|G|} \,. \tag{99}$$

The next step is to find some classical divergence-free frame $e^{\dot{\alpha}\mu}$ which gives rise to (23)

$$\rho^2 G^{\mu\nu} = \gamma^{\mu\nu} = \eta_{\dot{\alpha}\dot{\beta}} e^{\dot{\alpha}\mu} e^{\dot{\beta}\nu} \,. \tag{100}$$

Without the constraint, there are of course many frames (in fact a 6-dimensional orbit of $SO(3,1)$) which achieve that. The 4 divergence constraints are fairly easy to take into account in Cartesian coordinates $x^\mu$: for any given space-like components $e^{\dot{\alpha}j}$, the time components $e^{\dot{\alpha}0}$ are determined by

$$\partial_0(\rho_M e^{\dot{\alpha}0}) = -\partial_j(\rho_M e^{\dot{\alpha}j}) \,. \tag{101}$$

This can be viewed as an ordinary differential equation in $x^0$, which is solved by

$$e^{\dot{\alpha}0} = -\rho_M^{-1}\int_{\xi_0}^{x^0} d\xi\, \partial_j(\rho_M e^{\dot{\alpha}j}) \;+ e^{\dot{\alpha}0}(\xi_0) \tag{102}$$

where the value $e^{\dot{a}0}(\xi_0)$ at any given time $\xi_0$ can be chosen as desired. This means that we can freely choose the 12 space-like $e^{\dot{a}j}$, which should allow to reproduce the 10 dof in $\gamma^{\mu\nu}$ even if the divergence constraint is imposed.

A more systematic, iterative way to determine the frame is as follows: choose some reference point $\bar{x}$. After a global $SO(3,1)$ transformation on the frame indices, we can assume that $\gamma^{\mu\nu}|_{\bar{x}} = c\eta^{\mu\nu}$, and we assume $c = 1$ for simplicity. Then choose the diagonal elements as $e^{\dot{a}a} = \eta^{\dot{a}a}$, and off-diagonal frame elements which vanish at $\bar{x}$, such that the frame reproduces $\gamma^{\mu\nu}$. To satisfy the divergence constraint, we define a correction of the diagonal frame elements by

$$\delta e^{\dot{a}a} = -\rho_M^{-1} \int_{\bar{x}^a}^{x^a} d\xi^a \sum_{\mu\neq\dot{a}} \partial_\mu(\rho_M e^{\dot{a}\mu}) \,, \tag{103}$$

which vanishes at $\bar{x}$. Then the improved frame $e^{\dot{a}a} \to e^{\dot{a}a} + \delta e^{\dot{a}a}$ satisfies the divergence constraint, and reproduces $\gamma^{\mu\nu}$ to a good approximation near $\bar{x}$. Now we repeat this procedure iteratively by correcting the off-diagonal elements of the frame such that $\gamma^{\mu\nu}$ is reproduced, and correcting the diagonal elements again with (103), and so on. Since the corrections vanish at $\bar{x}$, this procedure will converge to a divergence-free frame which reproduces $\gamma^{\mu\nu}$ exactly at least in some neighborhood of $\bar{x}$. This could presumably be proved e.g. using the Banach fixed point theorem, but we leave it as a plausibility argument here and accept the statement as true.

We conclude that there are always divergence-free frames $e^{\dot{a}\mu}$ which realize (100) for any $\gamma^{\mu\nu}$. As explained in section 4.2, we can then find a corresponding matrix background which implements the frame in the sense of (88), and the higher spin contributions are negligible as long as the geometry is within the weak gravity regime. Therefore generic $3+1$-dimensional space-time geometries can indeed be implemented as backgrounds of the matrix model with an ansatz of the form (66), leading to a covariant quantum space-time.

Moreover, the above analysis shows that 2 of the 12 dof in $e^{\dot{a}\mu}$ remain undetermined even if the divergence constraint is imposed. They can be used to restrict the totally antisymmetric components of the torsion (114), which define a vector field via $\tilde{T}_\kappa \propto T^{(AS)\nu\sigma\mu}\varepsilon_{\nu\sigma\mu\kappa}$. For example, it is plausible that the frame can be chosen such that

$$\tilde{T}_\mu = \psi^{-1}\partial_\mu\tilde{\rho} \tag{104}$$

in terms of an axion $\tilde{\rho}$; this is a consequence of the (semi-classical) matrix model equations of motion [2]. This question and its implications should be addressed elsewhere.

