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Integrability of the modular vector field and quantization

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We present an approach to the problem of quantization of Poisson manifolds based on Poisson-Nijenhuis (PN) structures of symplectic type. This geometry describes the integrability of the flux of the modular vector fields of all Poisson structures appearing in the PN hierarchy. The integrable models can be lifted to *multiplicative* integrable models on their symplectic groupoid and are regarded as a singular real polarization. The output of the construction is the convolution algebra of the groupoid of Bohr-Sommerfeld leaves; such algebra takes into accout the topology of the space of symplectic leaves. We sketch here the main ingredients of the construction and discuss as an example the case of a PN structure on $\mathbb{S}^2$, the simplest case of a class of PN structures on compact hermitian symmetric spaces.

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1. Introduction

The mathematical formalization of the quantization procedure at the basis of quantum mechanics is one century old and has a multitude of incarnations. The simplest one is *deformation quantization*. A deformation quantization of a Poisson manifold (M, π) is an associative $\star_\hbar$-product defined on $C^\infty(M)[[\hbar]]$ that deforms the commutative product of functions and recovers the Poisson bracket in the semiclassical limit. Since the expansion is formal and no notion of topology is required, this is regarded as a formal quantization. The problem of finding such $\star_\hbar$-product for every Poisson manifold has been solved by Kontsevich formula [14]. A quantum field theoretical interpretation has been given in [6]. Much more elusive is the problem of non formal quantization, *i.e.* a procedure that associates to the classical observables operators on an Hilbert space. There are many different approaches, here we are interested in those having a geometrical flavour.

Geometric quantization generalizes the construction of canonical quantization to a symplectic manifold (M, ω) (see [19]). The first step is *prequantization*: provided ω takes integer values when evaluated on closed 2-cycles, one can consider *the prequantization line bundle*, *i.e.* a line bundle $L \to M$ endowed with a connection ∇ whose curvature is ω. The second step is *polarization*: one has to choose a lagrangian distribution $F \subset T_{\mathbb{C}}M$ and define the Hilbert space as the space of covariantly constant sections of L. In order this procedure to work, several regularity assumptions are required to F and suitable polarizations exist in general only for *cotangent manifolds* and for *Kahler manifolds*.

In the case of Poisson manifolds a straightforward generalization of geometric quantization is not satisfactory, since it misses the Casimirs. Weinstein and Karasev indipendently introduced the notion of *symplectic groupoid* having in mind the quantization problem (see [21],[12]). A symplectic groupoid is a Lie groupoid $\mathcal{G} \rightrightarrows M$ with a compatible symplectic form $\Omega_{\mathcal{G}}$. Since the space of units M naturally inherits a Poisson bracket, the symplectic groupoid is said to *integrate* the underlying Poisson manifold. Compatibility between the groupoid structure and $\Omega_{\mathcal{G}}$ can be stated by saying that the graph of the multiplication is Lagrangian inside $\mathcal{G} \times \mathcal{G} \times \bar{\mathcal{G}}$, where the bar indicates that the last factor is taken with opposite symplectic form. Quantization of $\mathcal{G}$ must take into account the compatibility with the groupoid structure so that the space of states carries the structure of associative algebra, regarded as the algebra of operators quantizing the underlying Poisson manifold. If the quantization scheme is geometric quantization, this compatibility amounts to use compatible prequantization and *multiplicative polarizations* (see [11]). In particular, the prototype example of polarization is given by a real polarization F such that the space $\mathcal{L}_F = \mathcal{G}/F$ of lagrangian leaves inherits a Lie groupoid structure. In this case, the output of the quantization is the convolution algebra $\mathcal{A}_{\mathcal{F}}$ of $\mathcal{L}_F$. The request of compatibility makes the task of finding polarizations for sufficiently wide classes of examples even harder.

In this note we review a variation to this approach that allows to consider also very singular polarizations whose space of leaves $\mathcal{L}_F$ is just a topological groupoid. The basic observation is that, if $\mathcal{L}_F$ allows an Haar system, it is still possible to define the convolution algebra $\mathcal{A}_{\mathcal{F}}$. We consider in particular the case where F is defined by a *multiplicative integrable model*. A particular rich source of examples can be found in *Poisson-Nijenhuis manifolds of symplectic type*, an example of bihamiltonian geometry. In this case, the integrable flux is given by the *modular vector field*, an intrinsic dynamical system on M defined by the Poisson brackets. The modular vector field can be lifted to the *modular cocycle* of the symplectic groupoid $\mathcal{G}$ together with the hamiltonians in involution, giving rise to an integrable model on $\mathcal{G}$ that is compatible with the groupoid structure.

Although this approach cannot be seen as a general one, there are indeed very interesting examples. Indeed, a class of PN structures defined on compact symmetric spaces was introduced in [13]. Here the degenerate Poisson structure is the *Bruhat-Poisson* structure, that is the semiclassical limit of homeogeneous spaces of quantum groups. The properties of the integrable models have been studied in [4] for the classical cases and in [5] for the exceptional cases. In particular, in the case of Grassmannians the integrable model corresponds to the much studied *Gelfand-Tsetlin system*. The quantization program has been discussed for $\mathbb{C}P_n$ in [2], where we recover as convolution algebras the groupoid C^*-algebras described in [18] for the quantum projective spaces. The quantization of Bruhat-Poisson for Grassmannians is currently under investigation.

