

The Nicolai map for super Yang–Mills theory and application to the supermembrane

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The nonlocal bosonic theory obtained from integrating out all anticommuting and auxiliary variables in a globally supersymmetric theory is characterized by the Nicolai map. We present a universal formula for the latter in terms of an ordered exponential of the integrated coupling flow operator, which can be canonically constructed. Also for supersymmetric gauge theories, this allows us to perturbatively construct the Nicolai map explicitly in terms of tree diagrams. For off-shell supersymmetry this works in any gauge, in the on-shell case the Landau gauge is required. The dimensional reduction of $D=10$ super Yang–Mills to maximally supersymmetric $SU(N)$ matrix mechanics (the BFSS model) is known to provide a regularization of the $D=11$ supermembrane in light-cone gauge via its incarnation as a one-dimensional gauge theory of area-preserving membrane diffeomorphisms. We show how a well-defined corresponding Nicolai map perturbatively linearizes the supermembrane in the small-tension regime and points the way for a computation of vertex-operator correlators.

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1. Definition and construction of the Nicolai map

The key idea is best illustrated by an example. Let us look at the Wess–Zumino model in 3+1 dimensional Minkowski space, consisting of a complex scalar ϕ , a Weyl fermion ψ and a complex auxiliary F , characterized by a superpotential $W(\phi)$ and featured in the off-shell lagrangian ¹

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi + F^* F + \frac{i}{2} \bar{\psi} \vec{\sigma} \cdot \partial \psi - \frac{i}{2} \psi \sigma \cdot \partial \bar{\psi} + W'(\phi) F + W'(\phi)^* F^* - \frac{1}{2} \psi W''(\phi) \psi - \frac{1}{2} \bar{\psi} W''(\phi)^* \bar{\psi}, \quad (1)$$

where $\sigma = (\mathbb{1}, \vec{\sigma})$ and $\bar{\sigma} = (\mathbb{1}, -\vec{\sigma})$ with Pauli matrices $\vec{\sigma}$. Integrating out the auxiliary fields yields $F^* = -W'(\phi)$ and

$$\mathcal{L}_{\text{SUSY}} = |\partial\phi|^2 - |W'(\phi)|^2 + \left(\frac{i}{2} \bar{\psi} \vec{\sigma} \cdot \partial \psi - \frac{1}{2} \psi W''(\phi) \psi + \text{h.c.} \right). \quad (2)$$

Integrating out the fermions $(\psi, \bar{\psi})$ produces a functional determinant $\det M = \exp\{\frac{i}{\hbar} \cdot (-i\hbar \text{tr} \ln M)\}$ so that the action becomes

$$S_g[\phi] = \int d^4x \{ |\partial\phi|^2 - |W'|^2 \} - i\hbar \text{tr} \ln \begin{pmatrix} W'' & i\sigma \cdot \partial \\ -i\bar{\sigma} \cdot \partial & W''^* \end{pmatrix} =: S_g^b[\phi] + \hbar S_g^f[\phi]. \quad (3)$$

Here, g denotes some coupling constant(s) or parameter(s) inside the superpotential $W(\phi)$. The objects of desire are quantum correlators

$$\langle Y[\phi] \rangle_g = \int \mathcal{D}\phi e^{\frac{i}{\hbar} S_g[\phi]} Y[\phi] \quad \text{with} \quad \langle \mathbb{1} \rangle = 1 \quad (4)$$

for any bosonic (local or nonlocal) functional Y .

The path integral in (4) describes a purely bosonic nonlocal field theory. What is characteristic of its supersymmetric origin? In other words: given such a nonlocal action S_g , how could one infer its hidden supersymmetric root? This question was answered in 1980 by Hermann Nicolai [1–3]: Such hiddenly supersymmetric theories admit a nonlocal and nonlinear invertible map

$$T_g : \phi \mapsto \phi'[\phi; g] \quad \text{such that} \quad \langle Y[\phi] \rangle_g = \langle Y[T_g^{-1}\phi] \rangle_0 \quad \forall Y, \quad (5)$$

relating correlators in the interacting theory ($g \neq 0$) to (more complicated) correlators in the free theory ($g=0$). For the path integrals, this is equivalent to

$$\mathcal{D}\phi \exp\left\{\frac{i}{\hbar} S_g[\phi]\right\} = \mathcal{D}(T_g \phi) \exp\left\{\frac{i}{\hbar} S_0[T_g \phi]\right\} = \mathcal{D}\phi \exp\left\{\frac{i}{\hbar} S_0[T_g \phi] + \text{tr} \ln \frac{\delta T_g \phi}{\delta \phi}\right\}. \quad (6)$$

