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Quantum matrix geometry in the lowest Landau level and higher Landau levels

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One of the most celebrated works of Professor Madore is the introduction of fuzzy sphere. I briefly review how the fuzzy two-sphere and its higher dimensional cousins are realized in the (spherical) Landau models in non-Abelian monopole backgrounds. For extracting quantum geometry from the Landau models, we evaluate the matrix elements of the coordinates of spheres in the lowest and higher Landau levels. For the lowest Landau level, the matrix geometry is identified as the geometry of fuzzy sphere. Meanwhile for the higher Landau levels, the obtained quantum geometry turns out to be a nested matrix geometry with no classical counterpart. There exists a hierarchical structure between the fuzzy geometries and the monopoles in different dimensions. That dimensional hierarchy signifies a Landau model counterpart of the dimensional ladder of quantum anomaly.

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1. Introduction

The non-commutative geometry is a promising mathematical framework for describing a quantized space-time. Typical and well-studied examples of the non-commutative spaces are non-commutative plane, fuzzy sphere and fuzzy hyperboloid. They exhibit mathematically unique structures and also exemplify solutions of the Matrix models of string theory.

Usually, a non-commutative structure is postulated when defining models such as in the non-commutative field theory, and their physical properties are discussed withing the given framework. Here, we will take an almost inverse approach: We will introduce Landau models at first not assuming any non-commutative structure and subsequently exploit the non-commutative structure from the models. The Landau models are simple models that describe quantum mechanics of electrons in magnetic fields. In the planar Landau model, the center-of-mass coordinates of electron are given by $X = x + i\frac{1}{B}(\partial_y + iA_y)$ and $Y = y - i\frac{1}{B}(\partial_x + iA_x)$ that satisfy the Heisenberg-Weyl algebra:

$$[X, Y] = i\frac{1}{B}\mathbf{1}, \qquad [X, \mathbf{1}] = [Y, \mathbf{1}] = 0. \tag{1}$$

The center-of-mass coordinates obey the non-commutative algebra due to the existence of the magnetic field and the electrons behave as if they live on a non-commutative plane. As for curved non-commutative spaces, the fuzzy sphere introduced by Madore [1] (see Refs.[2, 3] also) provides a typical example. The mathematics of fuzzy two-sphere is very simple: The matrix coordinates satisfy the $SU(2)$ algebra and the sum of their squares should be a constant (times an identity matrix):

$$[X_i, X_j] = i\epsilon_{ijk}X_k, \qquad \sum_{i=1}^{3} X_iX_i = \text{const}\cdot\mathbf{1}. \tag{2}$$

After a few years of the work of Madore, Grosse and his collaborators succeeded to generalize the concept of the fuzzy-sphere in the four dimension [6], which is now known as the fuzzy four-sphere. The fuzzy four-sphere was rediscovered in the context of string theory and studied in detail [7–10]. The fuzzy four-sphere is defined as

$$[X_a, X_b, X_c, X_d] = \epsilon_{abcde}X_e, \qquad \sum_{a=1}^{5} X_aX_a = \text{const}\cdot\mathbf{1}. \tag{3}$$

Obviously, (3) is a straightforward generalization of (2) replacing the usual commutator with the Nambu four-bracket $[X_{\sigma_1}, X_{\sigma_2}, X_{\sigma_3}, X_{\sigma_4}] \equiv \text{sgn}(\sigma)\ X_{\sigma_1}X_{\sigma_2}X_{\sigma_3}X_{\sigma_4}$ [11–13]. Meanwhile, the fuzzy three-sphere [14–19] is introduced as

$$[[X_\mu X_\nu, X_\rho]] = i\epsilon_{\mu\nu\rho\sigma}X_\sigma, \qquad \sum_{\mu=1}^{4} X_\mu X_\mu = \text{const}\cdot\mathbf{1}. \tag{4}$$

Note that, in (4), the "three bracket" $[[X_\mu X_\nu, X_\rho]] \equiv [X_\mu, X_\nu, X_\rho, G_5]$ with $G_5 \equiv P_{R_+} - P_{R_-} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is used.