We sketch all the ingredients needed for the construction without giving proofs, that can be found in the literature. Only exception is Proposition 5.1 where the description of the lift of the integrable model differs from that of [1]. Another novelty is the explicit description of the groupoid of lagrangian leaves for all the Poisson structures of the hierarchy.

2. Poisson manifolds

Let (M,π) be a Poisson manifold with Poisson tensor given by $\pi\in\Gamma(\wedge^2 TM)$; we denote with $\{,\}_\pi$ the Poisson bracket defined by π. The Jacobi identity of $\{,\}_\pi$ is encoded in the relation

$$[\pi,\pi]=0\,, \tag{1}$$

where $[,]$ denotes the Schouten bracket among multivector fields $\Gamma(\wedge TM)$. We can extend the antisymmetric bracket to $\Omega^1(M)$ as

$$\{\alpha,\beta\}_\pi = L_{\pi(\alpha)}(\beta)-L_{\pi(\beta)}(\alpha)-d\langle\pi,\alpha\wedge\beta\rangle \quad \alpha,\beta\in\Omega^1(M)\,, \tag{2}$$

where we denote with the same symbol the bivector π and the antisymmetric map $\pi: T^*M\to TM$. This antisymmetric bracket satisfies the Jacobi identity if and only if π satisfies (1).

The complex $(\Gamma(\wedge^\bullet TM),d_\pi)$, where $d_\pi=[\pi,-]$ squares to zero as a consequence of (1), computes the *Lichnerowicz-Poisson cohomology* $H^\bullet_{LP}(M,\pi)$. For any $f\in C^\infty(M)$, $d_\pi(f)=v_f$, where $v_f=\pi(df)$ denotes the *hamiltonian vector field* of f. Moreover, a vector field $v\in\Gamma(TM)$ satisfies $d_\pi(v)=0$ if it is a derivation of the bracket $\{,\}$; in this case it is called a *Poisson vector field*. We then see that $H^1_{LP}(M,\pi)$ is the space of Poisson modulo hamiltonian vector fields

2.1 The modular class

Let us fix a volume form $\nu\in\Gamma(\wedge^{top}T^*M)$. For each $f\in C^\infty(M)$, since ν is a volume form, we compute

$$L_{v_f}(\nu)=\chi_\nu(f)\nu$$

for some $\chi_\nu(f)\in C^\infty(M)$. It is immediate to see that $\chi_\nu(fg)=f\chi_\nu(g)+g\chi_\nu(f)$, *i.e.* χ_ν is a vector field, that we call the *modular vector field* with respect to the volume form ν.

The modular vector field is a Poisson vector field, *i.e.* it is a derivation of the Poisson bracket. Moreover, if we change the volume form to $\nu'=e^\sigma\nu$ for some $\sigma\in C^\infty(M)$ then the modular vector field changes as $\chi_{\nu'} = \chi_\nu + v_\sigma$. The class $[\chi_\nu] \in H^1_{LP}(M,\pi)$ is then independent on the choice of the volume form. We call this class the *modular class*. By construction the modular class vanishes if and only if there exists a volume form that is invariant with respect to the hamiltonian flow. In this case (M,π) is said to be *unimodular*.

In the non degenerate case, *i.e.* when π is the inverse of a symplectic form $\omega \in \Omega^2(M)$, the symplectic volume $\nu = \omega^k$ ($2k = \dim M$) is invariant under the hamiltonian flux: symplectic manifolds are unimodular.

2.2 The symplectic groupoid

A symplectic groupoid is a couple $(\mathcal{G},\Omega_{\mathcal{G}})$ where $\mathcal{G} \rightrightarrows M$ is a Lie groupoid and $\Omega_{\mathcal{G}}$ is a symplectic form on $\mathcal{G}$ such that the graph of the multiplication graph$(\mathcal{G}) \subset \mathcal{G}\times\mathcal{G}\times\bar{\mathcal{G}}$ is Lagrangian (where the bar on the third factor means that the symplectic form is $\Omega_{\mathcal{G}}\oplus\Omega_{\mathcal{G}}\ominus\Omega_{\mathcal{G}}$). We will denote with s,t the source and target maps. We will denote with $\mathcal{G}_{(k)} = \{(\gamma_1\ldots\gamma_k) \in \mathcal{G}\times\ldots\times\mathcal{G},\ t(\gamma_i) = s(\gamma_{i+1})\}$ the space of k-composable arrows. The face maps are $d_i : \mathcal{G}_{(k)} \to \mathcal{G}_{(k-1)}$, $i = 0,\ldots k$, defined for $k > 1$ as

$$d_i(\gamma_1,\ldots\gamma_k) = \begin{cases} (\gamma_2,\ldots\gamma_k) & i = 0 \\ (\gamma_1,\ldots\gamma_i\gamma_{i+1}\ldots) & 0 < i < k \\ (\gamma_1,\ldots\gamma_{k-1}) & i = k \end{cases} \tag{3}$$

and for $k = 1$ as $d_0(\gamma) = s(\gamma)$, $d_1(\gamma) = t(\gamma)$. The simplicial coboundary operator $\partial^* : \Omega^r(\mathcal{G}_{(k)}) \to \Omega^r(\mathcal{G}_{(k+1)})$ is defined as

$$\partial^*(\omega) = \sum_{i=0}^{s}(-)^i d_i^*(\omega)\ ,$$

and $\partial^{*2} = 0$. The cohomology of this complex for $r = 0$ is the real valued groupoid cohomology; k-cocycles are denoted as $Z^k(\mathcal{G},\mathbb{R})$.

