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Fuzzy de Sitter and anti-de Sitter spaces

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We review models of quantum spaces of maximal symmetry, obtained by quantisation of two-dimensional sphere and of de Sitter and anti-de Sitter spaces in two and four dimensions, using the noncommutative formalism of moving frames.

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1. Introduction

Idea that spacetime becomes granular/discrete or fuzzy at very small scales or very high energies encapsulates much of physicists intuition about the spacetime structure, and it is usually related to quantisation. One aspect of this idea is the existence of the lower limit on position measurements. Effective or strict, such limit can (in principle) cure the problem of singularities of classical configurations of the gravitational field; it can also provide the UV regulator for momentum integrals in quantum field perturbation theory.

One way to quantise spacetime is to replace coordinates by noncommuting operators. Another possibility is to represent coordinates as quantum fields (that describe elementary constituents of matter) of finite extent. In both cases, spacetime granularity/fuzziness is obtained from uncertainty relations. Therefore it is reasonable to expect that, as we approach the Planck scale, noncommutative spacetime emerges. However, the scale of spacetime quantisation (or noncommutativity), denoted here by $\ell_{NC} \equiv \sqrt{\bar{k}}$, is not necessarilly equal to the scale of quantum gravity, i.e. the Planck scale ℓ_{Pl}. Following the logic that spacetime emerges from quantum fields, one commonly assumes that $\ell_{NC} \geq \ell_{Pl}$. Also, if quantum spacetime is emergent, it need not have intrinsically geometric properties. Yet in general relativity, geometry gives one of equivalent descriptions of the gravitational field: thus the geometric description should appear in the classical limit of quantum gravity. One might expect/explore the possibility that it exists on the quantum level as well.

In order to to formulate the quantum spacetime regime one builds models of noncommutative geometry and noncommutative gravity. The common ingredient is noncommutative space, defined as an algebra $\mathcal{A}$ generated by coordinates x^μ that obey some (specified) commutation relations, and a set of classical fields, i.e. functions $f(x^\mu) \in \mathcal{A}$. Further, to introduce gauge fields, a differential geometry on $\mathcal{A}$ is defined. For a description of gravity, one introduces gravitational field $g_{\mu\nu}$, or tetrad (moving frame) e^μ_α and spin connection $\omega^\alpha{}_{\gamma\beta}$. In addition one attempts to preserve symmetries, or deform them in a controlled way: either the symmetries of general relativity (diffeomorphisms and local Lorentz transformations), or the symmetries of specific spacetime configurations.

A frequent framework for noncommutative gravity is to fix the noncommutativity of spacetime (via the star-product, abelian twist, etc.), that is, the algebra $\mathcal{A}$ and its representation, and generalise the description of gravity as a classical field with a defined Lagrangian and equations of motion [1, 2]. In this case we have a dynamical gravitational field that evolves on a predefined noncommutative spacetime. Gravity is then not in a direct relation to algebraic properties of spacetime (as one might, using geometric intuition, expect), but the modes of gravitational field are adjusted to the 'noncommutative spacetime lattice' through equations of motion. By construction, expansion in $\bar{k}$ in zeroth order gives the Einstein equations while their solutions, and higher orders give noncommutative corrections. Clearly, spacetime symmetries of a particular classical solution that are not compatible with noncommutativity will be broken in the full noncommutative extension.

The approach that we use is the noncommutative frame formalism, [3]. In its core is a quantisation of geometry, that is not (necessarily) the symplectic quantisation. Coordinates and momenta are quantised using a general-relativistic version of the correspondence principle,

$$[p_\alpha, x^\mu] = e^\mu_\alpha(x) \ . \tag{1}$$

On commutative manifolds, this relation gives components of the freely falling frame, a set of locally orthonormal vector fields e_α. In the noncommutative version momenta p_α can, and typically do belong to the spacetime $\mathcal{A}$. Clearly, noncommutativity and gravity are here intricately related. But on the other hand, neither is dynamical, that is, determined by equations aside from consistency conditions (which can indeed include time derivatives). In the view of this, noncommutative frame formalism is a noncommutative generalisation of Cartan geometry, and its solutions can perhaps be viewed only as ground states of a more complete quantum theory. The approach is well suited to describe geometry of finite-matrix spaces as well as spaces with high degree of symmetry, where momenta are represented by elements of a Lie algebra: therefore, formalism gives naturally quantisation of the solutions to Einstein equations with cosmological constant.

We shall in the following discuss some details of quantisation of the spaces of maximal symmetry: the fuzzy sphere, fuzzy anti-de Sitter space in two dimensions and fuzzy de Sitter space in four dimensions; we will at the end comment on quantisation of anti-de Sitter spaces in four and three dimensions. Before that, we shortly review some aspects of the noncommutative frame formalism, applied in particular to cases when momenta are generators of a Lie algebra.

2. Noncommutative frames

In Cartan's moving frame formalism one can single out two aspects of geometry:

∘ basic algebraic structure, that is, properties of the manifold described by coordinates x^μ as: range of variables, boundaries, horizons and singularities; diffeomorphism symmetry, and

∘ differential-geometric structure, that is, properties of linear spaces of vector fields and p-forms. The tangent space is spanned by partial derivatives ∂_μ and geometric properties of the manifold can be described by a freely falling frame of derivations e_α that are orthonormal at each spacetime point. Components of the frame and the structure functions are defined as

$$e_\alpha = e^\mu_\alpha(x)\,\partial_\mu\,, \qquad [e_\alpha, e_\beta] = C^\gamma{}_{\alpha\beta}(x)\,e_\gamma \ . \tag{2}$$

When spacetime is torsionless, the $C^\gamma{}_{\alpha\beta}(x)$ define the (Levi-Civita) connection $\omega^\gamma{}_{\alpha\beta}(x)$. The de Rham calculus of p-forms is defined via the differential d and the wedge product $\wedge$,

$$df(x) = (e_\alpha f)\,\theta^\alpha\,, \qquad \theta^\alpha\wedge\theta^\beta \equiv \theta^\alpha\theta^\beta = \tilde P^{\alpha\beta}{}_{\gamma\delta}\theta^\gamma\theta_\delta\ , \tag{3}$$

where θ^α are 1-forms dual to e_α, $\theta^\alpha(e_\beta) = \delta^\alpha_\beta$, and $\tilde P$ is antisymmetrization, $2P^{\alpha\beta}{}_{\gamma\delta} = \delta^\alpha_\gamma\delta^\beta_\delta - \delta^\alpha_\delta\delta^\beta_\gamma$. Tangent and cotangent spaces have local Lorentz symmetry.