Separating powers of \hbar in the exponent, this splits into two properties,

$$S_0^b[T_g \phi] = S_g^b[\phi] \quad \text{“free action condition”}, \quad (7a)$$

$$S_0^f - i \text{tr} \ln \frac{\delta T_g \phi}{\delta \phi} = S_g^f[\phi] \quad \text{“determinant matching condition”}. \quad (7b)$$

Every Nicolai map has to fulfil these two conditions, which originally were taken as its definition. The reason for the name of (7b) is that its exponentiation gives an equality of the functional fermion determinant $\det M$ with the Jacobian of the transformation (the first term is a constant since S_0^f does not depend on ϕ). From now on we put $\hbar=1$.

¹ A multi-field generalization is straightforward.

In 1984, the author derived (for his dissertation) an infinitesimal version [4–7] of the Nicolai map by considering the g -derivative of (5),

$$\begin{aligned} \partial_g \langle Y[\phi] \rangle_g &\stackrel{(5)}{=} \partial_g \langle Y[T_g^{-1} \phi] \rangle_0 \\ &= \langle \partial_g Y[\phi] \rangle_g + \langle f(\partial_g T_g^{-1} \phi) \cdot \frac{\delta Y}{\delta \phi}[T_g^{-1} \phi] \rangle_0 \\ &\stackrel{(5)^{-1}}{=} \langle \partial_g Y[\phi] \rangle_g + \langle f(\partial_g T_g^{-1} \circ T_g) \phi \cdot \frac{\delta Y}{\delta \phi}[\phi] \rangle_g =: \langle (\partial_g + R_g[\phi]) Y[\phi] \rangle_g \end{aligned} \quad (8)$$

with a “flow operator”²

$$R_g[\phi] = \int dx (\partial_g T_g^{-1} \circ T_g) \phi(x) \frac{\delta}{\delta \phi(x)} \quad (9)$$

representing a functional differential operator derived from T_g .

Nothing is gained, however, by these formal considerations, unless we can reverse the logic and somehow obtain R_g and exponentiate it in order to create a finite flow T_g from $g'=0$ to $g'=g$, by inverting

$$(T_g^{-1} \phi)(x) = \exp\{g (\partial_{g'} + R_{g'}[\phi])\} \phi(x) \Big|_{g'=0} = \sum_{n=0}^{\infty} \frac{g^n}{n!} (\partial_{g'} + R_{g'}[\phi])^n \phi(x) \Big|_{g'=0}. \quad (10)$$

At this stage two remarks are in order. Firstly, R_g is a derivation, and hence T_g^{-1} acts distributively,

$$R_g Y[\phi] = \int \frac{\delta Y}{\delta \phi} \cdot R_g \phi \quad \Leftrightarrow \quad T_g^{-1} Y[\phi] = Y[T_g^{-1} \phi]. \quad (11)$$

Secondly, by moving the map “to the other side”,

$$\langle Y[\phi] \rangle_0 = \langle Y[T_g \phi] \rangle_g, \quad (12)$$

choosing $\partial_g Y = 0$ and differentiating with respect to g , we learn that

$$0 = \partial_g \langle Y[T_g \phi] \rangle_g \stackrel{(8)}{=} \langle (\partial_g + R_g[\phi]) Y[T_g \phi] \rangle_g = \langle f(\partial_g + R_g[\phi]) T_g \phi \cdot \frac{\delta Y}{\delta \phi}[T_g \phi] \rangle_g \quad (13)$$

for any (not explicitly g -dependent) functional Y , and therefore

$$(\partial_g + R_g[\phi]) T_g \phi(x) = 0. \quad (14)$$

This “fixpoint property” of the Nicolai map under the infinitesimal flow allows us to directly construct $T_g \phi$ from R_g without invoking the inverse first.