As a consequence of compatibility, the manifold of units M inherits a Poisson structure. If a Poisson manifold (M,π) is the space of units of some symplectic groupoid then it is said to be *integrable*. There are obstructions to the integrability (see [8]), but here we assume that all Poisson structures are integrable; in this case, there exists a unique source simply connected (ssc) symplectic groupoid $\mathcal{G}(M,\pi)$.

In [7], $\mathcal{G}(M,\pi)$ has been constructed as the symplectic reduction of the phase space of the Poisson sigma model with target (M,π). Let $\{x^\mu\}$ denote a set of coordinates on a chart of M. A cotangent path is a bundle map $(X,\eta) : T[0,1] \to T^*M$ satisfying

$$\dot{X}^\mu + \pi^{\mu\nu}\eta_\nu = 0\ .$$

Let us call C the space of cotangent paths. For each $\beta \in X^*(T^*M)$ with $\beta(0) = \beta(1) = 0$, let us consider the vector field δ_β on C defined as

$$\delta_\beta X^\mu = \pi^{\mu\nu}\beta_\nu\ ,\quad \delta_\beta\eta_\mu = \dot{\beta}_\mu + \partial_\mu\pi^{\rho\sigma}\eta_\rho\beta_\sigma\ .$$

It is proven in [7] that, when smooth, $\mathcal{G} = C/\sim$ is finite dimensional and carries the structure of symplectic groupoid $(\mathcal{G} \rightrightarrows M,\Omega_{\mathcal{G}})$. The obstruction to make $\mathcal{G}$ smooth, and then to identify it as the (ssc) source simply connected groupoid $\mathcal{G}(M,\pi)$ integrating (M,π) are studied in [8].

Let us consider now $v \in \Gamma(TM)$ be a Poisson vector field. Then

$$h_v[X,\eta] = \int_0^1 v^\mu(X)\eta_\mu$$

descends to $h_v \in C^\infty(\mathcal{G})$ and satisfies

$$(\partial^* h_v)(\gamma_1,\gamma_2) = h_v(\gamma_1) + h_v(\gamma_2) - h_v(\gamma_1\gamma_2) = 0$$

for each multiplicable $(\gamma_1,\gamma_2) \in \mathcal{G}_{(2)}$, so that $h_v \in Z^1(\mathcal{G},\mathbb{R})$. In particular if $v = v_f$ for some $f \in C^\infty(M)$ then $h_f = \partial^* f$, where

$$\partial^* f = s^*(f) - t^*(f)$$

is the groupoid coboundary operator $\partial^* : C^\infty(M) \to C^\infty(\mathcal{G})$.

3. Poisson-Nijenhuis structures

We recall here basic facts of Poisson-Nijenhuis geometry (see [16] and [15]). A $(1,1)$-tensor $N : TM \to TM$ is called a *Nijenhuis* tensor if its Nijenhuis torsion $T(N)$ vanishes, *i.e.* for any couple (v_1,v_2) of vector fields on M we have

$$T(N)(v_1,v_2) = [Nv_1,Nv_2] - N([Nv_1,v_2] + [v_1,Nv_2] - N[v_1,v_2]) = 0\,. \tag{4}$$

Let ι_N be the degree 0 derivation on multivector fields defined as $\iota_N(f) = 0$ and $\iota_N(v) = N(v)$ for $f \in C^\infty(M)$ and $v \in \Gamma(TM)$. Let ι_{N^*} be the dual derivation on $\Omega^1(M)$. The algebroid differential is the degree one derivation d_N on $\Omega^1(M)$ defined as

$$d_N = [\iota_{N^*}, d]\ , \tag{5}$$

that squares to zero. It is clear that $[d,d_N] = dd_N + d_N d = 0$. The *hamiltonian forms* are defined as

$$\Omega^1_{ham}(M,N) = \{\alpha \in \Omega^1(M) |\, d\alpha = d_N\alpha = 0\}\ . \tag{6}$$

Definition 3.1. *A triple (M,P,N), where (M,P) is a Poisson manifold and N a Nijenhuis tensor, is called a Poisson-Nijenhuis* (PN) *manifold if P and N are compatible, i.e.*

$$NP = PN^*\ , \quad \{\alpha,\beta\}_{NP} = \{N^*\alpha,\beta\}_P + \{\alpha,N^*\beta\}_P - N^*\{\alpha,\beta\}_P \quad , \tag{7}$$

for $\alpha,\beta \in \Omega^1(M)$.

Among the consequences of the above definition, there exists a hierarchy of compatible Poisson structures. Indeed, for all $r > 0$ and $j \geq 0$, (M,P_j,N^r) where $P_j = N^j P$, is a PN manifold. Moreover, they are compatible, *i.e.* $[P_j,P_s] = 0$.

The proof of the following Proposition can be found in [1].

Proposition 3.2. *The space $\Omega^1_{ham}(M)$ of Hamiltonian forms is N-invariant.*

If we assume that $H^1(M) = 0$, then hamiltonian forms appear in hierarchies of smooth functions, as $f_\bullet = \{f_n\}_{n\geq 0}$, where $f_n \in C^\infty(M)$, such that

$$N^* df_n = df_{n+1} \ . \tag{8}$$

We call $f_\bullet$ a Lenhart hierarchy.