∘ gravity connects the two structures: components of the frame and the metrics are

$$e^\mu_\alpha(x) = e_\alpha x^\mu\,, \qquad g^{\mu\nu} = \eta_{\alpha\beta}\,e^\mu_\alpha\,e^\nu_\beta\ . \tag{4}$$

In the flat space we have $e_\alpha = \delta^\mu_\alpha\, \partial_\mu\,,\ \theta^\alpha = \delta^\alpha_\mu\, dx^\mu\,,\ e^\mu_\alpha(x) = \delta^\mu_\alpha\,,\ g^{\mu\nu} = \eta^{\mu\nu}\,.$

In the noncommutative version of the frame formalism all structures are more general:

∘ spacetime $\mathcal{A}$ can be a commutative algebra of functions, but also a finite-matrix algebra or an algebra of operators; coordinates obey a general position algebra,

$$[x^\mu, x^\nu] = i\bar{k} J^{\mu\nu}(x)\ . \tag{5}$$

∘ vector fields are not necessarily outer i.e. generated by partial derivatives: they can be inner, generated by momenta $p_\alpha \in \mathcal{A}$,

$$e_\alpha f = [p_\alpha, f]\ . \tag{6}$$

The linear space of vector fields is infinite-dimensional, because for two functions $h, f \in \mathcal{A}$, $(he_\alpha)f \neq h(e_\alpha f)$. Furthermore, the differential calculus for given algebraic structure (5) is not uniquely defined, as it is in the commutative case.

To obtain finite-dimensional tangent and cotangent spaces one has to restrict the space of vector fields to a finite subspace. The moving frame gives a natural (albeit not unique) choice for this restriction, normally related to the classical limit of the given noncommutative spacetime. It is furthermore assumed that the dual 1-forms θ^α freely generate the algebra of 1-forms. In order to ensure that the frame components of metric be constant, relation

$$[f, \theta^\alpha] = 0 \tag{7}$$

is imposed for all $f \in \mathcal{A}$. The differential calculus based on this noncommutative moving frame can the be defined generalising formulas (3) and (4). However, in order to preserve the linear structure of the space of p-forms $\Omega^*(\mathcal{A})$ and the consistency of the structure, additional constraints appear. Perhaps the most important one is that momenta must satisfy an algebra of the form

$$2P^{\alpha\beta}{}_{\gamma\delta}\, p_\alpha p_\beta - F^\alpha{}_{\gamma\delta}\, p_\alpha - K_{\gamma\delta} = 0\,, \tag{8}$$

where $P^{\alpha\beta}{}_{\gamma\delta}$, $F^\alpha{}_{\gamma\delta}$ and $K_{\gamma\delta}$ are constants.

In the moving frame basis, covariant derivative, connection and curvature are defined using the connection 1-form $\omega^\alpha{}_\beta$ and the curvature 2-form $\Omega^\alpha{}_\beta$ by

$$D\theta^\alpha = -\omega^\alpha{}_{\gamma\beta}\, \theta^\gamma\theta^\beta, \qquad \Omega^\alpha{}_\beta = d\omega^\alpha{}_\beta + \omega^\alpha{}_\gamma\omega^\gamma{}_\beta = \frac{1}{2}R^\alpha{}_{\beta\gamma\delta}\theta^\gamma\theta^\delta\ . \tag{9}$$

These are the same as classical expressions, except that obviously $\omega^\alpha{}_{\gamma\beta}$ and $R^\alpha{}_{\beta\gamma\delta}$ are elements of a noncommutative algebra.

The Riemannian Laplace-Beltrami operator may be constructed, as in Riemannian geometry, from the differential d and the Hodge star operation $*$ using the the co-differential δ,

$$\Delta = d\delta + \delta d\ . \tag{10}$$

The co-differential acts on p-forms as $\delta = (-1)^{p-1} * d*$. For a rigorous treatment in abstract and index notation and more details we refer to [3].

the noncommutative frame calculus works very well on a range of examples like the Moyal space, the fuzzy sphere and the $SU(n)$ matrix algebras, where one can more or less identify the sets $\{x^\mu\}$ and $\{p_\alpha\}$. In cases when momenta lie in a Lie algebra, the differential calculus simplifies: the structure functions $C^\gamma{}_{\alpha\beta} = F^\gamma{}_{\alpha\beta}$ coincide with the Lie-algebra structure constants, the central charges vanish, $K_{\alpha\beta} = 0$, and $P^{\alpha\beta}{}_{\gamma\delta} = \tilde{P}^{\alpha\beta}{}_{\gamma\delta}$ becomes the usual antisymmetrisation. In particular, frame 1-forms θ^α anticommute. This implies that the structure of the algebra of differential forms $\Omega^*(\mathcal{A})$, up to noncommutativity of functions, is the same as in commutative differential geometry.

3. Fuzzy S_2, dS_2, AdS_2

We will study quantisation of spaces of maximal symmetry, that is, solutions to the Einstein equations with cosmological constant. The interest for this study stems from cosmology and the fact that the de Sitter geometry describes fairly well the present state of the universe as well as its inflatory phase. Another motive is that the usual Moyal quantisation of the flat Minkowski space breaks rotational symmetry, so it cannot be used as a noncommutative ground state for spherically symmetric perturbations. To keep a certain set of symmetries, one chooses momenta that belong to the Lie algebra of the corresponding symmetry group.