In the following, I will discuss the Landau model realization of the fuzzy pheres.[1] Before going to details, I briefly mention the underlying idea. Suppose that we would like to construct a

[1]In the same spirit, the fuzzy $\mathbb{C}P^n$ is analyzed in [4]. See [5] as a review.

fuzzy manifold whose classical geometry is a coset, $M \simeq G/H$. The corresponding fuzzy manifold will be made of "quantum elements" each of which occupies some quantum finite area on M. As M returns to itself under the transformation G, the set of those quantum elements should return to itself under the transformation. We may adopt the states of an irreducible representation of G as such quantum elements, since irreducible representation denotes a closed set under the group transformations. Thus, the fuzzy manifold made of the states of the irreducible representation is necessarily symmetric under the transformation G. The stabilizer group H of M will be translated as the gauge group in the quantum mechanical side. In short, in passing from the classical geometry $M \simeq G/H$ to its fuzzy version M_F, we need to utilize the irreducible representations of the global symmetry group G with the gauge symmetry H. Such a quantum mechanical model is nothing but the Landau model constructed on M with the gauge symmetry H.

2. 2D Landau model and fuzzy two-sphere

To fuzzify a two-sphere $S^2 \simeq SO(3)/SO(2)$, we consider the Landau model with the $SO(2) \simeq U(1)$ gauge symmetry on S^2. The Landau model Hamiltonian is given by [20, 21]

$$H = -\frac{1}{2M}\sum_{i=1}^{3}(\partial_i + iA_i)^2|_{r=\text{const}} \tag{5}$$

where A_i denote the Dirac's monopole gauge field [22]

$$A_{\mu=1,2} = -\frac{I}{2r(r+x_3)}\epsilon_{\mu\nu}x_\nu, \quad A_3 = 0. \tag{6}$$

The corresponding Landau levels are $E_n = \frac{1}{2M}(I(n+\frac{1}{2}) + n(n+1))$ with $n = 0, 1, 2, \cdots$ being the Landau level index, and the Landau level degeneracy is counted as $d_n = 2n + I + 1$. The nth Landau level eigenstates are known as the monopole harmonics $Y^{(I/2)}_{l=n+\frac{I}{2},m}(\theta,\phi)$ [20] which constitute the $SU(2)$ irreducible representation of the spin index $l = n + \frac{I}{2}$.

While there are several methods for deriving the non-commutative geometry from the Landau model, the most straightforward way is to evaluate the matrix elements of the coordinates in each Landau level:

$$(X_i^{(n)})_{mm'} = \langle Y_{l,m}|x_i|Y_{l,m'}\rangle|_{l=n+\frac{I}{2}}. \tag{7}$$

Using the formulas of the monopole harmonics, we can explicitly evaluate (7) as [23]

$$X_i^{(n)} = \frac{2I}{(I+2n)(I+2n+2)}S_i^{(l=n+\frac{I}{2})}, \tag{8}$$

where $S^{(l)}$ denote the $SU(2)$ spin matrices with spin magnitude l. The $X_i^{(n)}$ (8) obviously satisfy the algebra of the fuzzy two-sphere (2):

$$[X_i^{(n)}, X_j^{(n)}] = i\epsilon_{ijk}\frac{(I+2n)(I+2n+2)}{2I}X_k^{(n)}, \qquad \sum_{i=1}^{3}X_i^{(n)}X_i^{(n)} = \frac{I^2}{(I+2n)(I+2n+2)}\mathbf{1}_{2n+I+1}. \tag{9}$$

The $U(1)$ quantum number m appears as the $2l+1$ diagonal components of $X_z^{(n)} \propto S_z^{(l=n+\frac{I}{2})}$ and specifies the position of the latitudes (Fig.1). While the non-commutative structure of the lowest Landau level ($n = 0$) is usually focused in literature, the non-commutative geometry also appears in the higher Landau levels ($n \neq 0$) as shown by (8).

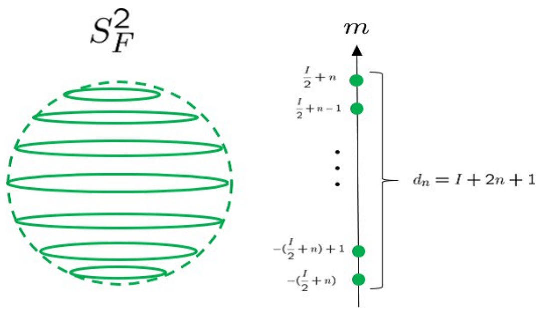

Figure 1: The nth Landau level eigenstates (monopole harmonics) with the $SU(2)$ Casimir index $l = n + \frac{I}{2}$ constitute the fuzzy two-sphere $X_i^{(n)}$.