Proposition 3.3. *Let $f_\bullet$ be a Lenhart hierarchy; then*

$$\{f_n, f_m\}_{P_j} = 0$$

for each $n, m, j \geq 0$.

Example 3.4. *Toda lattice* (see [10]). Let us consider $M = \mathbb{R}^{2n}$ and let $\omega = \sum_i dq^i \wedge dp_i$ the canonical symplectic form and

$$\Omega = \sum_{i=1}^{n} \left(e^{-(q^{i+1}-q^i)} dq^i \wedge dq^{i+1} - p_i dq^i \wedge dp_i \right) - \frac{1}{2}\epsilon_{ij} dp_i \wedge dp_j$$

where $\epsilon_{ij} = 1$ if $i < j$. They are compatible symplectic structures and define the Nijenhuis tensor $N = \Omega^{-1} \circ \omega$. One computes $I_1 = \mathrm{Tr} N = \sum_{i=1}^n p_i$ and

$$I_2 = \frac{1}{2}\mathrm{Tr} N^2 = \frac{1}{2}\sum_{i=1}^{2} p_i^2 + \sum_{i=1}^{n-1} e^{-(q^{i+1}-q^i)}$$

is the hamiltonian of the Toda lattice.

3.1 Lenhart hierarchies and Poisson cohomology

We are particularly interested in the cohomological interpretation of Lenhart hierarchies.

Lemma 3.5. *For each $n \geq 0$, the vector field Pdf_n is P_j-Poisson for all $j \geq 0$ and P_j-hamiltonian for all $n \geq j$.*

Proof. By applying j-times N to (8) we get that $N^j df_n = df_{n+j}$; if we apply P to both sides we get

$$P_j df_n = P df_{n+j} \ ,$$

which implies that Pdf_n is P_j-hamiltonian for $n \geq j$. Moreover,

$$[P_j, Pdf_n] = [P, P_j df_n] = [P, Pdf_{n+j}] = 0 \ ,$$

where the first step follows because P and P_j are compatible. □

Let us define $I_k = \mathrm{Tr} N^k / k$; it satisfies

$$N^* dI_k = dI_{k+1} \ , \tag{9}$$

and, as a consequence,

$$\{I_n, I_m\}_{P_j} = 0 \ .$$

We call $I_\bullet = \{I_k\}_{k\geq 0}$ the canonical hierarchy: it is a canonical set of functions in involution with respect to all Poisson structures P_j.

We will be interested in the case when $P = \omega^{-1}$ is the inverse of a symplectic form ω. We call this case a *symplectic PN structure*. In this case PN structures are completely characterized by compatible Poisson structures. Indeed if a Poisson structure π and a symplectic form ω are compatible, *i.e.* $[\pi, \omega^{-1}] = 0$ then $(M, \omega^{-1}, N = \pi \circ \omega)$ is a symplectic PN-structure.

The following result is proven in [9].

Proposition 3.6. *The vector field $\omega^{-1} dI_1$ is the modular vector field of π with respect to the symplectic volume form.*

By applying this statement to the symplectic PN structure (M, ω^{-1}, N^j) we get that $\omega^{-1} d(jI_j)$ is the modular vector field of P_j with respect to the symplectic volume, for each j.

We say that N *is of maximal rank* if there exist an open dense M_0 where there are defined $\dim M/2$ independent functions $\lambda_\alpha \in C^1(M_0)$ such that $\lambda_\alpha(x)$ is an eigenvalue of N at $x \in M_0$. We call such functions λ_α the *Nijenhuis eigenvalues*. They satisfy the following equation

$$N^* d\lambda_\alpha = \lambda_\alpha d\lambda_\alpha \ . \tag{10}$$

One can prove that hamiltonian forms are in involution with respect to all the Poisson structures P_j of the hierarchy. We then have an integrable model and the Nijenhuis eigenvalues make a canonical system of action variables. Based on the above discussion, we can say that the flux of the modular vector fields of all the Poisson structures of the PN hierarchy is integrable. Several classical integrable system give examples of the above geometry.

4. Quantization from singular polarizations

Let us consider the geometric quantization of $(\mathcal{G}, \Omega_{\mathcal{G}})$ (see [11]). The step of *prequantization* has been studied in [22]: if $\mathcal{G}$ is prequantizable as a symplectic manifold then there exist a unique prequantization line bundle $(L \to \mathcal{G}, \nabla)$ such that the flat connection $\nabla|_M$ obtained by restriction to the (lagrangian) space of units has trivial holonomy. Equivalently, this prequantization comes with a prequantization cocycle $\phi \in Z^2(\mathcal{G}, L)$. Let Θ be a (local) primitive of $\Omega_{\mathcal{G}}$ and let ∂^* the simplicial coboundary defined by the nerve of $\mathcal{G}$. Then $\phi \in C^\infty(\mathcal{G}_{(2)}, \mathbb{S}^1)$ is defined as

$$d\phi + \partial^* \Theta \phi = 0 \ . \tag{11}$$

We recall that a polarization $F \subset T_{\mathbb{C}}\mathcal{G}$ is an involutive, lagrangian distribution. It is said to be *multiplicative* if it is a subgroupoid of the tangent groupoid $T_{\mathbb{C}}\mathcal{G} \rightrightarrows TM$. Roughly speaking, the basic idea of [11] is to define an associative product (twisted by ϕ) on the space of polarized sections $\mathcal{A} = \Gamma_F(L)$. We refer to [11] for its definition, here it is enough to say that the paradigmatic example is the case when F is real and $\mathcal{L} = \mathcal{G}/F$ is a smooth Lie groupoid. In this case, the algebra $\mathcal{A}$ coincides with the convolution algebra $C_c(\mathcal{L})$ (twisted by ϕ). Of course, real polarizations are very rare in general, typically they can be found for cotangent bundles, and are not in general multiplicative.

