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Getting hot without accelerating: vacuum thermal effects from conformal quantum mechanics

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The generators of radial conformal symmetries in Minkowski space-time can be put in correspondence with the generators of time evolution in conformal quantum mechanics. Within this correspondence it is shown that in conformal quantum mechanics the state corresponding to the inertial vacuum for a conformally invariant field in Minkowski spacetime has the structure of a thermofield double. The latter is built from a bipartite "vacuum state" corresponding to the ground state of the generators of hyperbolic time evolution, which cover only a portion of the time domain. When such generators are the ones of conformal Killing vectors mapping a causal diamond in itself and of dilations, the temperature of the thermofield double reproduces, respectively, the diamond temperature and the Milne temperature. This result indicates that, for conformally invariant fields, the fundamental ingredient at the basis vacuum thermal effects in flat-space time is the non-eternal nature of the lifetime of observers rather than their acceleration.

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1. Introduction

In quantum field theory the notion of vacuum state has no universal meaning. To put it in the words of P. C. W. Davies [1] "Before the 1970s nobody thought very much about "for whom" the vacuum state appears devoid of "stuff"...". The well known example in flat Minkowski space-time is that of the Poincaré invariant vacuum state for inertial observers which is perceived as a thermal state by uniformly accelerating observers. This fact is at the basis of the celebrated *Unruh effect* [2] (see [3] for an extensive review) according to which a uniformly accelerated detector in Minkowski space-time responds as if immersed in a thermal bath of particles at a temperature proportional to the magnitude of its four-acceleration.

Focusing on a non-interacting field, one can trace the origin of the ambiguity in the definition of the vacuum state to the existence of different possible choices of time-like Killing vectors which one can use to decompose the solutions of the equation of motion of the field into positive and negative frequency components [4]. The positive frequency subspace of solutions defines the one-particle Hilbert space of the quantum field from which one can construct the Fock space, the full multi-particle Hilbert space of the theory [5]. In this picture the vacuum state is the element of the Fock space which does not contain particles seen as positive frequency excitations of the field and, as such, it depends on the notion of time evolution we choose. In the case of the Unruh effect the time evolution for inertial observers is determined by the time translation Killing vector of the Poincaré algebra while time evolution for uniformly accelerated observers is generated by the boost Killing vector. The latter is time-like only within the left and right Rindler wedges and becomes null on their boundaries given by the light-cone passing through the origin. Such light-cone acts as a causal horizon for uniformly accelerating observers within each wedge.

The key observation which is at the basis of the analysis we present in this contribution is that for conformally invariant fields the range of choices for generators of time evolution extends to *time-like conformal Killing vectors*. For example one could use the generator of dilations in Minkowski space-time to quantize a field according to the Milne time-evolution, the time evolution associated to a hyperbolic slicing of the future-cone of Minkowski space-time: the Milne universe (see [6–8]). Another instance in which quantization using a notion of time evolution determined by a time-like conformal Killing vector is found when considering uniformly accelerated observers with a finite lifetime [9]. The trajectories of such observers are confined to a region of Minkowski space-time known as the causal diamond, the intersection of a past and future light-cone, related to a Rindler wedge by a conformal map [10]. It has been suggested that, in analogy with accelerated observers in a Rindler wedge, for such "diamond observers" the inertial Minkowski vacuum state should appear as a thermal state at a temperature related to the size of the causal diamond [9, 11].

In this contribution we provide further evidence for the fact that the inertial vacuum should appear as a thermal state for both Milne and diamond observers by exploiting a correspondence between radial conformal Killing vectors in Minkowski space-time and the generators of time evolution in conformal quantum mechanics, a $0+1$-dimensional conformal field theory [12, 19]. We will show that in such simple one-dimensional model one can construct states which are the analogue of the vacuum states for inertial observers (which have access to the whole geometrical domain of the theory) and for observers whose time evolution is determined by a conformal Killing vector whose orbits cover a finite or semi-infinite region of the (space)-time (diamond and Milne observers, respectively). For such non-eternal observers the inertial vacuum is a thermal state at the diamond and Milne temperature respectively [13, 14]. The results we present provide a unified group-theoretical description of such temperatures and show that the essential ingredient the basis of these "vacuum thermal effects" is the existence of boundaries for the causal domain of the observers rather than their acceleration.

