Recent Progress On Membrane Theory

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Various lines of progress concerning relativistic extended objects are presented
(including some insight concerning quite general supersymmetrizable theories)

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1. Quantization of some non–compact polynomial minimal surfaces

Non–commutative analogues of a class of infinitely extended 2 dimensional time–dependent surfaces that sweep out in space time 3–manifolds of vanishing mean curvature described by polynomial equations are constructed.

As found in [1]

$$\begin{aligned}
x &= \frac{\sqrt{2}}{\tau}\sqrt{\mu^2+\varepsilon}\cos\varphi = \frac{\sqrt{2}}{\tau}\bar{x}(\mu,\varphi) = R(\tau,\mu)\cos\varphi \\
y &= \frac{\sqrt{2}}{\tau}\sqrt{\mu^2+\varepsilon}\sin\varphi = \frac{\sqrt{2}}{\tau}\bar{y}(\mu,\varphi) = R(\tau,\mu)\sin\varphi \\
\zeta &:= t-z = \frac{-\mu^2-\frac{\varepsilon}{3}}{\tau^3} = \frac{-1}{\tau^3}\bar{\zeta},\ \ \tau = \frac{t+z}{2},
\end{aligned} \tag{1}$$

satisfying

$$\begin{aligned}
&\ddot{x} = \{\{x,y\},y\},\ \ \ddot{y} = \{\{y,x\},x\} \\
&\ddot{\zeta} = \{\{\zeta,x\},x\} + \{\{\zeta,y\},y\}\ (=:\Delta\zeta) \\
&\ddot{\tau} = \Delta\tau (= 0)
\end{aligned} \tag{2}$$

(where $\{f,g\} := \frac{\partial f}{\partial\mu}\frac{\partial g}{\partial\varphi} - \frac{\partial g}{\partial\mu}\frac{\partial f}{\partial\varphi}$ and $\cdot = \frac{\partial}{\partial\tau}$), and resulting from a separation Ansatz for

$$\ddot{R} = R(RR')', \tag{3}$$

and solving

$$\{\{\bar{x},\bar{y}\},\bar{y}\} = \bar{x},\ \ \{\{\bar{y},\bar{x}\},\bar{x}\} = \bar{y} \tag{4}$$

and

$$\zeta' = \dot{R}R',\quad 2\dot{\zeta} = \dot{R}^2 + R^2R'^2, \tag{5}$$

as well as parametrizing

$$(t^2+x^2+y^2-z^2)(t+z)^2 = \frac{16}{3}\varepsilon \in \mathbb{R}, \tag{6}$$

describe 3 manifolds Σ_3 of vanishing mean curvature in $\mathbb{R}^{1,3}$ (see [5], [6] for other polynomial ones). As (4) may be written as

$$\{\bar{x},\bar{y}\} = \bar{\mu},\ \ \{\bar{y},\bar{\mu}\} = -\bar{x},\ \ \{\bar{\mu},\bar{x}\} = -\bar{y} \tag{7}$$

it is easy to see that hermitean operators X, Y, H satisfying

$$[X,Y] = iH,\ \ [Y,H] = -iX,\ \ [H,X] = -iY \tag{8}$$

(i.e. representations of $so(1,2)$; note that these do not necessarily have to give rise to group–representations, in contrast to (5.29) of [4]; so, e.g., k in (**) on p.27 of [3] need not be restricted to half–integers; any $k > 0$ would do) via $X_1 := \frac{\sqrt{2}}{\tau}X$, $X_2 := \frac{\sqrt{2}}{\tau}Y$ will then solve the 'membrane–matrix–model' [2] equations

$$\ddot{X}_i = -\big[[X_i,X_j],X_j\big],\ \ \sum_{i=1}^{2}[X_i,\dot{X}_i] = 0. \tag{9}$$

Just as

$$\bar{\mu}^2 - \bar{x}^2 - \bar{y}^2 = -\varepsilon, \tag{10}$$

the left hand side being a Casimir function of (7), $H^2 - X^2 - Y^2 = -Q = -C_2$ will be the standard Casimir operator, i.e. for irreducible representations of (8) (cp. [4], [3]) be proportional to the identity. Note that $\varepsilon < 0$, $\mu \geqslant \sqrt{-\varepsilon}$ will correspond to Σ_3 being time–like. Interestingly ζ, which in the classical theory is needed to (re)construct Σ_3 (once x and y are known) and usually difficult to 'quantize' (leading to the non–commutative 'membrane–matrix–model' often believed to *not* be Lorentz–invariant), in the above example does satisfy

$$\ddot{\hat{\zeta}} = -\big[[\hat{\zeta}, X], X\big] - \big[[\hat{\zeta}, Y], Y\big] =: \hat{\Delta}\zeta \tag{11}$$

for the obvious choice

$$\hat{\zeta} := \frac{-1}{\tau^3}(H^2 + \frac{\varepsilon}{3}) = \frac{-1}{\tau^3}(X^2 + Y^2 - \frac{2}{3}\varepsilon), \tag{12}$$

just as X and Y (and τ) do, so that one may think of

$$\begin{aligned} X^0 &= T = \tau + \frac{\hat{\zeta}}{2} \\ \text{and} \quad X^3 &= Z = \tau - \frac{\hat{\zeta}}{2} \end{aligned} \tag{13}$$

as the quantizations of t and z in this model (and could try to let Lorentz–transformations act on $X^\mu = (X^0, X^1, X^2, X^3)$).