The best example of this kind of quantisation is the fuzzy sphere. An additional idea incorporated in its definition is to interpret the Casimir relation of the $SO(3)$ group as the embedding condition: then the unitary irreducible representations define the fuzzy sphere.

Let us recall the main identifications for the fuzzy sphere, [4]. Denote by J_a the hermitian generators of the $\mathfrak{so}(3)$ algebra, $[J_a, J_b] = i\epsilon_{abc}J_c$. (The signature is Euclidean so there is no need to make difference between upper and lower indices; we will do it anyway in some formulas for clarity. The $a, b = 1, 2, 3$ are the frame indices and $i, j = 1, 2, 3$ the coordinate indices.) Define coordinates as

$$x^i = \ell J^i = \frac{\kbar}{r}\,\delta^i_a J_a\,. \tag{11}$$

The $SO(3)$ Casimir relation, $C_2 = J_a J_a = j(j+1) = (n^2-1)/4$, valid in n-dimensional unitary irreducible representations, can be interpreted as condition defining the embedding of two-dimensional noncommutative sphere in three flat dimensions, $x^i x^i = r^2$. The radius r of the sphere is quantised, $4r^4 = \kbar^2(n^2-1)$; in the large-n limit, $r^2 = \kbar n/2$.

In order to keep the spherical symmetry one introduces momenta as

$$p_a = \frac{1}{ir}\,J_a\,, \tag{12}$$

$C_{abc} = \epsilon_{abc}/r$. The Lie-algebra structure of momentum algebra implies a differential geometry very similar to the commutative one: the torsionless connection can be defined as

$$\omega_{acb} = \frac{1}{2}\left(C_{abc} + C_{bca} - C_{cab}\right)\,, \tag{13}$$

and it gives the Riemann curvature tensor

$$R^a{}_{bcd} = \frac{1}{4r^2}\,(\delta^a_c\delta_{bd} - \delta^a_d\delta_{bc})\ . \tag{14}$$

On the other hand, the frame

$$e^i_a = \frac{1}{r}\,\epsilon^i{}_{ja}x^j \tag{15}$$

gives, for the spatial components of the metric

$$g^{ij} = \frac{1}{r^2}\left(r^2\delta^{ij} - \frac{1}{2}\,\{x^i, x^j\} + \frac{i\bar{k}}{2r}\,\epsilon^{ijk}x_k\right)\ . \tag{16}$$

In the limit $\bar{k} \to 0$ this expression reduces to the projector to the sphere. The fuzzy sphere gives a finite, discrete model of spherically symmetric surface: in the limit $n \to \infty$ it tends to the smooth sphere.

Generalisation from sphere to hyperboloid, i.e. from $SO(3)$ to $SO(2,1)$, is in many aspects straightforward. Classically, two-dimensional de Sitter and anti-de Sitter spaces are defined via the embeddings

$$(x^0)^2 - (x^1)^2 - (x^2)^2 = -\alpha^2\,,\quad \text{and} \tag{17}$$

$$(x^0)^2 + (x^1)^2 - (x^2)^2 = -\alpha^2\ . \tag{18}$$

The symmetry groups of the two spaces, $SO(1,2)$ and $SO(2,1)$ are equivalent: the embedding relations are the same up to replacement of temporal and spatial coordinates $x^0 \leftrightarrow x^2$ and the sign of α^2. We review the construction following reference [5].

It is quite clear that construction of noncommutative extensions of dS_2 and AdS_2 spaces can be done in analogy with the fuzzy sphere, except that the metric is not Euclidean but Minkowskian, η_{ab} (the signature is $(+--)$ for de Sitter and $(++-)$ for anti-de Sitter). The $\mathfrak{so}(2,1)$ algebra commutation relations

$$[K_a, K_b] = i\epsilon_{ab}{}^c\,K_c \tag{19}$$

now have different signs for different components. We identify, as before,

$$x^i = \ell K^i\,,\qquad p_a = \frac{1}{i\alpha}\,K_a\,, \tag{20}$$

and find that differential geometry of fuzzy two-dimensional de Sitter and anti-de Sitter spaces, defined essentially by relations in the momentum algebra is, up to the spacetime signature, analogous to that that we found in case of the fuzzy sphere, (14),(16).

However, situationa are not completely identical. The classical de Sitter and anti-de Sitter spaces are non-compact and so is the symmetry group $SO(2,1)$, which means that its unitary irreducible representations are infinite-dimensional. The $SO(2,1)$ has three series of unitary irreducible representations: principal, complementary and discrete. They can be labeled by the quantum number h which gives value of the Casimir operator, $C_2 = h(h-1)$. We have

◦ principal continuous series T_ρ, $h = \frac{1}{2} + i\rho$, $\rho \in \mathbb{R}$, $C_2 < 0$,

∘ complemenary continuous series T_σ , $h = \frac{1}{2} + \sigma$, $\sigma \in (0, \frac{1}{2})$, $C_2 < 0$, and

∘ discrete series $T_l^\pm$, $h = -l = 1, \frac{3}{2}, 2, \dots,$ $C_2 > 0$.

Therefore, fuzzy dS_2 space is defined as the principal series of the $SO(2,1)$ while fuzzy AdS_2 corresponds to the discrete series.

There is a comment to be added. In cases discussed there is a (kind of) discrepancy in numbers of coordinates and momenta. Namely, three coordinates that are introduced are not independent: for fuzzy sphere for example $(x^3)^2 = r^2 - (x^1)^2 - (x^2)^2$ and $x^3 \sim [x^1, x^2]$, therefore we may say that we are dealing with a two-dimensional space. On the other hand the corresponding cotangent space is, by construction, three-dimensional linear space, as it is freely generated by three frame 1-forms θ^a induced by momenta p_a. In the noncommutative framework, one usually assigns the notion of dimensionsionality to dimension of tangent and cotangent spaces.