Indeed, (14) is formally solved by a path-ordered exponential,

$$T_g \phi = \mathcal{P} \exp\left\{-\int_0^g dh R_h[\phi]\right\} \phi = \sum_{s=0}^{\infty} (-1)^s \int_0^g dh_s \dots \int_0^{h_3} dh_2 \int_0^{h_2} dh_1 R_{h_s}[\phi] \dots R_{h_2}[\phi] R_{h_1}[\phi] \phi, \quad (15)$$

providing a “universal formula” for the Nicolai map in terms of the infinitesimal coupling flow [8]. It is often useful to expand the flow operator in powers of the coupling,

$$R_g[\phi] = \sum_{k=1}^{\infty} g^{k-1} r_k[\phi] = r_1[\phi] + g r_2[\phi] + g^2 r_3[\phi] + \dots \quad (16)$$

² We write dx for the spacetime volume differential as long as its dimension remains unspecified.

from which one easily computes a power series expansion for the map itself,

$$T_g \phi = \sum_{\mathbf{n}} g^n c_{\mathbf{n}} r_{n_s}[\phi] \dots r_{n_2}[\phi] r_{n_1}[\phi] \phi \quad \text{with } \mathbf{n} = (n_1, n_2, \dots, n_s), \quad n_i \in \mathbb{N}, \quad \sum_i n_i = n, \quad (17)$$

where $1 \leq s \leq n$ and the $n=0$ term is the identity. The numerical coefficients are computed as

$$c_{\mathbf{n}} = (-1)^s \int_0^1 dx_s x_s^{n_s-1} \dots \int_0^{x_3} dx_2 x_2^{n_2-1} \int_0^{x_2} dx_1 x_1^{n_1-1} = (-1)^s [n_1 \cdot (n_1+n_2) \cdot \dots \cdot (n_1+n_2+\dots+n_s)]^{-1} \quad (18)$$

and related to the Stirling numbers of the second kind. Writing out the first few terms, the perturbative Nicolai map reads

$$T_g \phi = \phi - g r_1 \phi - \frac{1}{2} g^2 (r_2 - r_1^2) \phi - \frac{1}{6} g^3 (2r_3 - r_1 r_2 - 2r_2 r_1 + r_1^3) \phi - \frac{1}{24} g^4 (6r_4 - 2r_1 r_3 - 3r_2 r_2 + r_1^2 r_2 - 6r_3 r_1 + 2r_1 r_2 r_1 + 3r_2 r_1^2 - r_1^4) \phi + O(g^5). \quad (19)$$

For computing correlation functions à la (5) we need the inverse map. It possesses an analogous universal representation in terms of an anti-path-ordered exponential, which gives rise to a different power series expansion,

$$T_g^{-1} \phi = \sum_{\mathbf{n}} g^n d_{\mathbf{n}} r_{n_s}[\phi] \dots r_{n_2}[\phi] r_{n_1}[\phi] \phi \quad \text{with } c_{\mathbf{n}} = [n_s \cdot (n_s+n_{s-1}) \cdot \dots \cdot (n_s+n_{s-1}+\dots+n_1)]^{-1} \quad (20)$$

whose first terms are

$$T_g^{-1} \phi = \phi + g r_1 \phi + \frac{1}{2} g^2 (r_2 + r_1^2) \phi + \frac{1}{6} g^3 (2r_3 + 2r_1 r_2 + r_2 r_1 + r_1^3) \phi + \frac{1}{24} g^4 (6r_4 + 6r_1 r_3 + 3r_2 r_2 + 3r_1^2 r_2 + 2r_3 r_1 + 2r_1 r_2 r_1 + r_2 r_1^2 + r_1^4) \phi + O(g^5). \quad (21)$$