## 7. Quantization, extra dimensions and induced gravity

Even though the semi-classical matrix model action defines a dynamical theory of space-time geometry, it is expected that a (near-) realistic theory of gravity can be obtained only from the Einstein-Hilbert action. Remarkably, this arises indeed in the 1-loop effective action under certain assumptions, in the spirit of induced gravity [22, 23]. The quantization of the matrix model is defined non-perturbatively through a matrix path integral

$$\mathcal{Z} = \int dT d\Psi e^{iS} \,.$$

The oscillatory integral becomes absolutely convergent for finite-dimensional $\mathcal{H}$ upon implementing the regularization

$$S \to S + i\varepsilon \sum_{\dot\alpha} Y_{\dot\alpha} Y_{\dot\alpha} \,, \tag{105}$$

which amounts to a Feynman $i\varepsilon$ term in the noncommutative gauge theory. For recent results of numerical simulations of such models see e.g. [24, 25].

In general, the quantization of matrix models on some noncommutative background leads to highly non-local action due to UV/IR mixing, *except* in the maximally supersymmetric IKKT model. This phenomenon was shown first identified in [26], but it is most transparent in terms of string states $|x\rangle\langle y| \in \mathrm{End}(\mathcal{H})$, which govern the deep quantum (or extreme UV) regime of noncommutative functions [27, 28]. These states are also extremely useful to compute the 1-loop effective action of the IKKT matrix model on generic backgrounds. It was indeed show in [5] that the Einstein-Hilbert action arises at 1 loop, *provided* the transversal 6 matrices $T^{\dot\alpha}$ of the IKKT model assume some non-trivial background given by some compact fuzzy space:

$$T^{\dot k} \sim t^{\dot k}: \quad \mathcal{K} \hookrightarrow \mathbb{R}^6, \qquad \dot k = 4, \ldots, 9 \,. \tag{106}$$

This describes a quantized compact symplectic space $\mathcal{K}$ embedded along the transversal directions, which plays the role of fuzzy extra dimensions. Together with the space-time brane $\mathcal{M}^{3,1}$, the overall background geometry then has a product structure

$$\mathcal{M}^{3,1} \times \mathcal{K} \quad \hookrightarrow \mathbb{R}^{9,1} \,. \tag{107}$$

The detailed structure of $\mathcal{K}$ will be irrelevant[5]; we only require that the internal matrix Laplacian $\Box_{\mathcal{K}} = [T^{\dot k}, [T_{\dot k}, .]]$ has positive spectrum,

$$\Box_{\mathcal{K}} Y_\Lambda = m_\Lambda^2 Y_\Lambda \,, \qquad m_\Lambda^2 = m_{\mathcal{K}}^2 \mu_\Lambda^2 \tag{108}$$

with a finite number of (Kaluza-Klein KK) eigenmodes $Y_\Lambda \in \mathrm{End}(\mathcal{H}_{\mathcal{K}})$ enumerated by some label $\Lambda$. Here $m_{\mathcal{K}}^2$ determines the radius of $\mathcal{K}$ and sets the scale of the KK modes, which will play an important role below.

Computing the 1-loop effective action on such a background then leads in particular to the following term [5]

$$\Gamma_{\text{1loop}}^{\mathcal{K}-\mathcal{M}} = -\frac{c_{\mathcal{K}}^2}{(2\pi)^4} \int\limits_{\mathcal{M}} d^4x \sqrt{G}\, \rho^{-2} m_{\mathcal{K}}^2 T^\rho{}_{\sigma\mu} T_\rho{}^\sigma{}_\nu G^{\mu\nu} \tag{109}$$

which describes the effective interaction between $\mathcal{M}^{3,1}$ and $\mathcal{K}$. Here

$$c_{\mathcal{K}}^2 = \frac{\pi^2}{8} \sum_{\Lambda,s} \frac{(2s+1) C_\Lambda^2}{\mu_\Lambda^2 + \frac{m_s^2}{m_{\mathcal{K}}^2}} > 0 \tag{110}$$

[5] $\mathcal{K}$ could be a fuzzy sphere $S_N^2$, or some richer fuzzy space leading to interesting low-energy gauge theories, cf. [29].