2. Composite dynamical symmetry of M–branes

It is shown that the previously noticed internal dynamical $SO(D-1)$ symmetry [7] for relativistic M–branes moving in D–dimensional space–time is naturally realized in the (extended by powers of $\frac{1}{p_+}$) enveloping algebra of the Poincaré algebra.

In the common light–cone derivation of the critical dimension for bosonic strings it is hidden in the calculation that the identification of terms in M_{i-} *not* involving zero–modes does not only require subtracting $X_i P_- - P_i X_-$ but also terms that are linear (!) in the transverse total momenta, implicit in the longitudinal oscillators (see e.g. [8][9]). The purely internal parts of (P_+times) the longitudinal Lorentz–generators M_{i-}, and the M_{ij} (generators of $SO(D-2)$), satisfy

$$\begin{aligned} \{\mathbb{M}_{jk}, \mathbb{M}_{i-}\} &= -\delta_{ik}\mathbb{M}_{j-} + \delta_{ij}\mathbb{M}_{k-} \\ \{\mathbb{M}_{i-}, \mathbb{M}_{j-}\} &= \mathbb{M}^2 \cdot \mathbb{M}_{ij} \\ \{\mathbb{M}_{ij}, \mathbb{M}_{kl}\} &= -\delta_{jk}\mathbb{M}_{il} \pm 3\,\text{more} \end{aligned} \tag{14}$$

with $\mathbb{M}^2 = 2P_+P_- - \vec{P}^2$ the internal (Mass)^2, very similar to the dynamical symmetry of the hydrogen atom – which gives hope [7] that it may be possible to obtain purely algebraically the spectrum

of $\mathbb{M}^2$ (when quantized), with the dimension and the topology of the extended object being encoded in the dimensions and multiplicities of the occurring irreducible finite dimensional representations of $SO(D-1)$ given by (14) via $\mathbb{L}_{i-} := \frac{\mathbb{M}_{i-}}{\sqrt{\mathbb{M}^2}}$ and $\mathbb{L}_{ij} := \mathbb{M}_{ij}$.
Attempts to quantize (14), using the constrained phase–space of transverse internal degrees of freedom are hindered by the constraints (that are reflecting residual invariance of the theory under volume–preserving diffeomorphismus, resp. solvability for the longitudinal degrees of freedom in terms of the transverse ones) – making even classical calculations, like the proof [10] of Poisson commutativity of the M_{i-} formidable. In [11], on the other hand, it was noticed that in the codimension one case (to which we intermediately restrict) relativistic M(em)–branes can be described as an isentropic inviscid irrotational gas. Taking proper care of (cp.[13])

$$P_+ := \int \sqrt{\frac{g}{2\dot{\zeta} - \dot{\vec{x}}^2}}\, d^M\varphi = \eta \int \rho\, d^M\varphi = \eta = \int q\, d^M x \tag{15}$$

when performing the hodograph–transformation

$$\begin{aligned}
\varphi^\alpha = (\tau, \varphi^1, \ldots, \varphi^M) &\to x^\alpha = (\tau, x^1(\tau,\varphi), \ldots, x^M(\tau,\varphi)) \\
|\frac{\partial x^\alpha}{\partial \varphi^\beta}| = |\frac{\partial x^i}{\partial \varphi^b}| &= \rho\{x_1, \ldots, x_M\} =: \frac{\eta\rho}{q(\vec{x},\tau)} \\
1 = \int \rho\, d^M\varphi &= \int \frac{\rho}{|\frac{\partial x}{\partial \varphi}|}\, d^M x = \frac{1}{\eta}\int q\, d^M x \\
\int f(\varphi^\alpha)\rho\, d^M\varphi &= \frac{1}{\eta}\int \hat{f}(x^\alpha) q\, d^M x \\
\frac{\vec{p}}{\eta\rho}(\varphi^\alpha) &= (\vec{\nabla}p)_{(x(\varphi))}, \\
X_- = \zeta_0 = \int \zeta\rho\, d^M\varphi &\stackrel{!}{=} \frac{1}{\eta}\int pq\, d^M x = -\frac{L_{+-}}{P_+} + \tau\frac{P_-}{P_+} \\
X_i = \int x_i\rho\, d^M\varphi &= \frac{1}{\eta}\int x_i q = \frac{L_{i+}}{P_+} + \tau\frac{P_i}{P_+} \\
P_- = \frac{1}{2\eta}\int (\frac{\vec{p}^2}{\rho^2} + \{,\ldots,\}^2)\rho\, d^M\varphi &= \frac{1}{2}\int ((\vec{\nabla}p)^2 + \frac{1}{q^2}) q\, d^M x
\end{aligned} \tag{16}$$