3. 4D Landau model and fuzzy four-sphere

Next I discuss the fuzzy four-sphere geometry. For $S^4 \simeq SO(5)/SO(4)$, we need to consider the Landau model on S^4 in the $SO(4)$ monopole background. The $SO(4)$ group is a direct product of two $SU(2)$ groups, and then we will adopt one of them.[2] With the $SU(2)$ monopole gauge field [25]

$$A_{\mu=1,2,3,4} = -\frac{1}{2r(r+x_5)}\eta^i_{\mu\nu}x_\nu S_i^{(I/2)}, \quad A_5 = 0, \qquad (\eta^i_{\mu\nu} = \epsilon_{\mu\nu i4} + \delta_{\mu i}\delta_{\nu 4} - \delta_{\mu 4}\delta_{\nu i}), \tag{10}$$

the 4D Landau Hamiltonian is constructed as [26].

$$H = -\frac{1}{2M}\sum_{a=1}^{5}(\partial_a + iA_a)^2|_{r=\text{const.}}. \tag{11}$$

The Landau levels are given by $E_N = \frac{1}{2M}(I(N+1)+N(N+3))$ $(N = 0, 1, 2, \cdots)$. The corresponding eigenstates are known as the $SU(2)$ monopole harmonics $Y_{p,q;\ j,m_j;k,m_k}$ [27] with the $SO(5)$ Casimir indices

$$(p, q) = (N + I, N) \quad (N = 0, 1, 2, \cdots). \tag{12}$$

Note that the $SU(2)$ monopole harmonics carry the $SO(4) \simeq SU(2) \otimes SU(2)$ quantum numbers $(j, m_j;\ k, m_k)$, which brings $D(N, I) \equiv \frac{1}{6}(N + I + 2)(N + 1)(2N + I + 3)(I + 1)$ degeneracy to the Nth Landau level, and the Nth Landau level eigenstates consist of $N + 1$ sets of the $I + 1$ $SO(4)$ irreducible representations (see Fig.2). The $SO(4) \simeq SU(2) \times SU(2)$ decomposition of the $SO(5)$ irreducible representation is given by

$$D(N, I) = \sum_{n=0}^{N}\sum_{j+k=n+\frac{I}{2}}(2j + 1)(2k + 1). \tag{13}$$

With those Landau level eigenstates, we derive the matrix coordinates

$$(X_a^{(N)})_{j,m_j;k,m_k} = \langle Y_{N+I,N;j,m_j;k,m_k}|x_a|Y_{N+I,N;j,m_j;k,m_k}\rangle, \tag{14}$$

which will become $D(N, I) \times D(N, I)$ matrices.

[2]Recently, the author performed a full analysis for the $SO(4)$ case [24].

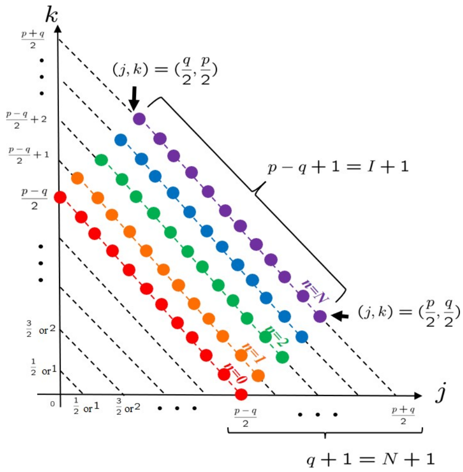

Figure 2: The $SO(5)$ irreducible representation for $(p, q) = (I+N, N)$. The $SO(5)$ irreducible representation consists of the $N + 1$ oblique lines on which $SO(4)$ irreducible representations are aligned. (Taken from [28].)