4. Fuzzy dS_4

As we saw in the previous section, the number of coordinates and momenta on a fuzzy space does not have to be the same: the moving-frames quantisation is not always simplectic. Difference in numbers becomes (in some sense) more pronounced in higher dimensions. For example, the de Sitter space in four dimensions is described by five flat coordinates x^0, x^1, x^1, x^3, x^4 that are constrained by the embedding relation

$$(x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 - (x^4)^2 = -\alpha^2 . \tag{21}$$

On the other hand, its symmetry group $SO(1,4)$ has ten generators. How can we quantise de Sitter, in a natural or physical manner? Or, reformulating the problem: starting from the de Sitter algebra $\mathfrak{so}(1,4)$ generated by $M_{\alpha\beta}$, $\alpha, \beta = 0, 1, 2, 3, 4$ (signature is $(+----)$),

$$[M_{\alpha\beta}, M_{\gamma\delta}] = -i\,(\eta_{\alpha\gamma}M_{\beta\delta} - \eta_{\alpha\delta}M_{\beta\gamma} - \eta_{\beta\gamma}M_{\alpha\delta} + \eta_{\beta\delta}M_{\alpha\gamma}) , \tag{22}$$

how do we identify coordinates and momenta such that the resulting fuzzy dS_4 space provides a quantisation of the classical one, that is, it has the correct commutative limit?

The last condition is in fact not at all clearly defined and it can mean a number of (unrelated or weakly related) conditions of varying strength. The weakest of the classical-limit conditions is that the noncommutative metric (in some coordinates) be equal, or reduce to the classical metric in the limit; similar or equivalent condition can be imposed on the frame. The other possibility is to require that symmetries of fuzzy and commutative spaces be the same, or coincide in the limit. Finally, the most stringent condition is to require eqivalence of noncommutative and commutative algebras of functions, as discussed in [6]. We use here the first, weakest form of the classical limit.

The Pauli-Lubanski vector W^α of the de Sitter algebra is defined as $W^\alpha = \dfrac{1}{8}\epsilon^{\alpha\beta\gamma\delta\eta}M_{\beta\gamma}M_{\delta\eta}$. There are two Casimir operators of $SO(1,4)$, of second and fourth order,

$$C_2 = Q = -\frac{1}{2}\,M_{\alpha\beta}M^{\alpha\beta} , \qquad C_4 = \mathcal{W} = -\eta_{\alpha\beta}\,W^\alpha W^\beta , \tag{23}$$

they characterise the irreducible representations. The unitary irreducible representations of $SO(1,4)$ are infinite-dimensional, labeled by quantum numbers s and ρ. They are classified as:

∘ principal continuous series: $s = 0, \frac{1}{2}, 1, \dots, \ \rho \geq 0$, $\mathcal{Q} = -s(s+1)+\frac{9}{4}+\rho^2$, $\mathcal{W} = s(s+1)(\frac{1}{4}+\rho^2)$;

∘ complementary continuous series: $s = 0, 1, 2\dots, \ \nu = i\rho \in \mathbb{R}$, $|\nu| < \frac{3}{2}$, $\mathcal{Q} = -s(s+1)+\frac{9}{4}-\nu^2$, $\mathcal{W} = s(s+1)(\frac{1}{4}-\nu^2)$;

∘ discrete series: $s = \frac{1}{2}, 1, \frac{3}{2}\dots, \ \ q = \frac{1}{2}+i\rho = s, s-1, \dots 0$ or $\frac{1}{2}$, $\mathcal{Q} = -s(s+1)-(q+1)(q-2)$, $\mathcal{W} = s(s+1)q(1-q)$.

Analogously to the fuzzy sphere, the fuzzy de Sitter space can be described as noncommutative embedding in a five-dimensional noncommutative space generated by coordinates

$$x^\alpha = \ell\, W^\alpha\ . \tag{24}$$

The embedding is realized through the second Casimir relation,

$$\eta_{\alpha\beta}\, x^\alpha x^\beta = -\ell^2 \mathcal{W} = -\frac{3}{\Lambda}\ , \tag{25}$$

where Λ is the cosmological constant. Thus fuzzy de Sitter space is defined as a unitary irreducible representation of the de Sitter group, [7, 8]. Condition $\Lambda > 0$ singles out the principal continuous series.

One may ask whether coordinates $x^\alpha = \ell\, W^\alpha$, being quadratic in $M_{\alpha\beta}$, generate the whole noncommutative space $\mathcal{A}$. The answer is positive: it can be shown that, in an irreducible representation,

$$i\mathcal{W} M^{\rho\sigma} = [W^\rho, W^\sigma] + \frac{1}{2}\,\epsilon^{\alpha\mu\rho\sigma\tau} W_\tau\, [W_\alpha, W_\mu]\ . \tag{26}$$

Let us proceed to the momenta. There are two choices of momenta that give the correct, de Sitter form of the metric. If we wish to preserve complete $SO(1,4)$ symmetry, we choose momenta to be

$$i p_a = \sqrt{\zeta\Lambda}\, M_{\alpha\beta} \tag{27}$$

(ζ is an appropriate numerical factor), and then we obtain the metric with coordinate components equal to the projector $g^{\alpha\beta} = 3\eta^{\alpha\beta} - \Lambda x^\beta x^\alpha$, similar to (16). The tangent space is ten-dimensional.