is finite, determined by the dimensionless KK masses $\mu_\Lambda$ on $\mathcal{K}$ (108) and their cousins $C^2_\Lambda$, which also depend on the structure of $\mathcal{K}$. The mass scale of the internal $\mathfrak{hs}$ modes on $S^2_n$ is given by

$$m_s^2 = \frac{s(s-1)}{R^2} \ . \tag{111}$$

Using partial integration, one can rewrite the above effective action in terms of an Einstein-Hilbert term with effective Newton constant

$$\frac{1}{16\pi G_N} = \frac{c_\mathcal{K}^2}{14\pi^4}\rho^{-2} m_\mathcal{K}^2 \, . \tag{112}$$

However, this requires assuming some specific behavior of $m_\mathcal{K}^2$ or $G_N$. If we assume $G_N = const$, we can use the identity [5]

$$\int d^4x \frac{\sqrt{|G|}}{G_N}\mathcal{R} = -\int d^4x \frac{\sqrt{|G|}}{G_N}\Big(\frac{7}{8}T^\mu{}_{\sigma\rho}T_{\mu\sigma'}{}^\rho G^{\sigma\sigma'} + \frac{3}{4}\tilde T_\nu \tilde T_\mu G^{\mu\nu}\Big) \tag{113}$$

where $\mathcal{R}$ is the Ricci scalar of the effective metric $G_{\mu\nu}$, and

$$\tilde T_\mu dx^\mu = - \star (\frac{1}{2} G_{\nu\sigma} T^\sigma{}_{\rho\mu}\, dx^\nu dx^\rho dx^\mu) \tag{114}$$

is the Hodge-dual of the totally antisymmetric torsion. This gives

$$\boxed{\Gamma_{\rm 1loop}^{\mathcal{K}-\mathcal{M}} = \int\limits_\mathcal{M} d^4x\, \frac{\sqrt{G}}{16\pi G_N}\Big(\mathcal{R} + \frac{3}{4}\tilde T_\nu \tilde T_\mu G^{\mu\nu}\Big) \, .} \tag{115}$$

Using the eom of the matrix model, $\tilde T_\nu$ reduces to a gravitational axion $\tilde\rho$ [2]

$$\tilde T_\mu = \rho^{-2}\partial_\mu \tilde\rho \ . \tag{116}$$

Since $\tilde T_\mu$ vanishes exactly on the cosmic background, it is plausible that its effect is small, in which case we recover the Einstein-Hilbert action as desired.

However since $G_N$ depends on $\rho$ and $m_\mathcal{K}$, it is not evident that $G_N = const$. If we assume instead that $m_\mathcal{K} = const$ (which is reasonable as discussed below), then one can derive an analogous identity

$$\int d^4x \frac{\sqrt{|G|}}{G_N}\mathcal{R} = -\int d^4x \frac{\sqrt{|G|}}{G_N}\Big(\frac{1}{8}T^\mu{}_{\sigma\rho}T_{\mu\sigma'}{}^\rho G^{\sigma\sigma'} + \frac{1}{4}\tilde T_\nu \tilde T_\mu G^{\mu\nu}\Big) \tag{117}$$

based on results in [2]. This leads to a slightly modified gravitational action

$$\boxed{\Gamma_{\rm 1loop}^{\mathcal{K}-\mathcal{M}} = 7\int\limits_\mathcal{M} d^4x \frac{\sqrt{G}}{16\pi G_N}\Big(\mathcal{R} + \frac{1}{4}\tilde T_\nu \tilde T_\mu G^{\mu\nu}\Big)} \tag{118}$$

where the Newton constant is modified by a factor 7. The precise form of the gravitational action thus depends on the behavior of the compactification scale $m_\mathcal{K}^2$, which needs to be clarified in future work.

These results are remarkable in many ways. The first observation is that the Newton constant $G_N$ (112) is set by the compactification scale $m_\mathcal{K}$. This means that the Planck scale is related to the Kaluza-Klein scale for the fuzzy extra dimensions $\mathcal{K}$. Without the fuzzy extra-dimensional $\mathcal{K}$, no Einstein-Hilbert action is induced, and only some (rather obscure) higher-derivative action is obtained. It should be noted that no UV divergence arises in the loop computation, due to maximal supersymmetry of the matrix model and the fact that $\mathcal{K}$ supports only a finite number of modes.