2. Radial conformal symmetries in Minkowski space-time

We start by describing the radial conformal Killing vectors of Minkowski space-time. Such vectors were fully classified in [15]. The Minkowski metric in spherical coordinates is given by

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\,. \tag{1}$$

Let us consider a generic radial vector field

$$\xi = A(t,r,\theta,\phi)\,\partial_t + B(t,r,\theta,\phi)\,\partial_r \tag{2}$$

and impose that it is a conformal Killing vector i.e. that

$$\mathcal{L}_\xi \eta_{\mu\nu} \propto \eta_{\mu\nu} \tag{3}$$

where $\mathcal{L}_\xi$ denotes the Lie derivative and $\eta_{\mu\nu}$ is the Minkowski metric. Such condition implies [15] that ξ is independent of θ and ϕ and it has the general form

$$\xi = \left(a(t^2+r^2) + bt + c\right)\,\partial_t + r(2at+b)\,\partial_r \tag{4}$$

with a, b, c real constants. A key observation is that this conformal Killing vector can be written as

$$\xi = aK_0 + bD_0 + cP_0\,, \tag{5}$$

where P_0, D_0 and K_0 generate, respectively, time translations, dilations and special conformal transformations

$$P_0 = \partial_t\,, \qquad D_0 = r\,\partial_r + t\,\partial_t\,, \qquad K_0 = 2tr\,\partial_r + (t^2+r^2)\,\partial_t\,. \tag{6}$$

Their commutators close the $\mathfrak{sl}(2,\mathbb{R})$ Lie algebra

$$[P_0, D_0] = P_0\,, \qquad [K_0, D_0] = -K_0\,, \qquad [P_0, K_0] = 2D_0\,. \tag{7}$$

One can define three different families of conformal Killing vectors according to the sign of the determinat $\Delta = b^2 - 4ac$

- For $\Delta < 0$ we have generators of *elliptic transformations*; thinking of the relationship between $\mathfrak{sl}(2,\mathbb{R})$ and the Lie algebra of the three-dimensional Lorentz group $\mathfrak{so}(2,1)$ these correspond to generators of rotations. A representative element of this class is

$$R_0 = \frac{1}{2}\left(\alpha P_0 + \frac{K_0}{\alpha}\right)\,. \tag{8}$$

  Notice the introduction of the constant α with dimensions of length needed for dimensional reasons: it will play a crucial role throughout our analysis.

- For $\Delta = 0$ we have generators of *parabolic transformations* which in Lorentz group language correspond to null rotations. Representatives of this class are P_0 and K_0.
- For $\Delta > 0$ we have generators of *hyperbolic transformation* corresponding to generators of Lorentz boosts in terms of Lorentz transformations. Representatives of this class are D_0 and the combination
$$S_0 = \frac{1}{2}\left(\alpha P_0 - \frac{K_0}{\alpha}\right) . \tag{9}$$

Notice that while the Killing vectors corresponding to generators of elliptic transformations are always space-like, the ones corresponding to to parabolic and hyperbolic transformations can be time-like. In particular we have that P_0 is everywhere timelike and generates inertial time evolution. The generator of dilations D_0 is time-like within a light-cone centered at the origin and, in particular, in the future light cone it generates conformal time evolution in the Milne universe i.e. Minkowski space-time written in hyperbolic slicing coordinates
$$ds^2 = -d\bar{t}^2 + \bar{t}^2\left(d\chi^2 + \sinh\chi^2 d\Omega^2\right) \tag{10}$$
with $t = \bar{t}\cosh\chi$ and $r = \bar{t}\sinh\chi$ [8]. Like-wise the Killing vector S_0 is time-like in various regions of Minkowski space-time [15, 16] and,in particular, it maps a causal diamond of radius α into itself [17]. Within the diamond it can be seen as the generator of time evolution of accelerated observers with a finite life-time, the *diamond time*. Worldlines of such observers are orbits of the transformation generated by S_0 [18].