However, the tangent space of such a high dimension is unusual. Motivated by applications in cosmology, we can alleviate our symmetry requirements: instead of the full de Sitter symmetry, we impose that spacetime just be homogeneous and isotropic. This can be attained assuming that momenta are generators of conformal translations and dilation:

$$i\Pi_0 = \sqrt{\zeta\Lambda}\, M_{04}\ , \qquad i\Pi_i = \sqrt{\zeta\Lambda}\, (M_{i4} + M_{0i})\ , \quad i = 1,2,3\ . \tag{28}$$

The cotangent space is then four-dimensional. Calculating the frame, we find the line element,

$$ds^2 = d\tau^2 - e^{\frac{2\tau}{\ell}}\, (dx^i)^2\ , \tag{29}$$

where the new coordinate, cosmic time $\tau = \ell\,\log(W^0 + W^4)$, is introduced. Equivalently, using the conformal time $\eta = -\ell\, e^{-\tau/\ell}$, the line element is written as

$$ds^2 = \frac{\ell^2}{\eta^2}\left(d\eta^2 - (dx^i)^2\right) . \tag{30}$$

The momentum algebra gives the details of differential geometry. We have

$$[\Pi_0, \Pi_i] = \sqrt{\zeta\Lambda}\,\Pi_i\,, \qquad [\Pi_i, \Pi_j] = 0\,, \tag{31}$$

with nonvanishing structure constants $C^i{}_{0j} = -C^i{}_{j0} = \sqrt{\zeta\Lambda}\,\delta^i_j$. As already commented, the Lie-algebra structure implies that the frame 1-forms anticommute and connection can be taken to be Levi-Civita, (13). We find

$$\omega^0{}_0 = 0, \quad \omega^0{}_j = \sqrt{\zeta\Lambda}\,\eta_{ij}\theta^i\,, \quad \omega^i{}_0 = -\sqrt{\zeta\Lambda}\,\theta^i\,, \quad \omega^i{}_j = 0\,. \tag{32}$$

For the components of the Ricci curvature we obtain

$$R_{00} = -3\zeta\Lambda\,\eta_{00}\,, \quad R_{ij} = -3\zeta\Lambda\,\eta_{ij}\,, \qquad R = 6\zeta\Lambda\,. \tag{33}$$

The Ricci tensor satisfies relation $R_{ab} = -3\zeta\Lambda\,\eta_{ab}$, that is, fuzzy de Sitter space is a noncommutative Einstein manifold.

Let us in addition write the Laplacian Δ, calculated using the Hodge-dual $*$ and the codifferential δ defined in the noncommutative frame formalism, [3]. The action of Δ on scalar fields, i.e. scalar functions of noncommutative coordinates, is given by [9]

$$\Delta f = [\Pi_0, [\Pi_0, f]] + [\Pi_i, [\Pi^i, f]] - 3\sqrt{\zeta\Lambda}\,[\Pi_0, f]\,. \tag{34}$$

The action on wave functions, i.e. elements of the representation space, is

$$\Delta\Psi = \left(\Pi_0\Pi^0 + \Pi_i\Pi^i - 3\sqrt{\zeta\Lambda}\,\Pi_0\right)\Psi\,. \tag{35}$$

Neither of the two actions of Δ is hermitian: this property is perhaps interesting for quantum information as it implies nonunitarity. For our purposes, we can reorder expressions (34-35) to get the usual forms of the Laplacian, e.g.

$$\Delta\Psi = (\Pi_0\Pi^0 + \Pi_i\Pi^i)\Psi\,. \tag{36}$$

5. Moylan representation

The representation of the principal continuous series of $SO(1,4)$ that we used in calculations is given by Moylan, [10]. It is defined on the space of unitary irreducible representations of the Poincaré group of mass m and spin s, [11]: the corresponding Hilbert space is a direct sum of

Hilbert spaces of states with positive and negative energies, $\mathcal{H}(m,s,+)\oplus\mathcal{H}(m,s,-)$. After a specific mapping to $\mathcal{H}(m,s,+)\oplus\mathcal{H}(m,s,+)$, we find the $\mathfrak{so}(1,4)$ generators, [9]:

$$\mathcal{M}_{\mu\nu}\equiv\begin{pmatrix}M_{\mu\nu,\uparrow} & 0\\ 0 & M_{\mu\nu,\downarrow}\end{pmatrix}=\begin{pmatrix}M_{\mu\nu} & 0\\ 0 & M_{\mu\nu}\end{pmatrix},\quad \mathcal{M}_{\mu4}\equiv\begin{pmatrix}M_{\mu4,\uparrow} & 0\\ 0 & M_{\mu4,\downarrow}\end{pmatrix}=\begin{pmatrix}M_{\mu4} & 0\\ 0 & -M_{\mu4}\end{pmatrix} \tag{37}$$

with

$$\begin{aligned} M_{ij}&=i\Big(p_i\frac{\partial}{\partial p^j}-p_j\frac{\partial}{\partial p_i}\Big)+S_{ij}, & M_{i0}&=-ip_0\frac{\partial}{\partial p^i}+S_{i0},\\ M_{4j}&=-\frac{\rho}{m}\,p_j-\frac{1}{2m}\{p^0,M_{0j}\}-\frac{1}{2m}\{p^i,M_{ij}\}, & M_{40}&=-\frac{\rho}{m}\,p_0+\frac{1}{2m}\{p^i,M_{0i}\}\,. \end{aligned} \tag{38}$$

The $S_{\mu\nu}$, $\mu,\nu=0,1,2,3$, are the usual Poincaré spin operators: for $(\rho,s=0)$, $S_{\mu\nu}=0$; for $(\rho,s=1/2)$, $S_{\mu\nu}=\frac{i}{4}[\gamma_\mu,\gamma_\nu]$. To simplify, one can put $m=1$. The wave functions Ψ are in momentum representation,

$$\Psi(\vec{p})=\begin{pmatrix}\psi_\uparrow(\vec{p})\\ \psi_\downarrow(\vec{p})\end{pmatrix}, \tag{39}$$

where $\psi_{\uparrow,\downarrow}$ are scalar functions and Dirac bispinors for $(\rho,s=0)$ and $(\rho,s=1/2)$ respectively. The scalar product is given by

$$(\Psi,\Psi')=(\psi_\uparrow,\psi'_\uparrow)+(-1)^{2s}(\psi_\downarrow,\psi'_\downarrow)\,, \tag{40}$$

where (ψ,ψ') depends on the spin,

$$(\psi,\psi')=\int\frac{d^3p}{2|p_0|}\,\psi^*\psi'\quad(\rho,s=0)\,,\qquad (\psi,\psi')=\int\frac{d^3p}{2|p_0|}\,\psi^\dagger\gamma^0\psi'\quad(\rho,s=1/2)\,. \tag{41}$$