one obtains the hydrodynamic M–brane Poincaré–generators ([11][12][13])

$$\begin{aligned}
P_- &= \frac{1}{2}\int (q(\nabla p)^2 + \frac{1}{q}), \quad P_+ = \int q, \quad \vec{P} = \int q\vec{\nabla}p \\
L_{ab} &= \int q(x_a\partial_b p - x_b\partial_a p), \quad L_{a+} = \int q x_a - \tau P_a \\
L_{a-} &= \frac{1}{2}\int (x_a(q(\nabla p)^2 + \frac{1}{q}) - q\partial_a(p^2)) \\
L_{+-} &= -\int qp\, d^M x + \tau P_-
\end{aligned} \tag{17}$$

satisfying

$$
\begin{aligned}
\{L_{a\pm}, L_{+-}\} &= \mp L_{a\pm}, \quad \{L_{a-}, L_{b-}\} = 0 \\
\{L_{a\pm}, P_{\mp}\} &= P_a, \quad \{L_{+-}, P_{\pm}\} = \pm P_{\pm} \\
\{L_{ab}, L_{cd}\} &= -\delta_{bc} L_{ad} \pm 3\,\text{more} \\
\{L_{a+}, L_{b-}\} &= \delta_{ab} L_{+-} - L_{ab}.
\end{aligned}
\tag{18}
$$

Due to the zero–modes being ratios[1] of $SO(D-1,1)$ generators,

$$
\begin{aligned}
X_i &= \frac{L_{i+}}{P_+} + \tau \frac{P_i}{P_+}, \\
X_- = \zeta_0 &= -\frac{L_{+-}}{P_+} + \tau \frac{P_-}{P_+}
\end{aligned}
\tag{19}
$$

one may write the internal ('$SO(D-1)$') generators occurring in (14) as composite operators, solely as rational[2] expression in the generators of the original Poincaré algebra:

$$
\begin{aligned}
\mathbb{M}_{ij} &= L_{ij} - \frac{1}{P_+}(L_{i+}P_j - L_{j+}P_i) = L_{ij} - L'_{ij} \\
\mathbb{M}_{i-} &= P_+ L_{i-} - (\underbrace{L_{i+}P_- + L_{+-}P_i}_{=P_+ L'_{i-}} + \underbrace{\mathbb{M}_{ik}P_k}_{=P_+\tilde{L}'_{i-}}) \\
&=: P_+ L_{i-} - P_+ \tilde{L}_{i-};
\end{aligned}
\tag{20}
$$

and a tedious, but straightforward calculation, resp.

$$
\begin{aligned}
&\{L'_{ij}, L'_{kl}\} = -\delta_{jk} L'_{il} \pm 3\,\text{more}, \quad \{L_{ij}, L'_{kl}\} = -\delta_{jk} L'_{il} \pm 3\,\text{more}, \\
&\{L'_{i-}, L'_{j-}\} = 0, \\
&\{\tilde{L}'_{i-}, \tilde{L}'_{j-}\} = \frac{1}{P_+^2}(\vec{P}^2 \mathbb{M}_{ij} + P_i \mathbb{M}_{jk} P_k - P_j \mathbb{M}_{il} P_l) \\
&\{L'_{i-}, \tilde{L}'_{j-}\} - (i \leftrightarrow j) = -2\frac{P_-}{P_+}\mathbb{M}_{ij} - \frac{1}{P_+^2}(P_i \mathbb{M}_{jk} P_k - P_j \mathbb{M}_{il} P_l) \\
&\{\tilde{L}_{i-}, \tilde{L}_{j-}\} = -\frac{\mathbb{M}^2}{P_+^2}\mathbb{M}_{ij}, \quad \{\frac{\mathbb{M}_{i-}}{P_+}, \tilde{L}_{j-}\} - (i \leftrightarrow j) = 0,
\end{aligned}
\tag{21}
$$

note also

$$
\{\mathbb{M}_{ik}, P_j\} = 0, \quad \{\mathbb{M}_{ik}, L_{+-}\} = 0, \quad \{P_+, \mathbb{M}_{i-}\} = 0,
\tag{22}
$$

gives that (20) indeed satisfies (14).

[1] up to terms proportional to τ (which drop out in (20))

[2] were it not for the $\frac{1}{P_+}$ in the subtraction to M_{ij}, and eventual appearance of (commuting) factors of $\frac{1}{\sqrt{\mathbb{M}^2}}$, *polynomial*, i.e. elements of the enveloping algebra

3. On the r–matrix of M(embrane)–theory

Supersymmetrizable theories, such as M(em)branes and associated matrix–models related to Yang–Mills theory, possess r–matrices

While the Lax–pairs found in [14, 15], in contrast to standard integrable systems (see e.g. [16]), naively do not seem to provide any non–trivial conserved quantity, it is also unlikely that they will not be useful. As a start I would like to point out that Lax–pairs arising from supersymmetrizability generically do have an r–matrix associated with them (which in principle is not even particularly difficult to explicitly calculate), including the infinite–dimensional case of relativistic higher dimensional extended objects (see e.g. [13] for a review) such as Membrane theory, whose discretized version, a SU(N)–invariant matrix–model [2], is known to be subtle in several ways (e.g. possessing classical solutions extending to infinity, but quantum–mechanically purely discrete spectrum [17, 18], while when supersymmetrized [19] changing "again" to continuous[3] [20, 21]); in this sense making the existence of a rather special Lax–pair for them not too surprising.