Still, we have to establish the existence of the flow operator R_g and find an explicit expression for it. We shall do this now for the exemplary case of scalar theories (gauge theories will be treated in the following section). If supersymmetry is realized off-shell on the action S then there exists a functional $\hat{\Delta}_\alpha[\phi, \psi, F]$ such that

$$\partial_g S[\phi, \psi, F] = \delta_\alpha \hat{\Delta}_\alpha[\phi, \psi, F] \quad (22)$$

for the supersymmetry transformations δ_α , where α denotes a Majorana spinor index. Integrating out the auxiliary F one has that

$$\partial_g S_{\text{SUSY}}[\phi, \psi] = \delta_\alpha \Delta_\alpha[\phi, \psi] \quad \text{with } \Delta_\alpha[\phi, \psi] = \hat{\Delta}_\alpha[\phi, \psi, -W'^*(\phi)] \quad (23)$$

for the on-shell action $S_{\text{SUSY}} = \int dx \mathcal{L}_{\text{SUSY}}$ with an anticommuting functional Δ_α . For our Wess–Zumino model example, it reads $\Delta_\alpha = \frac{1}{2} \int d^4x \psi_\alpha \partial_g W'(\phi)$. The construction of R_g employs the supersymmetry Ward identity:

$$\partial_g \int \mathcal{D}\phi \int \mathcal{D}\psi e^{iS_{\text{SUSY}}[\phi, \psi]} Y[\phi] = \int \mathcal{D}\phi \int \mathcal{D}\psi e^{iS_{\text{SUSY}}[\phi, \psi]} (\partial_g + i\Delta_\alpha[\phi, \psi] \delta_\alpha) Y[\phi], \quad (24)$$

Integrating out the fermions contracts bilinears to produce fermion propagators $\psi \underline{\psi}$ (in the ϕ background), hence [4]

$$R_g[\phi] = i \underline{\Delta_\alpha[\phi]} \delta_\alpha = i \int dx \underline{\Delta_\alpha[\phi]} \delta_\alpha \phi(x) \frac{\delta}{\delta \phi(x)} \quad (25)$$

For a simple example of the Wess–Zumino model with (massless) superpotential $W = \frac{1}{3}g\phi^3$, one finds that

$$R_g[\phi] = \frac{i}{2} \iint d^4x d^4y \{ \phi^2(x) \underbrace{\psi(x)\psi(y)} + \phi^{*2}(x) \underbrace{\bar{\psi}(x)\psi(y)} \}_{\alpha\alpha} \frac{\delta}{\delta\phi(y)} - \text{h.c.} \quad (26)$$

where the subscript on the curly brace indicates a spin trace. It is instructive to develop a diagrammatic shorthand notation. For the sake of illustration, here we oversimplify $(\phi, \phi^*) \sim \phi$ and write

$$R_g[\phi] \hat{=} \text{diagram} = \text{diagram} + g \text{diagram} + g^2 \text{diagram} + \dots$$

with the graphical rules [9]

$$\begin{aligned} \text{diagram} &\hat{=} \frac{1}{2}\phi^2 & \longrightarrow &\hat{=} \delta/\delta\phi & \equiv &\hat{=} (i\phi + 2g\phi)^{-1} \\ \text{diagram} &\hat{=} -2\phi & \text{diagram} &\hat{=} (i\phi)^{-1} & \dots &\hat{=} \sum_{\alpha} (\dots)_{\alpha\alpha} \end{aligned}$$

The linear tree for R_g exponentiates to a series of branched trees for $T_g\phi$,

$$\begin{aligned} T_g\phi &\hat{=} \text{diagram} - g \text{diagram} - \frac{1}{2}g^2 \left(\text{diagram} - \text{diagram} \right) \\ &\quad - \frac{1}{6}g^3 \left(2 \text{diagram} - \text{diagram} - 2 \text{diagram} + \text{diagram} - \text{diagram} + \text{diagram} \right) + \dots \end{aligned}$$

and likewise for the inverse $T_g^{-1}\phi$. Inserting the latter into (5) and performing the free-theory bosonic contractions, one obtains an alternative Feynman perturbation series for correlators, as displayed here for the two-point function:

$$\begin{aligned} \langle \phi(x) \phi^*(y) \rangle_g &= \langle T_g^{-1}\phi(x) T_g^{-1}\phi(y) \rangle \\ &= \text{diagram} + g^2 \left(2 \text{diagram} + \text{diagram} + \text{diagram} + \text{diagram} + \text{diagram} \right) + \dots \end{aligned}$$

Notably, the multiple action of R_g produces multiple spin traces (graphically separated by dots). The supersymmetric cancellation of the leading UV divergencies is automatically built in, as pure fermion loops are absent as well as boson tadpoles.