Our starting point is the observation that along $r = \text{const}$ worldlines and on the light cones $u = t - r = \text{const}$, $v = t + r = \text{const}$ the conformal Killing vector (4) can be written as
$$\xi = \left(a\,\tau^2 + b\,\tau + c\right)\partial_\tau \tag{11}$$
where τ is either t for observers at $r = 0$ or u or v. Written in this form ξ coincides with the generator of *conformal transformations of the real (time) line*. In particular $P_0 = \partial_\tau$ generates translations in "inertial time" τ covering the entire time line. The dilation Killing vector $D_0 = \tau\partial_\tau$ generates translation in "Milne time" ν defined by
$$D_0 = \alpha\partial_\nu . \tag{12}$$
One can easily derive that
$$\tau = \pm 2\alpha\ \exp\frac{\nu}{\alpha} \tag{13}$$
and thus the Milne time covers only half of of the time line (the regions $\tau > 0$ or $\tau < 0$). Finally the conformal Killing vector $S_0 = \frac{1}{2\alpha}\left(\alpha^2 - \tau^2\right)\partial_\tau$ generates translation in "diamond time" σ such that
$$S_0 = \alpha\partial_\sigma . \tag{14}$$
This time variable is related to inertial time by
$$\tau = \alpha\ \tanh\sigma/2\alpha$$
and thus it covers only the region $|\tau| < \alpha$ of the time line: the analogue of the diamond in Minkowski space-time. As it turns out these three types of time evolution are precisely the ones allowed in conformal quantum mechanics as we discuss in the following section.

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3. Conformal quantum mechanics

Conformal quantum mechanics can be seen as an $SL(2,\mathbb{R})$-invariant 0+1-dimensional quantum field theory [12]. One way to describe the model is through the conformally invariant Lagrangian

$$\mathcal{L} = \frac{1}{2}\left(\dot{q}(t)^2 + \frac{g}{q(t)^2}\right), \tag{15}$$

where $g > 0$ is a dimensionless coupling constant. The generators of conformal transformations belonging to the $\mathfrak{sl}(2,\mathbb{R})$ Lie algebra can be canonically realized as

$$H = iP_0 = \frac{1}{2}\left(p^2 + \frac{g}{q^2}\right) \tag{16}$$

$$D = iD_0 = t\,H - \frac{1}{4}\,(pq + qp) \tag{17}$$

$$K = iK_0 = -t^2\,H + 2t\,D + \frac{1}{2}q^2 \tag{18}$$

and the most general generator of time evolution for the model is given by

$$G = i\,(a\,K_0 + b\,D_0 + c\,P_0)$$

formally identical to $i\xi$ with ξ given by (11). The conformal quantum mechanics model of [12] can be interpreted as one dimensional conformal quantum field theory CFT_1 [19, 20]. The two-point functions of such theory are built from the kets $|\tau\rangle$ labelled by the time variable τ first introduced in [12] on which the Hamiltonian acts as a derivative

$$H\,|\tau\rangle = -i\,\partial_\tau\,|\tau\rangle\,.$$

To describe such kets one starts from the irreducible representations of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$. These can be constructed introducing ladder operators

$$L_\pm = \frac{1}{2}\left(\frac{K}{\alpha} - \alpha\,H\right) \pm i\,D\,, \qquad L_0 = \frac{1}{2}\left(\frac{K}{\alpha} + \alpha\,H\right) \tag{19}$$

whose commutators are given by

$$[L_-, L_+] = 2L_0\,, \quad [L_0, L_\pm] = \pm L_\pm\,. \tag{20}$$

Irreducible representations are given by kets $|n\rangle$ such that

$$L_0\,|n\rangle = r_n\,|n\rangle\,, \qquad r_n = r_0 + n\,, \qquad r_0 > 0\,, n = 0, 1\ldots \tag{21}$$

The constant r_0 characterizes the representations and is related to the eigenvalue of the Casimir operator on the kets

$$C\,|n\rangle = \left(\frac{1}{2}\,(KH + HK) - D^2\right)\,|n\rangle = r_0(r_0 - 1)\,|n\rangle\,. \tag{22}$$

The action of the raising and lowering operators $L_\pm$ is given by

$$L_\pm\,|n\rangle = \sqrt{r_n\,(r_n \pm 1) - r_0\,(r_0 - 1)}\,|n \pm 1\rangle\,. \tag{23}$$

The $|\tau\rangle$ kets can be characterized by their overlap with $|n\rangle$ states given by

$$\langle\tau|n\rangle = (-1)^n \left[\frac{\Gamma(2r_0+n)}{n!}\right]^{\frac{1}{2}} \left(\frac{\alpha - i\tau}{\alpha + i\tau}\right)^{r_n} \left(1+\frac{\tau^2}{\alpha^2}\right)^{-r_0}, \tag{24}$$

which can be used to derive the inner product

$$\langle\tau_1|\tau_2\rangle = \frac{\Gamma(2r_0)\,\alpha^{2r_0}}{[2i\,(\tau_1-\tau_2)]^{2r_0}}\,. \tag{25}$$