Let me first illustrate the idea by considering[4]

$$\begin{gathered}\dot{L}_1 = [L_1, M], \quad \dot{L}_2 = [L_2, M] \\ L_1 = \sum_{a=1}^{N}(\gamma_a p_a - \gamma_{a+N}\partial_a w), \quad L_2 = \sum_a (\gamma_a \partial_a w + \gamma_{a+N} p_a) \\ M = -\frac{1}{2}\sum_{a,b=1}^{N} \gamma_a \gamma_{b+N} \partial^2_{ab} w\end{gathered} \tag{23}$$

where $w = w(x_1, x_2, \ldots, x_N)$ and the hermitean Clifford matrices $\gamma_{i=1\ldots 2N}$, satisfying

$$\gamma_i\gamma_j + \gamma_j\gamma_i = 2\delta_{ij}\cdot \mathbf{1}, \tag{24}$$

canonically realized as $2^N \times 2^N$ dimensional tensorproducts of Pauli–matrices. While in that canonical representation L_1 and L_2 anticommute and square to a multiple of the unit matrix, $\beta, \beta' = 1, 2,$

$$L_\beta L_{\beta'} + L_{\beta'} L_\beta = 2\delta_{\beta\beta'}(2H := \vec{p}^{\,2} + (\nabla w)^2)\mathbf{1} \tag{25}$$

(23), due to the polynomials of degree ≤ 2 in the γ_i closing under commutation, forming a (spinor–) representation of $so(2N+1)$, may also be considered in the *defining*, 'vector' representation of $so(2N+1)$, in which $\tilde{L}(\lambda) := \frac{1}{2}(\tilde{L}_1 + \lambda\tilde{L}_2)$ and $\tilde{M}$, instead of having non–zero elements distributed over many of the $2^N \times 2^N$ entries, take the simple form

$$\begin{gathered}\tilde{L}(\lambda) = i\begin{pmatrix} 0 & v^T \\ -v & 0_{2N\times 2N}\end{pmatrix} =: \sqrt{2H}\sqrt{\lambda^2+1}K(\lambda) \\ v = \begin{pmatrix}\vec{p} + \lambda\vec{\nabla}w \\ \lambda\vec{p} - \vec{\nabla}w\end{pmatrix} =: \sqrt{2H}\sqrt{\lambda^2+1}\; e(\lambda) \\ \tilde{M} = \begin{pmatrix} 0 & 0 \\ 0 & A\end{pmatrix}, \quad A = \begin{pmatrix} 0 & -w_{ab} \\ w_{ab} & 0\end{pmatrix}_{2N\times 2N},\end{gathered} \tag{26}$$

[3] by many first interpreted negatively, then [23, 24] positively

[4] cp.[14] (missing $\frac{1}{2}$ in eq.(12), dWHN: 1988/305.)

as when representing $\frac{1}{2}\gamma^{ij} = \frac{1}{4}(\gamma^i\gamma^j - \gamma^j\gamma^i)$ by $M_{ij} := E_{ij} - E_{ji}$ (generating $so(2N) \subset so(2N+1)$) $\frac{i\gamma_k}{2}$ will correspond to the generators $M_{0k} = E_{0k} - E_{k0}$ of $so(2N+1)$,

$$[M_{\mu\nu}, M_{\rho\lambda}] = \delta_{\nu\rho}M_{\mu\lambda} \pm 3\text{more} \tag{27}$$

$\mu, \nu, \rho, \lambda = 0, 1, \ldots, 2N$.

It is trivial to check that

$$\dot{\tilde{L}}(\lambda) = [\tilde{L}(\lambda), \tilde{M}] \Leftrightarrow \dot{e}(\lambda) = Ae, \quad e \in S^{2N-1} \tag{28}$$

are equivalent to the equations of motion $\dot{x}_a = p_a$, $\dot{p}_a = -w_{ab}\partial_b w$, ($w_{ab} := \partial^2_{ab}w$) being the Hessian of the 'superpotential' w. However, in contrast with L_1 and L_2 in the spinor representation each having N eigenvalues $+\sqrt{2H}$ and N eigenvalues $-\sqrt{2H}$, the Lax–matrix $\tilde{L}(\lambda)$, as given in (26), will have only *two* non–zero eigenvalues (which, diving by $\sqrt{2H}\sqrt{1+\lambda^2}$, i.e. considering the normalized matrix $K(\lambda)$, may be taken to be ± 1), with corresponding eigenvectors $\hat{e}_\pm$, i.e.

$$\begin{aligned} K(\lambda) &= U(\lambda)\begin{pmatrix} 1 & & & & \\ & -1 & & & \\ & & 0 & & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix} U^\dagger(\lambda), \\ U(\lambda) &= \frac{1}{\sqrt{2}}\begin{pmatrix} +i & -i & 0 & \ldots & 0 \\ e & e & \sqrt{2}n_1 & \ldots & \sqrt{2}n_{2N-1} \end{pmatrix} = (u_0 u_1 \ldots u_{2N-1}) \end{aligned} \tag{29}$$

with $(e, n_1, \ldots, n_{2N-1})$ forming an orthonormal basis of $\mathbb{R}^{2N}$.