3.1 Matrix geometry from the lowest Landau level

In the lowest Landau level ($N = 0$), the matrix coordinates (14) are obtained as [28]

$$X_a^{(0)} = \frac{1}{I+4}\Gamma_a, \tag{15}$$

where Γ_a denote I tensor product of the $SO(5)$ gamma matrices.[3] The $X_a^{(0)}$ (15) actually satisfy the relations of the fuzzy four-sphere (3):

$$[X_a^{(0)}, X_b^{(0)}, X_c^{(0)}, X_d^{(0)}] = 8\frac{(I+2)}{(I+4)^3}\epsilon_{abcde}X_e^{(0)}, \qquad \sum_{a=1}^{5} X_a^{(0)}X_a^{(0)} = \frac{I}{I+4}\mathbf{1}. \tag{16}$$

X_5 is a diagonal matrix whose diagonal elements represent the difference between the two $SU(2)$ Casimir indices, $j - k$, and specify the position of S^3 latitudes [Fig.3]. Each of the S^3-latitudes accommodates the degeneracy $(2j + 1)(2k + 1)$ due to the $SO(4) \simeq SU(2) \otimes SU(2)$ internal structure.

3.2 Nested matrix geometry from the higher Landau levels

The decomposition rule (13) from $SO(5)$ to $SO(4)$ implies a nested structure in the corresponding matrix geometry [28]: Each oblique line of the $SO(4)$ irreducible representations corresponds to a "fuzzy shell" in the matrix geometry side, and the $(N + 1)$ sets of the $SO(4)$ irreducible representations constitute the nested structure of the $(N + 1)$ shells [Fig.4].

Each of the fuzzy shells does not respect the $SO(5)$ symmetry, but the set of the fuzzy shells gives rise to the $SO(5)$ symmetric fuzzy manifold composed of the states of the $SO(5)$ irreducible representation. In Fig.4, the nested fuzzy manifold does not seem to have the $SO(5)$ rotational symmetry, but this is not the case. We have chosen the 5th axis as the quantization axis, but we can

[3] The authors [29] derived the result (15) in the context of the Berezin-Toeplitz quantization.

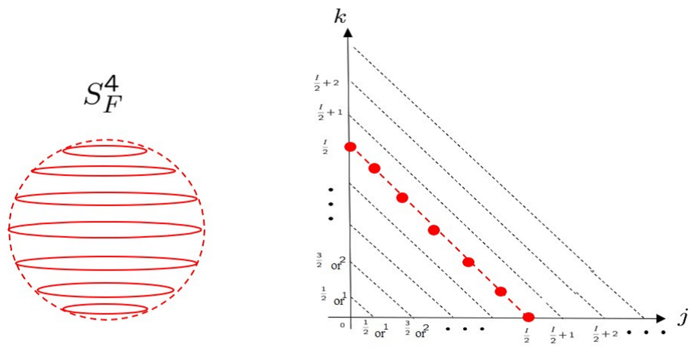

Figure 3: The S^3-latitudes on the fuzzy four-sphere (left) correspond to the $SO(4)$ irreducible representations (right).

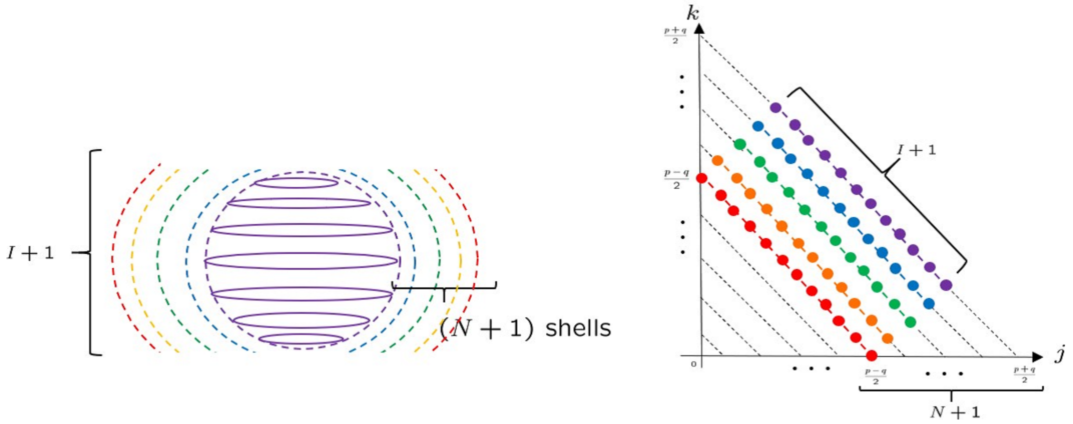

Figure 4: The $(N + 1)$ fuzzy shells realize the nested geometry of the Nth Landau level.