5.1 Coordinates

In the previous section, differential-geometric properties of fuzzy de Sitter space were determined: it has a homogeneous and isotropic FLRW-type metric and a constant curvature. These properties arose from the algebra of momenta and are independent of the concrete representation. To understand the space further, we should in addition investigate coordinates: their spectra, motion of the scalar particles and scalar fields, etc. We will first describe the spectral properties of components of the Pauli-Lubanski vector, $x^\alpha=\ell\,\mathcal{W}^\alpha$.

One characteristic of $(\rho,s=0)$ representations is $\mathcal{W}=0$, and also $\mathcal{W}^\alpha=0$. This means that the cosmological constant $\Lambda=3/(\ell^2\mathcal{W}^2)=\infty$, so that there is no direct physical interpretation of coordinates in this case. We thus calculate the spectra of coordinates for $(\rho,s=1/2)$. We obtain

$$\mathcal{W}_4=\begin{pmatrix}W_4 & 0\\ 0 & W_4\end{pmatrix},\quad \mathcal{W}_0=\begin{pmatrix}W_0 & 0\\ 0 & -W_0\end{pmatrix}, \tag{42}$$

with

$$W^0 = -\frac{1}{2}\begin{pmatrix} (\rho-\frac{i}{2})\,p_i\sigma^i + i\,p_0^2\,\frac{\partial}{\partial p^i}\sigma^i & \epsilon^{ijk}p_0p_i\,\dfrac{\partial}{\partial p^j}\,\sigma_k + \dfrac{3i}{2}\,p_0 \\ \epsilon^{ijk}p_0p_i\,\frac{\partial}{\partial p^j}\,\sigma_k + \dfrac{3i}{2}\,p_0 & (\rho-\dfrac{i}{2})\,p_i\sigma^i + i\,p_0^2\,\dfrac{\partial}{\partial p^i}\sigma^i \end{pmatrix}, \tag{43}$$

$$W^4 = -\frac{1}{2}\begin{pmatrix} ip_0\,\dfrac{\partial}{\partial p^i}\sigma^i & \epsilon^{ijk}p_i\,\dfrac{\partial}{\partial p^j}\,\sigma_k + \dfrac{3i}{2} \\ \epsilon^{ijk}p_i\,\dfrac{\partial}{\partial p^j}\,\sigma_k + \dfrac{3i}{2} & ip_0\,\dfrac{\partial}{\partial p^i}\sigma^i \end{pmatrix}. \tag{44}$$

Let us outline computation of the eigenvalues, for example for $\mathcal{W}_4$. Since $\mathcal{W}_4$ commutes with the generators of 3-rotations, the eigenfunctions (in one of two subspaces) can be written in the form

$$\psi_{\uparrow,\lambda jm}(p,\theta,\varphi) = \frac{f_{\uparrow,\lambda j}(p)}{p}\,\phi_{jm}(\theta,\varphi) + \frac{h_{\uparrow,\lambda j}(p)}{p}\,\chi_{jm}(\theta,\varphi)\,, \tag{45}$$

where p is the radial momentum, $p^2=(p_i)^2=p_0^2-m^2$, and ϕ_{jm} and χ_{jm} are the spin spherical harmonics,

$$\phi_{jm}(\theta,\varphi) = \begin{pmatrix} \sqrt{\frac{j+m}{2j}}\;Y^{m-1/2}_{j-1/2}(\theta,\varphi) \\ \sqrt{\frac{j-m}{2j}}\;Y^{m+1/2}_{j-1/2}(\theta,\varphi) \end{pmatrix}, \quad \chi_{jm}(\theta,\varphi) = \begin{pmatrix} \sqrt{\frac{j+1-m}{2(j+1)}}\;Y^{m-1/2}_{j+1/2}(\theta,\varphi) \\ -\sqrt{\frac{j+1+m}{2(j+1)}}\;Y^{m+1/2}_{j+1/2}(\theta,\varphi) \end{pmatrix}. \tag{46}$$

The radial part of the eigenvalue equation $\mathcal{W}_4\,\Psi=\lambda\,\Psi$ has the associated Legendre functions as solutions, or more precisely,

$$f_{\uparrow,\lambda j} = A\left(\frac{p}{m}\right)^{\frac12} P^{-j}_{-2i\lambda}\left(\frac{p_0}{m}\right), \quad h_{\uparrow,\lambda j} = A\,(2i\lambda-j-1)\left(\frac{p}{m}\right)^{\frac12} P^{-j-1}_{-2i\lambda}\left(\frac{p_0}{m}\right). \tag{47}$$

Solutions are normalised to the δ-function,

$$\left(\psi_{\uparrow,\lambda jm},\psi_{\uparrow,\lambda' j'm'}\right) = 2A^*A'\,\frac{\Gamma(\frac12-2i\lambda)\,\Gamma(\frac12+2i\lambda')}{\Gamma(j+1-2i\lambda)\,\Gamma(j+1+2i\lambda')}\,\delta_{mm'}\,\delta_{jj'}\,\delta(\lambda-\lambda')\,, \tag{48}$$

so the spectrum of $\mathcal{W}_4$ is continuous.

Similar calculations can be done for the embedding time $\ell\mathcal{W}_0$ and the cosmological time τ. We find that, regarding coordinates, fuzzy de Sitter space looks very much like the usual commutative de Sitter as [9, 12],

∘ the spectrum of spatial coordinates x^i, x^4 is continuous, the real line;

∘ the spectrum of the embedding time x^0 is discrete, and the spectra of the cosmic and conformal times τ,η are continuous.