We can justify the presence of $\mathcal{K}$ to some extent by studying how the 1-loop effective action depends on its radius, or equivalently on $m_\mathcal{K}$. This is obtained from the same computation as above: It turns out that (109)

$$\Gamma_{1\text{loop}}^{\mathcal{K}-\mathcal{M}} = c^2 m_\mathcal{K}^2 = -V_{1\text{loop}}(m_\mathcal{K}^2) > 0 \tag{119}$$

is positive for the covariant FLRW space-time in [1]. Combined with the bare matrix model action, the effective potential has the structure

$$V(m_\mathcal{K}^2) = -c^2 m_\mathcal{K}^2 + \frac{d^2}{g^2} m_\mathcal{K}^4 \tag{120}$$

at weak coupling. This clearly has a minimum for $m_\mathcal{K}^2 > 0$ with $V < 0$. Since $m_\mathcal{K}$ is essentially the radius of $\mathcal{K}$, this strongly suggests that $\mathcal{K}$ is stabilized by quantum effects, thus providing some justification for (107).

One may worry that the effective potential for $m_\mathcal{K}$ depends on the geometry of $\mathcal{M}^{3,1}$, which we have assumed to be the cosmic background brane. Thus gravitational deformations of the geometry should have some influence on the Newton constant. Nevertheless, $m_\mathcal{K}$ is expected to be constant to a very good approximation. Since $m_\mathcal{K}$ is essentially the radius of $\mathcal{K}$, its kinetic term $\int \partial^\mu m_\mathcal{K} \partial_\mu m_\mathcal{K}$ in the matrix model is huge, which would strongly suppress any local variations; note that $m_\mathcal{K}^2 \sim \rho^2 G_N^{-1}$ is a huge energy beyond the Planck scale. Therefore $m_\mathcal{K}$ should be almost constant, and hence governed by the large-scale cosmic background as assumed above.

On the other hand, this suggests that the Newton constant may change during the cosmic expansion. This may be a significant concern, since there are rather strong observational bounds on such a variation. Nevertheless, at this early stage such worries are presumably sub-leading, and the prime focus should be to gain a more detailed understanding of this new mechanism for gravity.

Furthermore, the above induced gravity *action* in 3+1 dimensions can be interpreted as a quasi-local *interaction* of $\mathcal{K}$ and $\mathcal{M}$ via 9+1-dimensional IIB supergravity, recalling that the 1-loop effective action is related to IIB supergravity [6, 28, 30, 31]. This provides additional confidence into the above rather formal computations, since $9+1$-dimensional supergravity is well established in string theory and expected to be recovered in the matrix model. A more detailed understanding of the relation with supergravity for backgrounds of the structure $\mathcal{M}^{3,1} \times \mathcal{K} \subset \mathbb{R}^{9,1}$ would be desirable.

Note that in contrast to orthodox string theory, target space $\mathbb{R}^{9,1}$ is not compactified here. This makes sense, since the perturbative physics on such backgrounds is restricted to the brane, and there are no bulk modes radiating off the brane at weak coupling. Hence the main problem of string theory - i.e. the need for compactification and the lack of preferred choices thereof - turns into a blessing, as there would be no induced gravity on space-time without the extra dimensions of target space.

**Vacuum energy due to $\mathcal{K}$.** The 1-loop contribution to the vacuum energy due to $\mathcal{K}$ is obtained using an analogous trace computation, leading to a result of the structure

$$\Gamma^{\mathcal{K}}_{1\mathrm{loop}} = \frac{3i}{4}\mathrm{Tr}\Big(\frac{V^{\mathcal{K}}_4}{\Box^4}\Big) \sim -\frac{\pi^2}{8(2\pi)^4}\int\limits_{\mathcal{M}}\Omega\,\rho^{-2}m^4_{\mathcal{K}}\sum_{\Lambda s}\frac{V_{4,\Lambda}}{\mu^4_\Lambda} \tag{121}$$

assuming $\frac{1}{R^2} \ll m^2_\Lambda \sim m^2_{\mathcal{K}}$. Here $V_{4,\Lambda}$ depends on the structure of $\mathcal{K}$. This is typically a large vacuum energy with scale set by $m_{\mathcal{K}}$ which was related to the Planck scale above, which could have either sign. However as the symplectic volume form $\Omega$ is independent of the metric, this 1-loop vacuum energy is *not* equivalent to a cosmological constant; its effect on the dilaton $\rho$ remains to be understood. The present framework can therefore be viewed as a realization of induced gravity in the spirit of Sakharov [22, 23], which is free of UV divergences, and appears to avoid the associated cosmological constant problem.

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