choose any axis in arbitrary direction [Fig.5]. In other words, that $SO(5)$ non-symmetric picture of the nested fuzzy manifold in Fig.4 is due to a "gauge artifact". This situation is somewhat similar to the covalent bond of benzene [Fig.5]. The covalent band of benzene respects the C6 rotational symmetry, but either of the two Kekulé structures does not have the C6 rotational symmetry. Only the quantum composite of the two Kekulé structures respects the C6 rotational symmetry as a whole. The covalent bond of benzene is a purely quantum mechanical structure with no classical counterpart. Back to the present fuzzy geometry, each of the fuzzy shells does not respect the $SO(5)$ rotational symmetry, but their composite nested structure does the $SO(5)$ symmetry. In a similar sense of benzene, it is fair to say that the higher Landau level geometry realizes a pure quantum geometry. See [30] for details.

4. 3D Landau model and fuzzy three- and four-spheres

As the last concrete example, we will discuss the fuzzy three-sphere. For $S^3 \simeq SO(4)/SO(3)$, the corresponding Landau model is constructed on S^3 in the $SO(3) \simeq SU(2)$ monopole background. That Landau model was first analyzed in [31] and subsequently in [32, 33]. The $SU(2)$ monopole

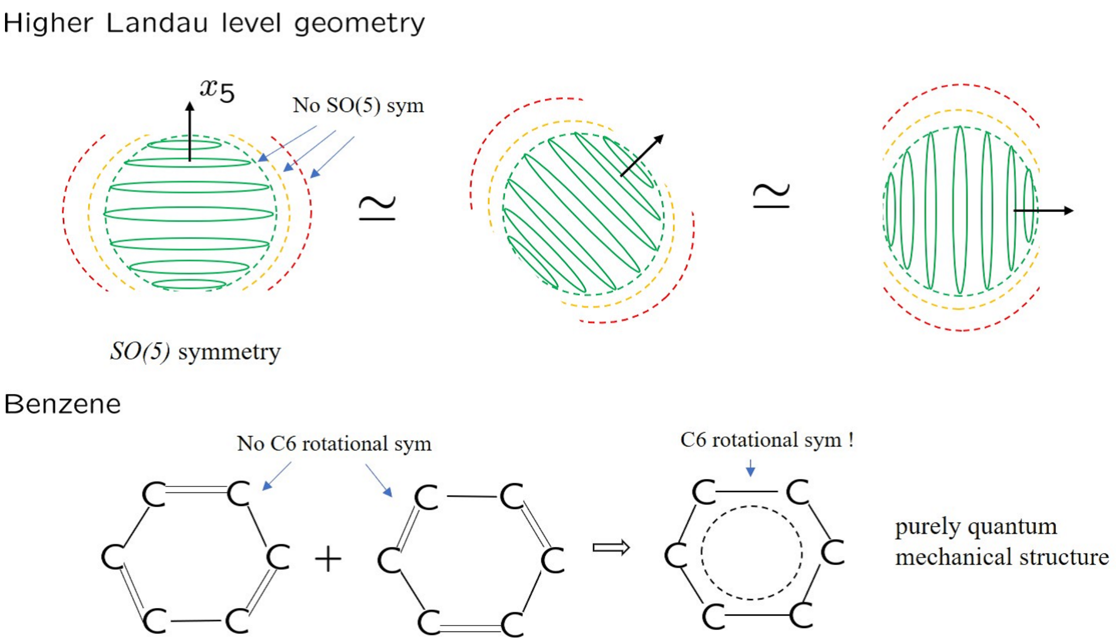

Figure 5: Analogies between the nested geometry of higher Landau level and the covalent bond of benzene.

gauge field is given by

$$A_i = -\frac{I}{2r(r+x_4)}\epsilon_{ijk}x_j S_k^{(I/2)}, \quad A_4 = 0, \tag{17}$$

and the Landau Hamiltonian is

$$H = -\frac{1}{2M}\sum_{\mu=1}^{4}(\partial_\mu{}^2 + iA_\mu)|_{r=\text{const.}}. \tag{18}$$

The Landau levels are derived as $E_{n,s} = \frac{1}{2M}(I(n+\frac{1}{2}) + n(n+2) + s^2)$, which depend both on the Landau level index $n = 0, 1, 2, \cdots$ and the sub-band index

$$s = \frac{I}{2}, \; \frac{I}{2} - 1, \; \frac{I}{2} - 2, \; \cdots, \; -\frac{I}{2}. \tag{19}$$