As the eigenvalues of $K(\lambda)$ are numerical constants $(= +1, -1, 0\ldots 0)$,hence Poisson–commuting with everything, it easily follows, with

$$\{X_1, Y_2\} := \{X \otimes \mathbf{1}, \mathbf{1} \otimes Y\} := \{X_{ij}, Y_{kl}\}E_{ij} \otimes E_{kl} \tag{30}$$

that $K_1 := K \otimes \mathbf{1}$ and $K_2 := \mathbf{1} \times K$ satisfy

$$\begin{aligned} \{K_1(\lambda), K_2(\lambda)\} &= \big[[U_{12}(\lambda), K_1], K_2\big] = \big[[U_{12}(\lambda), K_2], K_1\big] \\ &= [\mathring{r}_{12}, K_1] - [\mathring{r}_{21}, K_2] \\ U_{12} &:= \{U_1, U_2\}U_1^{-1}U_2^{-1} \\ \mathring{r}_{12} &= \frac{1}{2}[U_{12}, K_2],\ \mathring{r}_{21} = \frac{1}{2}[U_{21}, K_1] = -\frac{1}{2}[U_{12}, K_1]; \end{aligned} \tag{31}$$

and $J(\lambda) := \sqrt{H}K(\lambda)$ will therefore satisfy

$$\begin{aligned} \{J_1, J_2\} &= H\{K_1, K_2\} + \frac{1}{2}(\dot{K}_1K_2 - \dot{K}_2K_1) \\ &= [H\mathring{r}_{12} - \frac{1}{2}\tilde{M}_1K_2, K_1] - (1 \leftrightarrow 2) \end{aligned} \tag{32}$$

so that the r–matrix for the Lax–pair $(J(\lambda), \tilde{M} = \left(\begin{smallmatrix} 0 & 0 \\ 0 & A \end{smallmatrix}\right))$ with equations of motion $\dot{e} = Ae$, $e(\lambda) \in S^{2N-1}$ is

$$r_{12}(\lambda) = \sqrt{H}\mathring{r}_{12}(\lambda) - \frac{1}{2}\tilde{M}_1K_2(\lambda)\frac{1}{\sqrt{H}}. \tag{33}$$

Having gone through this simple example it is almost correct to say that one has understood *all* supersymmetrizable systems (from this perspective) whose supercharges are linear in the Clifford generators, resp. 'fermions' (and at this stage it would be tempting to see a relation to other, old and new - see e.g. [22], and references therein - statements about supersymmetric systems); one 'only' gets different (for field–theories: infinite dimensional) unit vectors e and different ('more complicated') antisymmetric matrices A (cp. (28)) satisfying $\dot{e}(\lambda) = Ae(\lambda)$.

Consider now the membrane–matrix model (cp.(24)$_{J_a=0}$ of [14]), i.e.

$$v = \begin{pmatrix} \sum_\beta \lambda_\beta P^\beta_{\alpha a} \\ \sum_\beta \lambda_\beta Q^\beta_{\alpha a} \end{pmatrix} = \begin{pmatrix} \sum_\beta \lambda_\beta \vec{p}^{\,\beta} \\ \sum_\beta \lambda_\beta \vec{q}^{\,\beta} \end{pmatrix} = \begin{pmatrix} \vec{p}(\lambda) \\ \vec{q}(\lambda) \end{pmatrix} = \begin{pmatrix} \vec{p} \\ \vec{q} \end{pmatrix} \tag{34}$$

$$P^\beta_{\alpha a} = \sum_{t=1}^{d} p_{ta}\gamma^t_{\beta\alpha}, \quad Q^\beta_{\alpha a} = \frac{1}{2}(\gamma^{st})_{\beta\alpha} f_{abc} x_{sb} x_{tc},$$

where the f_{abc} are totally antisymmetric (real) structure constants of $su(N)$, $a, b, c = 1 \dots N^2 - 1$, the x_{sb} and p_{tc} are canonically conjugate variables, the γ^t are real symmetric $\sigma \times \sigma$ matrices satisfying $\gamma^s\gamma^t + \gamma^t\gamma^s = \delta^{st}\mathbf{1}$, the time–evolution is given by