5.2 Energy

Another interesting observable is energy. In conformal field theory, energy $\mathcal{E}$ is often identified with dilation generator $\mathcal{M}_{04}$. We also have here

$$[i\mathcal{M}_{04}, \mathcal{W}_0 - \mathcal{W}_4] = \mathcal{W}_0 - \mathcal{W}_4 \,, \tag{49}$$

that is, $\mathcal{M}_{04}$ is canonically conjugate to cosmic time τ. Therefore we define

$$\mathcal{E} = \frac{\hbar}{\ell}\,\mathcal{M}_{04} = \frac{i\hbar}{l\sqrt{\zeta\Lambda}}\,\Pi_0 \,. \tag{50}$$

We will solve the energy eigenvalue equation in the simpler representation $(\rho, s = 0)$. On the subspace $\mathcal{H}_\uparrow$, $\mathcal{M}_{04}$ reduces to

$$M_{04,\uparrow} = M_{04} = p_0\left(\rho - \frac{3i}{2} - ip\,\frac{\partial}{\partial p}\right) \,, \tag{51}$$

and commutes with M_{ij}. We can thus separate angular variables in equation $M_{04,\uparrow}\,\psi_\uparrow = \lambda\,\psi_\uparrow$. Using the Ansatz

$$\psi_{\lambda lm,\uparrow}(\vec{p}) = \psi_{\lambda,\uparrow}(p)\,Y_l^m(\theta,\varphi) = \frac{f_{\lambda,\uparrow}(p)}{p}\,Y_l^m(\theta,\varphi)\,, \tag{52}$$

we obtain the radial equation

$$i\big(p_0^2 - 1\big)\,\frac{d\psi_{\lambda,\uparrow}}{dp_0} + \Big(\frac{3i}{2} - \rho\Big)p_0\,\psi_{\lambda,\uparrow} = -\lambda\,\psi_{\lambda,\uparrow}\,. \tag{53}$$

It has solution

$$\psi_{\lambda,\uparrow} = c_\lambda\,p^{-\frac{3}{2}-i\rho}\left(\frac{p_0 - 1}{p_0 + 1}\right)^{\frac{i\lambda}{2}} \,, \tag{54}$$

or, written in variable $z = \sqrt{\dfrac{p_0 - 1}{p_0 + 1}} \in (0,1)$,

$$f_{\lambda,\uparrow} = C_\lambda\,(1 - z^2)^{\frac{1}{2}+i\rho}\,z^{-\frac{1}{2}-i\rho+i\lambda} \,. \tag{55}$$

The radial equation is the same in the other subspace $\mathcal{H}_\downarrow$ with p_0 replaced by $-p_0$ or λ by $-\lambda$, i.e.

$$f_{\lambda,\downarrow}(p) = f_{-\lambda,\uparrow}(p)\,. \tag{56}$$

Radial solutions behave, upon integration, as plane waves in $\log z$. The eigenvalue λ is not restricted: $\lambda \in \mathbb{R}$, and $\mathcal{E}$ has a continuous spectrum. The eigenfunctions are normalized as

$$(\psi_\lambda, \psi_{\lambda'}) = (\psi_{\lambda,\uparrow}, \psi_{\lambda',\uparrow}) + (\psi_{\lambda,\downarrow}, \psi_{\lambda',\downarrow}) = (\psi_{\lambda,\uparrow}, \psi_{\lambda',\uparrow}) + (\psi_{-\lambda,\uparrow}, \psi_{-\lambda',\uparrow}) = \delta_{ll'}\,\delta_{mm'}\,\delta(\lambda' - \lambda)$$

for $C_\lambda = \sqrt{1/2\pi}$.

The form of the radial eigenfunctions in $(\rho, s = 1/2)$ representation is somewhat different, but the energy spectrum and normalization are the same.

5.3 Free scalar particles

Classical equation of motion for the scalar field f is in the curved spacetime given by

$$(\Delta + \mu^2 + \xi R)\, f = 0\,, \tag{57}$$

where μ is mass of the field and ξ is coupling to the curvature. Since the scalar curvature R is constant in de Sitter space, (57) has the form of the eigenvalue equation for the Laplacian,

$$(\Delta + M^2) f = 0, \quad \text{with} \quad M^2 = \mu^2 + \xi R\,. \tag{58}$$

On a Riemannian manifold, particular solutions to (57) constitute a basis to the Hilbert space of solutions $\mathcal{H}$ adapted for quantization: positive-energy solutions define the one-particle space of states that gives quantum-mechanical description of scalar particles.

In the noncommutative framework, classical equation of motion for the scalar field $f \in \mathcal{A}$,

$$[\Pi_0, [\Pi^0, f]] + [\Pi_i, [\Pi^i, f]] + M^2 f = 0\,, \tag{59}$$

differs from the quantum-mechanical equation for scalar particle described by wave function $\Psi \in \mathcal{H}$,

$$(\Pi_0\Pi^0 + \Pi_i\Pi^i)\Psi + M^2\Psi = 0\,. \tag{60}$$

Another important observation is that in fuzzy de Sitter space eigenstates of the Laplacian (36) do not have definite values of energy because the two observables are not compatible, $[\mathcal{E}, \Delta] \neq 0$. This obstructs direct interpretation of the positive-energy subspace as the space of one-particle excitations of the quantum field.