The fuzzy three-sphere geometry is realized at the minimum energy level of $(n, s) = (0, 1/2)$ and $(0, -1/2)$. In the lowest Landau level ($n = 0$), there are $I + 1$ sub-bands [Fig.6] labeled by s, and the matrix coordinates of S^3 are obtained as

$$X_{\mu=1,2,3,4} = \frac{1}{I+3}\Gamma_\mu, \tag{20}$$

where $\Gamma_{\mu=1,2,3,4}$ are the four matrices of (15). Aligning the fuzzy geometries in the sub-bands along the virtual 5th direction of s, we can reproduce the fuzzy four-sphere geometry [Fig.6] [32, 33]. Thus interestingly, the 3D Landau model "knows" the geometry of the one dimension higher Landau model. Such a hierarchical structure has been observed in the context of the fuzzy geometry [14–16, 18, 19], and the present Landau models are the physical models that nicely realize the structure.

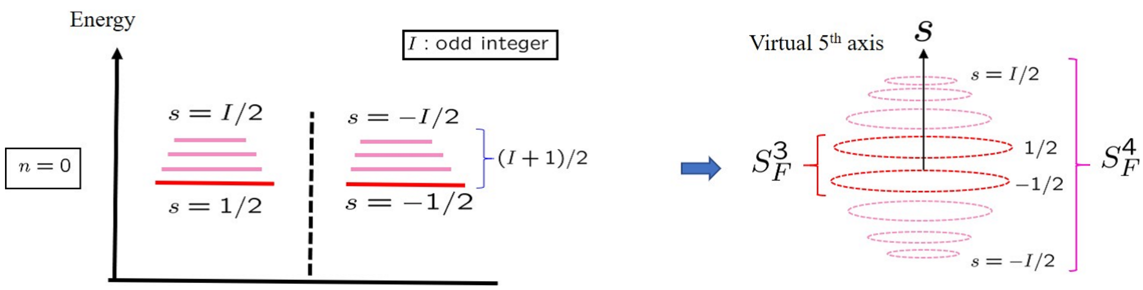

Figure 6: The lowest Landau level ($n = 0$) consists of $(I + 1)$ sub-bands labeled by s (left). We align the fuzzy 3D manifolds along the virtual 5th direction to reproduce the fuzzy four-sphere geometry (right).

5. Dimensional hierarchy

We can find a similar hierarchical structure with respect to the monopole gauge fields [28] [Fig.7]. One may wonder whether such a hierarchical relation is extended in even higher dimensions. Actually, it is straightforward to generalize the set-up of the Landau model in any dimensions, and we can show that the dimensional hierarchy ranges in all dimensions [34–36].

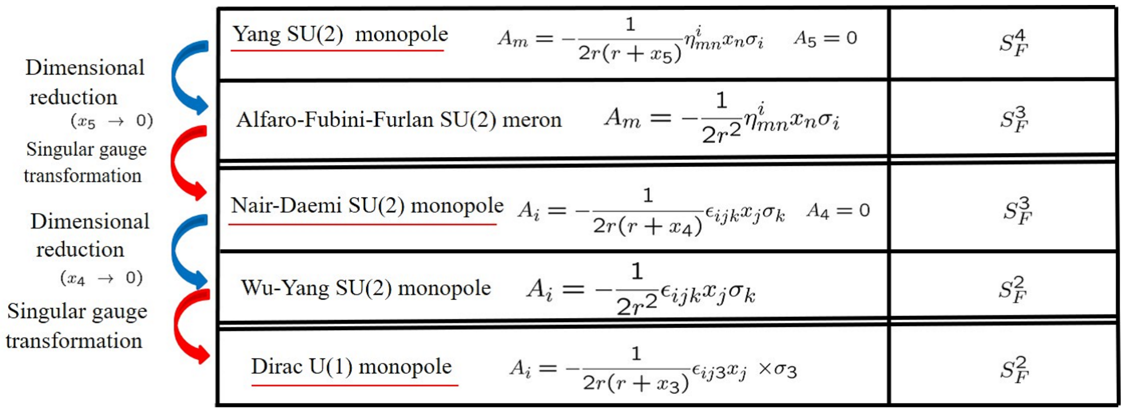

Figure 7: The dimensional hierarchy of the monopole gauge fields. Starting from Yang's $SU(2)$ monopole in 5D, we reach Dirac's $U(1)$ monopole in 3D by applying the dimensional reductions and the singular gauge transformations. (Taken from [28].)