$$H = \frac{1}{2}(P^\beta_{\alpha a}P^\beta_{\alpha a} + Q^\beta_{\alpha a}Q^\beta_{\alpha a}) = \frac{1}{2}(\vec{p}^{\,\beta}\vec{p}^{\,\beta} + \vec{q}^{\,\beta}\vec{q}^{\,\beta}), \tag{35}$$

which is independent of β, sum over $(\alpha a) = (11)\dots(\sigma, N^2 - 1)$ and

$$J_a = f_{abc} x_{sb} p_{sc} \stackrel{!}{=} 0, \tag{36}$$

which also implies $v^T v = (2H)(\vec{\lambda}^2)$; the equations of motion can be written in the form (cp.(6) of [14])

$$\dot{\vec{q}} = \Omega\vec{p}, \quad \dot{\vec{p}} = \Omega\vec{q}, \tag{37}$$

and the Lax–pair [14], when going to the defining vector representation of $so(2n+1)$, $n = \sigma(N^2 - 1) \in \mathbb{N}$, becomes –as explained above–

$$\begin{aligned} J(\lambda) &= i \begin{pmatrix} 0 & v^T \\ -v & 0 \end{pmatrix} \frac{1}{\sqrt{2\vec{\lambda}^2}} \\ \tilde{M} &= \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix} \\ A_{2n\times 2n} &= \begin{pmatrix} 0 & \Omega \\ \Omega & 0 \end{pmatrix} = -A^T \\ \Omega_{\alpha a, \alpha' a'} &= f_{aa'c} x_{tc} \gamma^t_{\alpha\alpha'} \end{aligned} \tag{38}$$

with

$$J = U \begin{pmatrix} \sqrt{H} & & & & \\ & -\sqrt{H} & & & \\ & & 0 & & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix} U^\dagger$$
$$U = \begin{pmatrix} \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0 & \dots & 0 \\ \frac{e}{\sqrt{2}} & \frac{e}{\sqrt{2}} & n_1 & \dots & n_* \end{pmatrix} \tag{39}$$

and the $2n$ unit vectors $(e, n_1 \dots n_*)$ being orthonormal, and

$$\dot{J}(\lambda) = [J(\lambda), \tilde{M}] \Leftrightarrow \dot{e} = Ae \tag{40}$$

being equivalent to the matrix–model equations of motion.

As shown above, the $su(N)$–invariant membrane matrix model of [2] therefore possesses an r–matrix,

$$\begin{aligned} \{J_1(\lambda), J_2(\lambda)\} &= [r_{12}(\lambda), J_1] - (1 \leftrightarrow 2) \\ r_{12} &= (\frac{1}{2}[U_{12}, J_2] - \frac{1}{2}\frac{\tilde{M}_1 J_2}{H}); \end{aligned} \tag{41}$$

note that the normalisation of J is chosen such that $\frac{1}{2}\mathrm{tr}J^2 = H$ (as a consistency check, one can calculate $-\mathrm{Tr}_2(r_{12}J_2) = \frac{1}{2}\tilde{M}\frac{\mathrm{tr}J_2^2}{H} + \frac{1}{4}\mathrm{tr}_2[U_{12}, J_2^2]$ which indeed gives $\tilde{M}$; that $\mathring{r}_{12}$ does not give any contribution means that it is in some sense 'trivial', i.e. not influencing the time–evolution; dimensionally $[x] \sim E^{\frac{1}{4}}$, $[p] \sim E^{\frac{1}{2}}$ so $[\frac{\partial}{\partial x}\frac{\partial}{\partial p}] = E^{-\frac{3}{4}} = [\frac{\tilde{M}}{H}] = \frac{E^{\frac{1}{4}}}{E}$).

What about the infinite–dimensional case of membrane–theory?

$$\begin{aligned} v^\beta_{\alpha a} &= \int Y_a(\varphi)(\frac{p_i}{\rho}\gamma^i_{\beta\alpha} + \frac{1}{2}\{x_i, x_j\}\gamma^{ij}_{\beta\alpha})\rho\, d^2\varphi \quad a \in \mathbb{N}_0 \\ \{x_i, x_j\}(\varphi) &:= \frac{\varepsilon^{rs}}{\rho}\partial_r x_i \partial_s x_j \\ \int Y_a Y_b \rho\, d^2\varphi &= \delta_{ab}, \quad \sum_{a=0}^{\infty} Y_a(\varphi)Y_a(\tilde{\varphi}) = \frac{\delta^2(\varphi, \tilde{\varphi})}{\rho}, \end{aligned} \tag{42}$$

$$\begin{aligned} \dot{J}(\lambda) &= [J(\lambda), \tilde{M}], \\ \Omega_{\alpha a, \alpha' a'} &= g_{aa'c} x_{ic} \gamma^i_{\alpha\alpha'}, \; g_{abc} = \int Y_a\{Y_b, Y_c\}\rho\, d^2\varphi \\ J &= i\begin{pmatrix} 0 & v^\dagger \\ -v & 0 \end{pmatrix} \frac{1}{\sqrt{2\lambda^\dagger\lambda}}, \\ \tilde{M} &= \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix}, \; A = \begin{pmatrix} 0 & \Omega \\ \Omega & 0 \end{pmatrix}. \end{aligned} \tag{43}$$