2. The case of gauge theories

Supersymmetric gauge theories present additional challenges. Firstly, one has to deal with the gauge redundancy necessitating a (supersymmetry-breaking) gauge fixing and, secondly, the g -derivative of the supersymmetric action cannot easily be expressed as a supervariation. We eliminate the auxiliary field (D -term), use a local gauge-fixing functional \mathcal{G} to fix a gauge $\mathcal{G}(A)=0$ with a parameter ξ and include the corresponding ghost fields to formulate a BRST-invariant on-shell action

$$S_{\text{SUSY}}[A, \lambda, c, \bar{c}] = \int dx \text{tr} \left\{ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\xi} \mathcal{G}(A)^2 - \frac{1}{2} \bar{\lambda} \not{D} \lambda + \bar{c} \frac{\partial \mathcal{G}}{\partial A_{\mu}} D_{\mu} c \right\} \quad (27)$$

for $su(N)$ -valued gluons $A_{\mu} = A_{\mu}^A T^A$, gluinos $\lambda_{\alpha} = \lambda_{\alpha}^A T^A$, a ghost $c = c^A T^A$ and an antighost $\bar{c} = \bar{c}^A T^A$, with $F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} + g[A_{\mu}, A_{\nu}] = F_{\mu\nu}^A T^A$ and group generators subject to $[T^A, T^B] = f^{ABC} T^C$

with $A, B, \dots = 1, 2, \dots, N^2-1$. The trace refers to the color degrees of freedom. We allow for various spacetime dimensionalities D by letting the fields live on $\mathbb{R}^{1,D-1}$ so that $\mu, \nu, \dots = 0, 1, \dots, D-1$ and $\alpha = 1, \dots, r$, where r is the complex dimension of the corresponding Majorana representation, i.e. $\lambda^A \in \mathbb{C}^r$. It essentially grows exponentially with D . In the following, we present two different attempts to emulate the successful scalar-field procedure.

In version A [10], from $F = dA + g A \wedge A$ we see that $g=0$ is the free theory. A quick computation shows things now are more involved than in (23),

$$\partial_g S_{\text{SUSY}} = \delta_\alpha \Delta_\alpha + q \int \text{tr} i \bar{\lambda} A \lambda + \int \text{tr} \bar{c} \frac{\partial \mathcal{G}}{\partial A_\mu} A_\mu c \quad \text{with} \quad q = \frac{D-1}{r} - \frac{1}{2} \quad (28)$$

where $(\gamma^{\mu\nu} = \frac{1}{2}[\gamma^\mu, \gamma^\nu])$

$$\Delta_\alpha = -\frac{1}{2r} \int \text{tr} (\gamma^{\mu\nu} \lambda)_\alpha A_\mu A_\nu \quad (29)$$

is the gauge-theory counterpart of the on-line functional in (23). However, we have to fight with a ghost contribution and a “mismatch” q in the construction of R_g . With the help of the broken-supersymmetry and BRST Ward identities one derives that

$$\partial_g \langle Y[A] \rangle_g = \langle (\partial_g + R_g[A] + Z_g[A]) Y[A] \rangle_g \quad (30)$$

where

$$R_g = i \underbrace{\Delta_\alpha}_{\delta_\alpha} - \underbrace{\Delta_\alpha (\delta_\alpha \Delta_{\text{gh}})}_s \quad \text{with} \quad \Delta_{\text{gh}} = \int \text{tr} \bar{c} \mathcal{G}(A), \quad (31)$$

$$Z_g = (s \underbrace{\Delta_\alpha}_{\delta_\alpha}) (\delta_\alpha \Delta_{\text{gh}}) - q \int \text{tr} \bar{\lambda} A \lambda + i \int \text{tr} \bar{c} \frac{\partial \mathcal{G}}{\partial A_\mu} A_\mu c, \quad (32)$$