The dimensional hierarchy finds its origin in the differential topology. In the lowest Landau level, the degenerate eigenstates constituting the fuzzy spheres are equal to the zero-modes of the Dirac-Landau operator of the relativistic Landau models. Meanwhile, the Atiyah-Singer index theorem signifies the equality between the number of the zero-modes and the Chern number which is the monopole charge in the present Landau models [35, 37]. This implies a close relationship between the non-commutative geometry and the differential topology, since the quantum space of the fuzzy space are spanned by the zero-modes while the mathematics of the monopole gauge fields is accounted for by the fibre-bundle theory of the differential topology [35, 36]. The similar hierarchical structure in the fuzzy geometry and the monopole gauge field is a consequence of this observation. In quantum field theory, the chiral anomaly is a manifestation of the Atiyah-Singer index theorem and exhibits a hierarchical structure referred to as the dimensional ladder. The dimensional hierarchy is said to be the Landau model counterpart of the dimensional ladder of quantum anomaly [Fig.8].

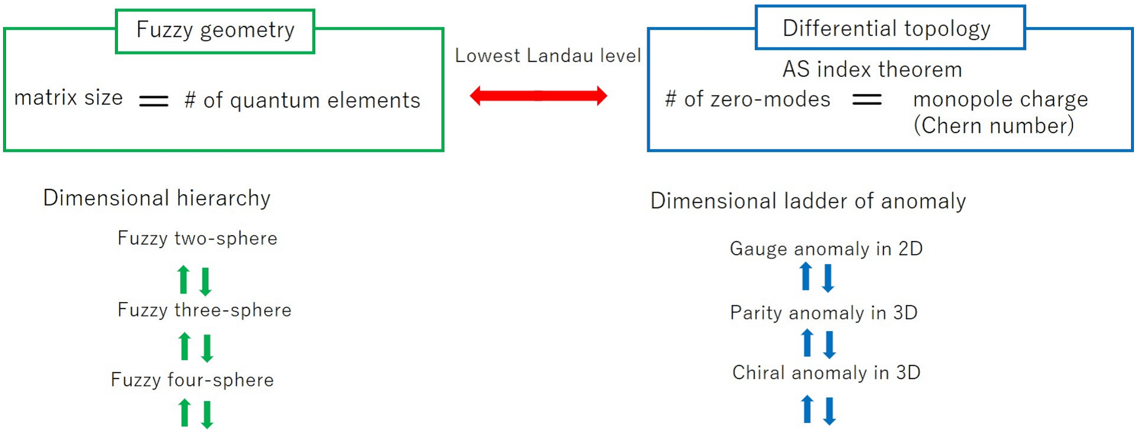

Figure 8: The correspondence between the fuzzy geometry and the differential topology.

Incidentally, the even D Landau models correspond to the A-class topological insulators and the odd D Landau models the AIII-class topological insulators [Fig.9]. In the topological table, it is known that there exists a dimensional relation [38, 39], which also supports the idea of the dimensional hierarchy of the Landau models.

AZ	T	C	TC	1	2	3	4	5	6	7	8
A	0	0	0	0	Z	0	Z	0	Z	0	Z
AIII	0	0	1	Z	0	Z	0	Z	0	Z	0

Figure 9: The topological table for the A class and the AIII class topological insulators [38, 39].

6. Summary

We discussed the Landau model realization of the fuzzy spheres through the concrete evaluation of the matrix coordinates. For the 2D Landau model, the matrix geometries were identified as the fuzzy spheres in any Landau levels. Similarly for the 4D Landau model, the lowest Landau level geometry is shown to be the fuzzy four-sphere geometry. Meanwhile, the higher Landau level geometry turned out to be the nested fuzzy structure with no classical counterpart. In the 3D Landau model, the fuzzy three-sphere geometry was realized at the lowest energy level, and the hidden one dimension higher fuzzy geometry was unveiled in the lowest Landau level. The dimensional hierarchy among the Landau models is the Landau model counterpart of the dimensional ladder of quantum anomaly and implies an intimate relationship between the non-commutative geometry and the differential topology.

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