While in principle having to worry about potentially diverging infinite sums, and Lie–algebraically one would have to identify a well–defined algebra, I think that (43), and the infinite–dimensional analogue of (41), should be fine, for various reasons: $v^\dagger v = 2\lambda^\dagger\lambda H$ implies that for fixed (trivially conserved) energy all components of v are finite, and $\frac{v}{\sqrt{2\lambda^\dagger\lambda H}} =: e$ will be a unit vector; the norm of each row of Ω (or column, given by (αa)) is $\int\{Y_a, x_i\}^2\rho\, d^2\varphi$, which is clearly finite, as one integrates over a compact manifold, and if $\{Y_a, x_i\}$ was infinite for some a, the potential term of H could not be finite; the arising scalar products of infinite–dimensional vectors (corresponding to $\int \rho\, d^2\varphi$ of the product of corresponding square–integrable functions) therefore involve only vectors of finite norm. One may also write (43), resp. the equations of motion, in the following compact suggestive forms:

$$\begin{aligned}\dot{q}_\alpha^\beta &= \gamma^i_{\alpha\alpha'}\{p^\beta_{\alpha'}, x_i\},\ \dot{p}^\beta_\alpha = \gamma^i_{\alpha\alpha'}\{q^\beta_{\alpha'}, x_i\}, \text{ or as}\\ \dot{V} &= \{V, X\} := \{V_{\beta\alpha}, X_{\delta\varepsilon}\}E_{\beta\alpha}E_{\delta\varepsilon} = -\{X, V^T\}^T\end{aligned}$$

resp.

$$\begin{aligned}\dot{Q}_{\beta\alpha} &= \{P_{\beta\alpha'}, X_{\alpha'\alpha}\} = -\{X_{\alpha\alpha'}, P_{\alpha'\beta}\} = -\dot{Q}_{\alpha\beta}\\ \dot{P}_{\beta\alpha} &= \{Q_{\beta\alpha'}, X_{\alpha'\alpha}\} = +\{X_{\alpha\alpha'}, Q_{\alpha'\beta}\} = +\dot{P}_{\alpha\beta}\end{aligned} \tag{44}$$

(using that, as finite matrices, P is symmetric, $Q = \frac{1}{2}\{X, X\}$ antisymmetric and $X_{\alpha\alpha'} := \gamma^i_{\alpha\alpha'}x_i = X_{\alpha'\alpha}$, $\dot{X} = P$, $\ddot{X} = \frac{1}{2}\{X, \{X, X\}\}$).

Finally, note [25] and that $L = P + iQ$, respectively $\dot{L} = i\{L^*, X\}$ could be considered a generalization to arbitrary d of (39) in the first reference of [11], turning into a real Lax–pair for the Wick–rotated/Euclidean equation of motion (though care is needed for the definition of a Lie–algebra involving matrix–valued functions on the membrane).

4. Commuting signs of infinity

Discrete minimal surface algebras and Yang Mills algebras may be related to (generalized) Kac Moody algebras, just as Membrane (matrix) models and the IKKT model - including a novel construction technique for minimal surfaces.

I would like to mention some aspects of two kinds of double commutator equations[5]

$$\big[[X^\mu, X^\nu], X_\nu\big] = 0 \tag{45}$$

$$\big[[M_i, M_j], M_j\big] = \mu_i M_i. \tag{46}$$

(45) appears e.g. in [26][27][3][29][30][31][32] (and references therein), describing non–commutative minimal surfaces, resp. a quantization of string theory (related to the Schild action [33]; stronger

[5]the bilinear antisymmetric bracket, assumed to satisfy the Jacobi identity, is not assumed to necessarily come from an underlying associative multiplication, i.e. could also be a Poisson–bracket; repeated indices are summed over, unless stated otherwise; the distinction between upper and lower indices could of course also be made in (46), and the distinction between (45) and (46) is equally "pragmatical"

conditions, including $\big[[X^\mu, X^\nu], X_\rho\big] = 0$, implying basic uncertainty relations for a non–commutative space–time, appear in [34]), resp. covariant derivatives of Yang-Mills connections (see e.g. [36] [35]); (46) e.g. in [2][36][37][1][38] [39][40][41] (and references therein). [37] implies that the maximal compact subalgebra of simply laced Kac Moody algebras (for GIM algebras, see [42]; note that [40] contains observations and ideas that may also be relevant for the general infinite dimensional case) is isomorphic to the quotient of a free Lie algebra generated by $Y_1 \dots Y_n$ subject to the relations

$$\big[[Y_i, Y_j], Y_j\big] = \pm Y_i \qquad (\textit{no} \text{ sum}) \tag{47}$$