Our observation is that the definition of the convolution algebra does not need that the groupoid $\mathcal{L}$ is a Lie groupoid. We recall that a lagrangian leaf $\ell \in \mathcal{L}$ is a *Bohr-Sommerfeld leaf* if the restriction of the (flat) prequantization connection $\nabla|_\ell$ has trivial holonomy. Let us denote with $\mathcal{L}^{BS} \subset \mathcal{L}$ the set of BS leaves. Under some topological hypothesys (see [2]), it is a subgroupoid of $\mathcal{L}$ that we call the BS groupoid $\mathcal{L}^{BS} \rightrightarrows \mathcal{L}^{BS}_0$.

In order to proceed, it is now enough that the BS groupoid $\mathcal{L}^{BS}$ admits a Haar system. Let us recall its definition from [17]. Let $C_c(\mathcal{L}^{BS})$ denote the space of continuous functions with compact support. A *left Haar system* for $\mathcal{L}^{BS}$ is a family of measures $\{{}_x\lambda, x \in \mathcal{L}^{BS}_0\}$ on $\mathcal{L}^{BS}$ such that

i) the support of ${}_x\lambda$ is ${}_x\mathcal{L}^{BS} = s^{-1}(x)$;

ii) for any $f \in C_c(\mathcal{L}^{BS})$, and $x \in \mathcal{L}^{BS}_0$, $\lambda(f)(x) = \int_{\mathcal{L}} f d_x\lambda$, defines $\lambda(f) \in C_c(\mathcal{L}^{BS}_0)$;

iii) for any $\gamma \in \mathcal{L}^{BS}$ and $f \in C_c(\mathcal{L}^{BS})$, $\int_{\mathcal{L}} f(\gamma\gamma') d_{t(\gamma)}\lambda(\gamma') = \int_{\mathcal{L}} f(\gamma') d_{s(\gamma)}\lambda(\gamma')$.

The composition of ${}_x\lambda$ with the inverse map will be denoted as λ_x; this family defines a *right Haar system*. Let $\phi \in Z^2(\mathcal{L}^{BS}, \mathbb{S}^1)$ be a 2-cocycle, hopefully the one descending from (11). We finally define the convolution product between $f, g \in C_c(\mathcal{L}^{BS}, \phi)$ as

$$f \star g(\gamma) = \int_{\mathcal{L}}^{BS} f(\gamma\gamma') g(\gamma') d_{t(\gamma)}\lambda(\gamma').$$

5. Polarization from PN structures

We showed in the previous section that it is possible to consider polarizations that are singular and useless from the point of view of ordinary geometric quantization. We are going to show that PN geometry is a source of such polarizations.

5.1 Integrating PN structures

A symplectic Nijenhuis groupoid is a symplectic groupoid $(\mathcal{G} \rightrightarrows M, \Omega_{\mathcal{G}})$ equipped with a multiplicative tensor $N_{\mathcal{G}} : T\mathcal{G} \to T\mathcal{G}$ that makes $(\mathcal{G}, \Omega^{-1}_{\mathcal{G}}, N_{\mathcal{G}})$ a PN structure of symplectic type (see [20]). Here multiplicative means that $N_{\mathcal{G}}$ is a groupoid endomorphism of the tangent groupoid $T\mathcal{G} \rightrightarrows TM$. As a consequence the unit space M inherits the structure of PN manifold; every PN structure (M, P, N) such that P is integrable integrates to a symplectic Nijenhuis groupoid.

Since given a PN structure (M, P, N), we have a hierarchy of PN structures (M, P_j, N) with $P_j = N^j P$, if we assume that P_j are all integrable Poisson structures we will have a hierarchy of symplectic Nijenhuis groupoids $(\mathcal{G}_j, N_{\mathcal{G}_j}, \Omega_{\mathcal{G}_j})$ for $j \geq 0$.

Let us consider a Nijenhuis eigenvalue λ of N. It is clear that since $N_{\mathcal{G}_i}$ commutes with source and target maps $s_*, t_* : T\mathcal{G}_i \to TM$, $\{s^*\lambda, t^*\lambda\}$ is a Nijenhuis eigenvalue of $N_{\mathcal{G}_i}$, *i.e.* it satisfies

$$N_{\mathcal{G}_i} ds^*\lambda = s^*\lambda ds^*\lambda\,, \quad N_{\mathcal{G}_i} dt^*\lambda = t^*\lambda dt^*\lambda\ .$$

Let us consider now a Lenhart hierarchy $f_\bullet = \{f_n\}_{n\geq 0}$. Let $(\mathcal{G}_j \rightrightarrows M, \Omega_{\mathcal{G}_j})$ be the symplectic groupoid integrating P_j with Nijenhuis tensor $N_{\mathcal{G}_j}$. Since $N_{\mathcal{G}_j} : T\mathcal{G}_j \to T\mathcal{G}_j$ is a groupoid morphism, it is clear that both $s^*(f)_\bullet = \{s^* f_n\}_{n\geq 0}$ and $t^*(f)_\bullet = \{t^* f_n\}_{n\geq 0}$ are Lenhart hierarchies.