As shown in [19] such inner product can be interpreted as the two-point function of the CFT_1

$$G_2(\tau_1,\tau_2) \equiv \langle\tau_1|\tau_2\rangle\,. \tag{26}$$

Notice how for $r_0 = 1$ such two-point function coincides with that of a massless scalar field in Minkowski space-time, evaluated along the worldline of an inertial observer sitting at the origin. In other words two-point correlators for a massless field along the worldline of static observers in Minkowski space-time are in correspondence with two-point functions of conformal quantum mechanics for the states $|\tau\rangle$. This is reminiscent of the $SL(2,\mathbb{R})$-invariant *wordline quantum mechanics* for static-patch observers in de Sitter space-time [21]. As shown in [19, 20] one can re-write the CFT_1 two-point function as

$$G_2(\tau_1,\tau_2) \equiv \langle\tau_1|\tau_2\rangle = \langle\tau=0|e^{-i(\tau_1-\tau_2)H}|\tau=0\rangle \tag{27}$$

where

$$|\tau=0\rangle = \exp(-L_+)|n=0\rangle \tag{28}$$

(from now on we restrict to the case $r_0 = 1$). We now come to a crucial observation made in [14] which is central for what follows. It is well known from quantum optics (see e.g. [22]) that the generators $L_\pm$ and L_0 can be realized in terms of pairs of creation and annihilation operators for simple harmonic oscillators, which we denote by L and R subscripts,

$$L_+ = a_L^\dagger a_R^\dagger\,,\quad L_- = a_L a_R\,,\quad L_0 = \frac{1}{2}\left(a_L^\dagger a_L + a_R^\dagger a_R + 1\right) \tag{29}$$

and thus, from (28),

$$|\tau=0\rangle = \exp\left[-a_L^\dagger a_R^\dagger\right]|n=0\rangle\,. \tag{30}$$

This shows that the ground state $|n=0\rangle$ has a *bipartite structure*

$$|n=0\rangle = |0\rangle_L \otimes |0\rangle_R \tag{31}$$

in terms of the ground states of the two harmonic oscillators $|0\rangle_L$ and $|0\rangle_R$. In what follows we will explore the interpretation of such bipartite structure in terms of the analogue of the bipartite structure of the quantum field vacuum states associated to observers whose causal domain covers only a portion of Minkowski space-time.

4. Vacuum states and thermal effects in CFT_1

We start by observing that the Lie algebra

$$[L_-, L_+] = 2L_0\,, \quad [L_0, L_\pm] = \pm L_\pm \tag{32}$$

can be realized via another combination of H, D and K, namely

$$L_0 = iS\,, \qquad L_+ = \frac{1}{2}(D - R)\,, \qquad L_- = 2(D + R) \tag{33}$$

This suggests that in CFT_1 we can identify two "vacuum-like" states:

- the state $|n = 0\rangle$, the ground state of the generator of diamond time evolution S, which is the analogue of the "Boulware vacuum", the vacuum state for generators with a bounded causal domain in Minkowski space-time, in this case observers restricted to a causal diamond;
- the state $|\tau = 0\rangle$ the "inertial vacuum" from which we build the two point function acting with the $0 + 1$-dimensional analogues of the field operator

  $$G_2(\tau_1, \tau_2) = \langle \tau = 0| e^{-i(\tau_1 - \tau_2)H} |\tau = 0\rangle\,,$$

  which can be seen as the analogue of the "Hartle-Hawking vacuum", the vacuum state for observers whose causal domain covers the entire space-time.

As in the "real world" case of four-dimensional Minkowski space-time the Hartle-Hawking vacuum we identified above is a thermofield double state built on its bipartite Boulware counterpart as we will now show. With simple manipulations from (30) we can write the inertial vacuum as

$$\begin{aligned} |\tau = 0\rangle &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(a_L^\dagger a_R^\dagger\right)^n |0\rangle_L \otimes |0\rangle_R = \sum_{n=0}^{\infty} (-1)^n |n\rangle_L \otimes |n\rangle_R \\ &= -\sum_{n=0}^{\infty} e^{i\pi L_0} |n\rangle_L \otimes |n\rangle_R \end{aligned} \tag{34}$$

and thus

$$|\tau = 0\rangle = -\sum_{n=0}^{\infty} e^{-\pi S} |n\rangle_L \otimes |n\rangle_R\,. \tag{35}$$