Fuzzy de Sitter Laplacian is block-diagonal

$$\Delta = \begin{pmatrix} \Delta_\uparrow & 0 \\ 0 & \Delta_\downarrow \end{pmatrix} . \tag{61}$$

We will determine its eigenvalues and the corresponding eigenstates. To simplify equations, we rescale $\Delta \to \zeta\Lambda\,\Delta$ and $\zeta\Lambda\, M^2 \to M^2$ and calculate in $(\rho, s = 0)$ representation. We have

$$\Delta_\uparrow = \rho^2 + \frac{1}{4} + \frac{1}{2}(1 + 2i\rho)p_0 + \big(2i\rho p_0 + 4p_0 + 2\big)\,\frac{\partial}{\partial p}\,p + \frac{1 + p_0}{1 - p_0}\,\vec{L}^2 - \frac{2p_0^2}{1 - p_0}\,p\,\frac{\partial}{\partial p}\,\frac{\partial}{\partial p}\,p\,, \tag{62}$$

and eigenfunctions of the form

$$\psi_{Mlm,\uparrow}(\vec{p}) = \frac{f_\uparrow(p)}{p}\,Y_l^m(\theta, \varphi)\,. \tag{63}$$

When rewritten in variable $z \in (0, 1)$, the corresponding radial equation becomes

$$(1 - z^2)\,\frac{d^2 f_\uparrow}{dz^2} + 2i\rho z\,\frac{df_\uparrow}{dz} + \left((\rho - \frac{i}{2})^2 - M^2 - \frac{l(l+1)}{z^2} + \frac{1 + 2i\rho}{1 - z^2}\right) f_\uparrow = 0\,. \tag{64}$$

It is a hypergeometric equation. Its physical solution, nonsingular at $z = 0$, is

$$f_{\uparrow,Ml} = C_{Ml}\, z^{l+1}\,(1-z^2)^{\frac{1-i\rho}{2}}\; F\left(\frac{3}{4}+\frac{l}{2}-\frac{i\rho}{2}-\frac{iM}{2}\,,\frac{3}{4}+\frac{l}{2}-\frac{i\rho}{2}+\frac{iM}{2}\,;\frac{3}{2}+l\,;z^2\right). \tag{65}$$

Extension of the solution to the other half of the Hilbert space $\mathcal{H}_\downarrow$ is obtained by continuation of $f_{\uparrow,Ml}$ to the interval $z\in(0,\infty)$, [9]. The scalar product becomes

$$(\Psi_{Mlm},\Psi_{M'l'm'}) = C^*C'\,\delta_{ll'}\,\delta_{mm'}\int_0^\infty \frac{dz}{|1-z^2|}\, f^*_{Ml}\, f'_{M'l'}\,. \tag{66}$$

Eigenfunctions (65) can be expressed in terms of the Jacobi functions,

$$f_{Ml} = f^*_{Ml} = C^*_{Ml}\, z^{l+1}\sqrt{|1-z^2|}\,(1-z^2)^{\frac{i\rho}{2}}\;\phi_\lambda^{(\alpha,\beta)}(t) \tag{67}$$

with $\alpha = l+\frac{1}{2}\,,\beta = i\rho,\ \lambda = M,\ i\sinh t = z$. Using the fact that Jacobi functions are continuously orthogonal [13], one can prove orthogonality of the eigenbasis and normalize appropriately, finding

$$C_{Ml} = \frac{\sqrt{2\pi}\;\Gamma\big(l+\frac{3}{2}\big)\,\Gamma\big(iM\big)}{\Gamma\Big(\frac{1}{2}\,(l+\frac{3}{2}+i\rho+iM)\Big)\,\Gamma\Big(\frac{1}{2}\,(l+\frac{3}{2}-i\rho+iM)\Big)}\,. \tag{68}$$

6. Future directions, conclusions

We found solutions to the curved Klein-Gordon equation and showed that they form a basis in representation space of fuzzy dS_4: these solutions describe the motion of free relativistic particles. We saw furthermore that energy of a particle is not conserved, which is to be expected in a nonstationary, expanding space. A number of further interesting questions can be considered, for example the free motion and dispersion of a wave packet, the geodesic motion, the energy spectra in typical potentials. More important and more difficult future problem is to study evolution of the scalar field, for example the evolution in potential typical for the inflatory phase of the universe. Results of this investigation will certainly have importance in applications to cosmology.

It is possible to construct noncommutative extensions of fuzzy anti-de Sitter spaces in dimensions higher than two as well. Indeed, the fuzzy AdS_4 can be defined by the same identifications of coordinates and momenta as fuzzy dS_4, just using the $SO(2,3)$ group. Thus on differential-geometric level we can easily obtain spacetime with negative cosmological constant, with symmetries realised (or broken) in a similar way as on fuzzy dS_4. However, the structure of spacetime 'itself' will not be the same: as in the case of dS_2/AdS_2, the unitary irreducible representations will differ. In fact, it is quite clear that the simple idea of Wick rotation, $x_4 \to ix_4$, does not work on the quantum level, as it changes the hermitian character of coordinates i.e. the unitarity. The set of relevant representations for the fuzzy AdS_4 will be different from those for dS_4. It will include singletons: in particular, it will be interesting to understand whether singleton representations have some special role in a noncommutative description of anti-de Sitter space.

Generalisation to odd dimensions is not straightforward, but possible. The first results for the fuzzy AdS_3 and the fuzzy BTZ black hole are obtained in [14].

To conclude: we hope that we have shown that applications of the noncommutative frame formalism can give interesting and viable models of quantum spaces. The geometry of these models is very well behaved, partly because the formalism was built in a close analogy to the classical one. A more difficult problem in this approach is to analyse behavior of classical and quantised fields. Although formalism defines the general framework for description of scalar, spinor and gauge fields (differential calculus, Laplace and Dirac operators, etc.), in concrete applications it can prove challenging. A nice analysis of quantisation of scalar field on the fuzzy sphere was done in [15]. However, in Lorentzian spaces one is dealing with infinite-dimensional representations, while natural symmetry-adapted bases are more complicated than the plane waves. But the latter is also the case in curved commutative spaces: we believe therefore that it is worth to put more effort in concrete calculations and obtain physical predictions of noncommutative geometry models.

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