There exists a third way to integrate $f_\bullet$. We know from Lemma 8 that Pdf_n is P_j-Poisson for all $j \geq 0$ and P_j-hamiltonian for $n \geq j$.

Proposition 5.1. *Let $h^{(j)}_{f_n}$ be the groupoid* 1*-cocycle of $\mathcal{G}_j \rightrightarrows M$ integrating Pdf_n. They satisfy*

$$N^*_{\mathcal{G}_j} dh^{(j)}_{f_n} = dh^{(j)}_{f_{n+1}} \ ,$$

i.e. $h^{(j)}_{f_\bullet} = \{h^{(j)}_{f_n}\}_{n\geq 0}$ *is a Lenhart hierarchy.*

Proof. Let us compute for each $\alpha_m \in T^*_m M$ and $n \geq 0$

$$\begin{aligned} \langle \overrightarrow{\alpha}, dh^{(j)}_{f_{n+1}} \rangle_m &= \langle \alpha_m, Pdf_{n+1} \rangle = \langle \alpha_m, PN^* df_n \rangle = \langle \alpha_m, NPdf_n \rangle \\ &= \langle N^*\alpha_m, Pdf_n \rangle = \langle \overrightarrow{N^*\alpha}_m, dh^{(j)}_{f_n} \rangle = \langle N_{\mathcal{G}_j} \overrightarrow{\alpha}_m, dh^{(j)}_{f_n} \rangle \\ &= \langle \overrightarrow{\alpha}_m, N^*_{\mathcal{G}_j} dh^{(j)}_{f_n} \rangle \ . \end{aligned}$$

This proves that the 1-form $\omega_n = dh^{(j)}_{f_{n+1}} - N^*_{\mathcal{G}_j} dh^{(j)}_{f_n} \in s^*(T^*M)^o \subset T\mathcal{G}_j$. Since it is multiplicative, *i.e.* $\partial^*\alpha_n = 0$, it vanishes when restricted to $M \subset \mathcal{G}_j$; moreover, we have that for each $\gamma \in \mathcal{G}_j$ and each $v_\gamma \in T_\gamma \mathcal{G}_j$ and $w_{t(\gamma)} \in T_{t(\gamma)}\mathcal{G}_j$ such that $t_*(v_\gamma) = s_*(w_{t(\gamma)})$, we have that

$$\langle \omega_n(\gamma), v_\gamma \cdot w_{t(\gamma)} \rangle = \langle \omega_n(\gamma), v_\gamma \rangle + \langle \omega_n(t(\gamma)), w_{t(\gamma)} \rangle = \langle \omega_n(\gamma), v_\gamma \rangle \ ,$$

where $\cdot$ denotes groupoid multiplication of $T\mathcal{G}_j \rightrightarrows TM$. This implies that $\omega_n(\gamma) = 0$. □

It is clear that the Lenhart hierarchies $s^*(f)_\bullet$, $t^*(f)_\bullet$ and $h^{(j)}_{f_\bullet}$ are not independent; indeed, since $Pdf_j = P_j df_0$ we have that

$$t^* f_n = s^* f_n - h_{f_{n+j}} \ . \tag{12}$$

We can say that every Lenhart hierarchy is doubled to two independent hierarchies of $(\mathcal{G}_j, N_{\mathcal{G}_j})$.

5.2 Groupoid of lagrangian leaves

An integrable model can be seen as a singular real polarization. Indeed there exists a dense open where the quotient map from the phase space to the contour level set of the hamiltonians in involution has lagrangian fibres. Of course in the closure of this open set, we do not have in general control and the fibres can have quite different behaviors.

Given the non degenerate PN structure (M, ω^{-1}, P), we can consider the topological quotient $M \to \Delta_N$ defined by the equivalence relation $m \sim m'$ when $f_n(m) = f_n(m')$ for each Lenhart hierarchy $f_\bullet$. The same quotient $\mathcal{G}_j \to \mathcal{L}_j$ is defined for the non degenerate symplectic PN strucure $(\mathcal{G}_j, \Omega_{\mathcal{G}_j}, N_{\mathcal{G}_j})$ for each $j \geq 0$.

It is easy to see that $\mathcal{L}_j$ inherits a topological groupoid structure over Δ_N. Indeed, (12) proves that source and target maps descend to $\mathcal{L}_j$; the multiplication is inherited since $h_\bullet$ are groupoid cocycles. With an abuse of notation, we denote with $(f_\bullet, h_\bullet)$ the value of the functions appearing in the hierarchies $s^*(f)_\bullet$ and $h^{(j)}_{f_\bullet}$. A point in $\mathcal{L}_j$ and Δ_N is described by assigning the values of all the Lenhart hierarchies. The groupoid structure of $\mathcal{L}_j \rightrightarrows \Delta_N$ is then described for each hierarchy as $f_\bullet$

$$s(f_\bullet, h_{f_\bullet}) = f_\bullet, \ t(f_\bullet, h_{f_\bullet}) = f_\bullet - h_{f_{\bullet+j}}$$

$$(f_\bullet, h_{f_\bullet})(f'_\bullet, h'_{f_\bullet}) = (f_\bullet, h_{f_\bullet} + h'_{f_\bullet}) \ , \quad (f_\bullet, h_{f_\bullet})^{-1} = (f_\bullet - h_{f_{\bullet+j}}, -h_{f_\bullet})$$

$$f_\bullet \to (f_\bullet, 0) \ .$$

It is clear that $\mathcal{L}_0 = \Delta_N \times \Delta_N \rightrightarrows \Delta_N$ is the pair groupoid (we suppose that M is simply connected so that $\mathcal{G}_0 = M \times M \rightrightarrows M$).