Before proceeding let us briefly recall the definition of a thermfofield double state (see e.g. [23] for more details). Let us consider the set of eigenstates of the Hamiltonian of a generic quantum system

$$H\,|k\rangle = E_k\,|k\rangle\,. \tag{36}$$

The *thermofield double state* is built by "doubling" the system's degrees of freedom and it consists of the superposition

$$|TFD\rangle = \frac{1}{Z(\beta)} \sum_{k=0}^{\infty} e^{-\beta E_k/2} |k\rangle_L \otimes |k\rangle_R\,, \tag{37}$$

where $Z(\beta) = \sum_{k=0}^{\infty} e^{-\beta E_k}$ is the partition function at inverse temperature β. The peculiarity of such systems is that it is a highly entangled pure state which, tracing over the degrees of freedom of one copy of the system, one obtains a thermal density matrix

$$Tr_L(|TFD\rangle\langle TFD|) = \frac{e^{-\beta H}}{Z(\beta)}$$

at a temperature $T = 1/\beta$. Going back to our model we see that the inertial vacuum (35) is formally[1] a thermofield double state at a temperature

$$T_S = \frac{1}{2\pi\alpha} \tag{38}$$

for the Hamiltonian S/α which generates diamond time evolution. The temperature T_S is precisely the diamond temperature perceived by observers sitting at the origin within a causal diamond in Minkwoski space-time, with worldlines given by orbits of the conformal Killing vector (9), whose existence was suggested in [9] and [11].

As discussed in Section 2, the generators S and D belong to the same class of Hamiltonians determining hyperbolic time evolution. We can thus find a $SL(2,\mathbb{R})$ transformation on the time axis which maps one generator into another [14]. Such map $\tau \longrightarrow \tau'$ is easily found by requiring that

$$S(\tau) \equiv D(\tau') \tag{39}$$

and is given by

$$\tau' = -2\alpha\,\frac{\tau+\alpha}{\tau-\alpha}\,. \tag{40}$$

Notice how such map coincides with the map from the causal diamond to the Rindler wedge in light-cone coordinates in Minkowski space-time used to derive the diamond modular Hamiltonian from the Rindler one [24]. Under such map we have the following identification of the ladder operators

$$L_0 = iD\,, \qquad L_+ = -\alpha H\,, \qquad L_- = \frac{K}{\alpha}\,. \tag{41}$$

Under such identification the ground state $|n = 0\rangle$ is seen as the CFT_1 analogue of the vacuum state associated to the generator of Milne time evolution D. Thus the inertial vacuum $|\tau = 0\rangle$ with such identification matches the thermofield double state for the Hamiltonian D/α at the Milne temperature

$$T_D = \frac{1}{2\pi\alpha} \tag{42}$$

as described e.g. in [6]. Finally we should stress that, as shown in [15], observers whose worldlines are integral curves of a time-like radial conformal Killing vector of the form

$$\xi = aK_0 + bD_0 + cP_0 \tag{43}$$

[1]The τ-vacuum (35) is not normalizable since the two-point function (25) is divergent in the limit of coincident points. To make the correspondence precise one should regolarize the expression by introducing an appropriate normalization factor. For our purposes it is sufficient to show that the superposition of states appearing in (35) is the same of that of a thermofield double.

are accelerated, with modulus of the four-acceleration given by

$$|\mathbf{a}| = \frac{2|a|}{\sqrt{\omega - \Delta}} \tag{44}$$

where $\Delta = b^2 - 4ac$ and

$$\omega = \frac{a(t^2 - a^2) + bt + c}{r} \,. \tag{45}$$

We see that for integral curves of the generator of dilations D, i.e. worldlines of Milne observers, $a = c = 0$ and thus the modulus of the four-acceleration vanishes $|\mathbf{a}| = 0$. The same holds for diamond observers sitting at the origin $r = 0$. Thus the correspondence between the generators of time evolution for such observers and the ones of conformal quantum mechanics supports the view that these observers experience the inertial vacuum as a thermal bath at temperatures (42) and (38), respectively, even though they have vanishing acceleration. This suggests that certain classes of observers can experience the inertial vacuum as a thermal state, as long as the field theoretic content of the theory is conformally invariant and thus we are allowed to consider time-like conformal Killing vectors as generators of time evolution.