if the (ij) entry of the generalized Cartan matrix A is non–zero, while

$$[Y_i, Y_j] = 0 \quad \text{if} \quad A_{ij} = 0 \tag{48}$$

(the sign in (47) is in principle fixed by the Cartan involution and reality properties; it would be interesting to see whether *vanishing* 'signs' could arise after summing over j, thus relating (47) also to (45)). The relation to (46) is apparent, as for each generalized Cartan matrix μ_i simply results from the number of non–zero elements $A_{j\neq i}$ in the i–th column resp. row (which e.g. for the affine Kac Moody algebra $\tilde{A}_l$ would be 2, independent of i). Note that in the relation of (46) with the membrane (matrix) model in D–dimensional space–time [2][13], with classical equations of motion

$$\ddot{X}_i = -\sum_{j=1}^{d} \big[[X_i, X_j], X_j\big], \tag{49}$$

the X_i being $d = D - 2$ time–dependent traceless hermitean $N \times N$ matrices, one would want the non-zero μ_i (> 0) to appear in pairs – because of the Ansatz (cp. e.g. [40])

$$X_i(t) = \big(e^{J(t-t_0)}\big)_{ij} M_j \tag{50}$$

with $J^T = -J$ real, J^2 diagonal – and the hermitean M_i to satisfy (46), as well as

$$J_{ij}[M_i, M_j] = 0 \tag{51}$$

(in order to satisfy the $SU(N)$–'Gauss–constraint' $\sum[X_i, \dot{X}_i] = 0$, which is the discrete analogue of the residual invariance under area–preserving diffeomorphisms [2], hence has to be satisfied for (49) to include membranes as $N \to \infty$). One[6] way to satisfy the constraint would be to choose half of the M's to be identically zero, while in view of the Berman–construction, (47)+(48), one could simply pair each node with one to which it is not connected; or consider (51) an analogue of the sum–condition in (6.3) of [31]. Of particular interest would be to identify the maximal compact subalgebra of E_{10} (E_9) in (45) ((46)/(49)).

In the simplest example, $\tilde{A}_l$, one gets

$$\sum_j \big[[Y_i, Y_j], Y_j\big] = 2Y_i, \tag{52}$$

[6] in the context of membrane (matrix) solutions 'practised' – though I always considered it as somewhat unnatural.

each simple root having 2 neighbours; and the simplest finite–dimensional generalized spin representation of (47)+(48), hence (52)/(46), in this case is (apparently [36] first noticed by A.Kent; though the connection with infinite dimensional Lie–algebras was not realized at that time)

$$\begin{aligned} &M_k := Y_k := \frac{i}{2}\gamma^k\gamma^{k+1} = \frac{i}{2}\gamma^{k\,k+1} \\ &M_k = M_k^\dagger, \quad M_k^2 = \frac{1}{4}, \qquad\qquad k = 1\dots K\ (K+1 \equiv 1) \end{aligned} \tag{53}$$

where the γ^k are traceless anti–commuting hermitean (Clifford) matrices squaring to $\mathbf{1}$. In the context of the $d = 9$ rotating membrane solutions example one could e.g. take $K = 8$, $\gamma_1 = \sigma_1 \times 1 \times 1 \times 1$, $\gamma_2 = \sigma_2 \times 1 \times 1 \times 1$, $\gamma_3 = \sigma_3 \times \sigma_1 \times 1 \times 1, \dots, \gamma_8 = \sigma_3 \times \sigma_3 \times \sigma_3 \times \sigma_2$, $N = 16$, resp. 8 (note that the Clifford–solutions of (46) in [40] naively would need the doubling mechanism, i.e. $K = 4$, $N = 4$). For the affine Kac–Moody algebra $\tilde{D}_l$ one would naturally get solutions of (46) where $\mu = 1$ has multiplicity 4, $\mu = 3$ multiplicity 2, and $\mu = 2$ multiplicity $l - 5$.

While in the physics context the most important aspect of realizing (47)+(48) is that it signals potential infinite symmetries for (46) (resp.(45); note the 'reconstruction algebra' [7][13]), including a possible relation to the area–preserving diffeomorphism algebra for relativistic extended objects [2] [13], there is another, equally interesting, aspect: (47) can (and does) describe discrete minimal surfaces (hence the name DMSA in [40]) embedded in spheres (once in each connected component the μ_i are equal and the constraint $\sum M_i^2 \sim \mathbf{1}$ is added); hence it is natural to conjecture that the generalized higher spin representations of (47)+(48) (cp. [41], and references therein) include series of finite dimensional representations (of increasing dimension) that for $N \to \infty$ converge to (new) minimal surfaces in spheres.

5. Outlook

Possible Multi-Hamiltonian structures (of hydrodynamic type) describing Membrane Theory should be investigated. In [43] new classes of exact M(em)brane solutions in M+2 dimensional Minkowski space are presented (some describing non-trivial topology changes, while others explicitly avoid finite-time singularity formation). In [44] Baecklund-type transformations in four-dimensional space-time and an intriguing reduced zero-curvature formulation for axially symmetric membranes are found, with diffeomorphism- resp. Lorentz-symmetries reappearing after orthonormal gauge-fixing.

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