In order to discuss $j \geq 1$, let us suppose that there exist global Nijenhuis eigenvalues. Thanks to (10), for each eigenvalue λ the Lenhart hierarchy $\lambda_\bullet$ is computed as $\lambda_n = \lambda^{n+1}/(n+1)$. In particular, it is completely identified by fixing the value of λ. Let us call $h_\bullet$ the Lenhart hierarchy of cocycles of $\mathcal{G}_j \rightrightarrows M$ integrating $\lambda_\bullet$.

Let us consider now the first Poisson structure of the hierarchy P_1. By applying P to (10), we obtain $Pd\lambda = P_1 d\lambda/\lambda$ that integrates to

$$t^*\lambda = e^{-h_0} s^*\lambda \ . \tag{13}$$

From (12) we conclude that the hierarchy $h_\bullet$ is identified by fixing h_0 and $s^*\lambda$.

Let us consider P_2. By repeating the same trick we get that

$$Pd\lambda = -P_2 d(1/\lambda) \tag{14}$$

that integrates to $h_0 = -\partial^*(1/\lambda)$, that we solve as

$$t^*\lambda = \frac{s^*\lambda}{1 + s^*\lambda h_0} \ . \tag{15}$$

Analogously we can write (14) as $P_2 d\lambda = \lambda P d\lambda_1$ that integrates to $h_1 = \partial^* \log\lambda$ so that $t^*\lambda = e^{-h_1} s^*\lambda$ which implies

$$e^{h_1} = 1 + s^*\lambda h_0 \ .$$

Also in this case the values of hierarchy $h_\bullet$ are fixed by fixing h_0 and $s^*\lambda$.

Let us consider P_{j+1}, for $j \geq 1$. By repeating the same trick we get that $Pd\lambda = -\frac{1}{j} P_{j+1} d\lambda^{-j}$ that integrates to $h_0 = -\frac{1}{j}\partial^*\lambda^{-j}$, that we solve as

$$t^*\lambda = \frac{s^*\lambda}{\sqrt[j]{1 + j s^*\lambda^j h_0}} \ .$$

Analogously, we can prove that h_k for $k \geq 1$ can be written as a function of $s^*\lambda$ and h_0 so that the values of the hierarchies $h_\bullet$ and $\lambda_\bullet$ are fixed by fixing λ and h_0.

If all Nijenhuis eigenvalues are globally smooth functions we obtain the following description of $\mathcal{L}_j \rightrightarrows \Delta_N$ for $j \geq 1$. Let $\dim M = 2r$ and let us consider an embedding $\Delta_N \hookrightarrow \mathbb{R}^r$ thanks to a choice of numbering of the eigenvalues. Then $\mathcal{L}_j \rightrightarrows \Delta_N$ is a subgroupoid of the restriction to Δ_N of the action groupoid $\mathbb{R}^r \ltimes \mathbb{R}^r \rightrightarrows \mathbb{R}^r$ with respect to the action of $\mathbb{R}^r$ on itself defined for $j = 1$ as

$$h \cdot_1 \lambda = e^{-h}\lambda \tag{16}$$

and for $j > 1$ as

$$h \cdot_j \lambda = \frac{\lambda}{\sqrt[j]{1 + j\lambda^j h}} \ . \tag{17}$$

6. Bruhat-Poisson

We are mainly interested in a class of PN structures of symplectic type defined on compact hermitian symmetric spaces. These were introduced in [13], where it was proved that the KKS symplectic form defined by looking at the M as a coadjoint orbit of a simple compact Lie group and the Bruhat-Poisson structure π obtained by considering M as a quotient of the standard Poisson-Lie group are compatible. The non degeneracy of these PN structures has been studied in [4] for the classical cases and in [5] where one can find partial results on the exceptional cases. The quantization program has been completed only for $M = \mathbb{C}P_n$ in [2]. Here, as an example, we briefly describe the simplest case of $\mathbb{C}P_1$.

By using the complex stereographic coordinates $z = 1/w$ the inverse of the symplectic form reads

$$\omega = \frac{idz\, d\bar{z}}{(1+|z|^2)^2} = \frac{idw\, d\bar{w}}{(1+|w|^2)^2}$$

and the Bruhat-Poisson structure reads

$$\pi = i(1+|z|^2)\partial_z \wedge \partial_{\bar{z}} = i|w|^2(1+|w|^2)\partial_w \wedge \partial_{\bar{w}}\ ,$$

so that $N(\partial_z) = 1/(1+|z|^2)\partial_z$. There is then only one global eigenvalue

$$I_1 = \lambda = \frac{1}{1+|z|^2} = \frac{|w|^2}{1+|w|^2}\ ,$$

so that $\Delta_N = [0,1]$. The Poisson structures of the hierarchy are for $j \geq 0$

$$P_j = \frac{i}{(1+|z|^2)^{j-2}}\partial_z \wedge \partial_{\bar{z}} = \frac{i|w|^{2j}}{(1+|w|^2)^{j-2}}\partial_w \wedge \partial_{\bar{w}}\ .$$