5. Summary

In this contribution it is argued that a one-dimensional conformal field theory (conformal quantum mechanics), despite its simplicity, is rich enough to reproduce the basic features which lead to vacuum thermal effects in space-times in which different classes of observers disagree on the particle content of the theory and on the notion of vacuum state. As in quantum field theoretic models such ambiguity in the choice of vacuum state is related to the freedom in the choice of time evolution. Our analysis provides further evidence for the existence of diamond and Milne temperatures whose existence has been discussed in scattered works in the literature and only for 1+1-d Minkwoski space-time. The correspondence between radial conformal flows in Minkowski space-time and time evolution in conformal quantum mechanics which we exploited in our analysis provides a group-theoretical ground for the existence of such temperatures and show how they are intimately related. Moreover our results show that the inertial vacuum appears as a thermal state for observers whose *time evolution is not eternal* despite of whether they are accelerating or not. Finally our analysis suggests that thermodynamic properties of the Milne "patch" of Minkowski space-time and of causal diamonds are deeply connected and we hope this might provide a new powerful tool for studying entanglement properties of quantum fields across regions of Minkowski and other maximally symmetric space-times [18].

References

[1] P. C. W. Davies, In *Christensen, S.m. (Ed.): Quantum Theory Of Gravity*, 66-77 (1984)

[2] W. G. Unruh, Phys. Rev. D **14**, 870 (1976).

[3] L. C. B. Crispino, A. Higuchi and G. E. A. Matsas, Rev. Mod. Phys. **80**, 787-838 (2008) [arXiv:0710.5373 [gr-qc]].

[4] A. Ashtekar and A. Magnon, Proc. Roy. Soc. Lond. A **346**, 375 (1975).

[5] R. M. Wald, "Quantum Field Theory in Curved Space-Time and Black Hole Thermodynamics," Chicago Lectures in Physics (1995)

[6] S. J. Olson and T. C. Ralph, Phys. Rev. Lett. **106**, 110404 (2011) [arXiv:1003.0720 [quant-ph]].

[7] A. Higuchi, S. Iso, K. Ueda and K. Yamamoto, Phys. Rev. D **96**, no.8, 083531 (2017) [arXiv:1709.05757 [hep-th]].

[8] R. M. Wald, Phys. Rev. D **100**, no.6, 065019 (2019) [arXiv:1908.06363 [gr-qc]].

[9] P. Martinetti and C. Rovelli, Class. Quant. Grav. **20**, 4919-4932 (2003) [arXiv:gr-qc/0212074 [gr-qc]].

[10] P. D. Hislop and R. Longo, Commun. Math. Phys. **84**, 71 (1982)

[11] D. Su and T. C. Ralph, Phys. Rev. D **93**, no.4, 044023 (2016) [arXiv:1507.00423 [quant-ph]].

[12] V. de Alfaro, S. Fubini and G. Furlan, Nuovo Cim. A **34**, 569 (1976)

[13] M. Arzano, JHEP **05**, 072 (2020) [arXiv:2002.01836 [gr-qc]].

[14] M. Arzano, JHEP **07**, 003 (2021) [arXiv:2103.07228 [hep-th]].

[15] A. Herrero and J. A. Morales, J. Math. Phys. **40**, 3499 (1999)

[16] T. De Lorenzo and A. Perez, Phys. Rev. D **97**, no. 4, 044052 (2018) [arXiv:1707.00479 [gr-qc]].

[17] T. Jacobson, Phys. Rev. Lett. **116**, no.20, 201101 (2016) [arXiv:1505.04753 [gr-qc]].

[18] T. Jacobson and M. Visser, SciPost Phys. **7**, no.6, 079 (2019) [arXiv:1812.01596 [hep-th]].

[19] C. Chamon, R. Jackiw, S. Y. Pi and L. Santos, Phys. Lett. B **701**, 503-507 (2011) [arXiv:1106.0726 [hep-th]].

[20] R. Jackiw and S.-Y. Pi, Phys. Rev. D **86**, 045017 (2012) Erratum: [Phys. Rev. D **86**, 089905 (2012)] [arXiv:1205.0443 [hep-th]].

[21] D. Anninos, S. A. Hartnoll and D. M. Hofman, Class. Quant. Grav. **29**, 075002 (2012) [arXiv:1109.4942 [hep-th]].

[22] M. Ban, J. Math. Phys. **33**, 3213-3228 (1992)

[23] S. Chapman, J. Eisert, L. Hackl, M. P. Heller, R. Jefferson, H. Marrochio and R. C. Myers, SciPost Phys. **6**, no.3, 034 (2019) [arXiv:1810.05151 [hep-th]].

[24] H. Casini, M. Huerta and R. C. Myers, JHEP **05**, 036 (2011) [arXiv:1102.0440 [hep-th]].