We see that for all $j \geq 1$ the north pole chart is symplectic; at $z = \infty$, P_j has an isolated zero of degree $2j$. The modular vector field of P_j with respect to the symplectic volume is

$$\chi_j = \omega^{-1} d\lambda^j = j\lambda^{j-1} i(z\partial_z - \bar{z}\partial_{\bar{z}}) = -j\lambda^{j-1} i(w\partial_w - \bar{w}\partial_{\bar{w}})\ .$$

The symplectic groupoid integrating $P_0 = \omega^{-1}$ is the pair groupoid $\mathbb{S}^2 \times \mathbb{S}^2 \rightrightarrows \mathbb{S}^2$ with symplectic form $\Omega_{\mathcal{G}_0} = \omega \ominus \omega$. The groupoid $\mathcal{L}_0^{BS}$ is easily computed as $\{0,1\} \times \{0,1\} \rightrightarrows \{0,1\}$, corresponding to the states of spin 1/2-representation of $SU(2)$. The convolution algebra is then $\mathcal{A}(\mathcal{L}_0^{BS}) = M_2(\mathbb{C})$.

The symplectic groupoid of $P_1 = \pi$ can be described as a symplectic reduction of the Lu-Weinstein groupoid integrating the standard Poisson structure on $SU(2)$ (see [3]). The integrability of (M, P_j) for $j > 1$ has not been studied. In order to compute the groupoid of BS leaves in this simple case it is not necessary to have such description; we simply suppose that $(\mathbb{S}^2, P_j)$ is integrable and we denote with $\mathcal{G}_j$ its (ssc) symplectic groupoid.

When restricted to the symplectic chart U_N, $\mathcal{G}_j|_{U_N} \sim U_N \times U_N$ and the symplectic form reads

$$\Omega_{\mathcal{G}_j} = d\partial^* \Theta_j \tag{18}$$

where

$$\Theta_1 = \log\lambda\, d\phi\ , \quad \Theta_j = -\frac{1}{j-1}\lambda^{1-j} d\phi\ , \quad j \geq 2$$

and $d\phi = (i/2)(dz/z - d\bar{z}/\bar{z})$. In the symplectic chart, the lagrangian leaf is defined by fixing $s^*\lambda \equiv \lambda$ and $t^*\lambda \equiv \rho$ so that a leaf is BS when there exist integers $n, m \geq 0$ such that for $j = 1$

$$\log \lambda = -n, \log \rho = -m \ . \tag{19}$$

From (13) the modular cocycle is $h_0 = n - m$. When $j \geq 2$ a leaf is BS if there exist integers $n, m \geq 1/(j-1)$ such that

$$\lambda^{1-j} = (j-1)n\,, \quad \rho^{1-j} = (j-1)m \tag{20}$$

so that from (15) the modular cocycle reads $h_0 = m - n$. We only have to add $\mathcal{G}_j|_{w=0}$ that corresponds to a single leaf $\lambda = \rho = h_0 = 0$.

We then conclude that the groupoid $\mathcal{L}^{BS}_j = \mathcal{L}^{BS}$ for each $j \geq 1$ can be described as follows. Let $\mathbb{Z}$ acting on $\bar{\mathbb{Z}} = \mathbb{Z} \cup \{\infty\}$, where ∞ is added as a fixed point of the action. Then $\mathcal{L}^{BS}$ is the restriction to $\bar{\mathbb{N}} = \mathbb{N} \cup \{\infty\} \subset \bar{\mathbb{Z}}$ of the action groupoid $\mathbb{Z} \ltimes \bar{\mathbb{Z}} \rightrightarrows \bar{\mathbb{Z}}$. This groupoid was used in [18] to describe the C^*-algebra of the Podles quantum 2-sphere.

The groupoid $\mathcal{L}^{BS} \rightrightarrows \bar{\mathbb{N}}$ admits a unique (up to costant multiplicative) Haar system, that is the counting measure. We can then consider its convolution algebra $\mathcal{A}(\mathcal{L}^{\mathcal{BS}})$ that is generated by $\{e_{m,n}, n \geq 0, m+n \geq 0\}$ and the identity $\mathrm{id} = \sum_{n\geq 0} e_{0,n}$ satisfying

$$e_{mn} * e_{m'n'} = \delta_{n,m'+n'} e_{m+m',n'} \qquad . \tag{21}$$

Remark 6.1.

i) The BS groupoid is the same for all $j \geq 1$. This means that it is only sensible to the topology of the space of symplectic leaves, which is the same for all P_j. These Poisson structures differ only by the order of the zero that is $2j$ for P_j. The different embeddings of the induced *BS* leaves inside $[0, 1]$ given by (19) and (20) feel this order.

ii) The output of the quantization is the convolution algebra $\mathcal{A}(\mathcal{L}^{\mathcal{BS}})$; since it does not contain $\hbar$, no notion of semiclassical limit exists. This probably can be added to the picture by enhancing it to the quantization of the Poisson structure $\hbar P_j$.

iii) The prequantization cocycle in (11) should be added to the description. So far we used $\phi = 1$ which is the right one only if there exists a multiplicative primitive Θ of $\Omega_{\mathcal{G}}$. The one used in (18) is obviously multiplicative but it is not global.

iv) In the computation of BS condition the Maslov correction is not included. This is equivalent to ignoring the addition of the partial Bott connection to the prequantization connection.

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