and s denotes the BRST (or Slavnov) variation. The contractions signify gaugino or ghost propagators. The multiplicative contribution Z_g destroys the derivation property of R_g and hence the distributivity of T_g , which is not acceptable. A somewhat lengthy computation reveals, however, that in the Landau gauge, $\mathcal{G} = \partial^\mu A_\mu$ with $\xi \rightarrow \infty$, the obstacle may be overcome,

$$Z_g = 0 \quad \text{if and only if} \quad q = \frac{1}{r} \Leftrightarrow r = 2(D-2) \Leftrightarrow D = 3, 4, 6, 10. \quad (33)$$

Amazingly, these are precisely the “critical spacetime dimensions” which admit super Yang–Mills theory to exist [11], demonstrating that the Nicolai map knows about them [12]!

For version B [4, 13, 14], we restrict to a linear gauge $\mathcal{G}(A) = n \cdot A$ or $\partial \cdot A$ and rescale all fields to tilded versions in order to pull out the gauge coupling. In particular,

$$g A =: \tilde{A} \Rightarrow S_{\text{SUSY}}[\tilde{A}, \tilde{\lambda}, \tilde{c}, \tilde{\bar{c}}] = \frac{1}{g^2} \int dx \text{tr} \left\{ -\frac{1}{4} \tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu} - \frac{1}{2\xi} \mathcal{G}(\tilde{A})^2 - \frac{i}{2} \tilde{\lambda} \tilde{D} \tilde{\lambda} + \sqrt{g} \tilde{c} \frac{\partial \mathcal{G}}{\partial \tilde{A}_\mu} \tilde{D}_\mu \tilde{c} \right\} \quad (34)$$

where the tilded quantities are g -independent (or evaluated at $g=1$). Since the g -derivative now is proportional to the action itself,³ we can use off-shell supersymmetry (only in $D \leq 4$ though) to obtain

$$\partial_g S_{\text{SUSY}} = -\frac{1}{g^3} \left\{ \delta_\alpha \tilde{\Delta}_\alpha - \sqrt{g} s \tilde{\Delta}_{\text{gh}} \right\} \quad (35)$$

where

$$\tilde{\Delta}_\alpha = -\frac{1}{2r} \int \text{tr} (\gamma^{\mu\nu} \tilde{\lambda})_\alpha \tilde{F}_{\mu\nu} \quad \text{and} \quad \tilde{\Delta}_{\text{gh}} = \int \text{tr} \tilde{c} \mathcal{G}(\tilde{A}). \quad (36)$$

³ except for the ghost term, which has to be scaled non-canonically

Now we may proceed using broken-supersymmetry and BRST Ward identities to get

$$\partial_g \langle Y[\tilde{A}] \rangle_g = \langle (\partial_g + \tilde{R}_g[\tilde{A}]) Y[\tilde{A}] \rangle_g \quad (37)$$

where

$$\tilde{R}_g = -i \overleftarrow{\Delta}_\alpha \delta_\alpha + \frac{i}{\sqrt{g}} \overleftarrow{\Delta}_{gh} s - \frac{1}{\sqrt{g}} \overleftarrow{\Delta}_\alpha (\delta_\alpha \overleftarrow{\Delta}_{gh}) s. \quad (38)$$

Yet, in this version, we cannot expand around $g=0$ but for perturbation theory must scale back to

$$A = \frac{1}{g} \tilde{A} \quad \Rightarrow \quad R_g[A] = \frac{1}{g} (\tilde{R}_g[\tilde{A}] - f \tilde{A} \frac{\delta}{\delta \tilde{A}}). \quad (39)$$

Note that $\tilde{R}_g[gA] \neq g R_g[A]$ but contains an Euler operator w.r.t. A . This is crucial to remove the formal $g \rightarrow 0$ singularity in the above expression, so that in fact $\lim_{g \rightarrow 0} R_g$ is finite. We can give an explicit expression for any gauge but limited to $D \leq 4$:

$$\overleftarrow{R}_g[A] = \frac{1}{2r} \int \text{tr} \frac{\overleftarrow{\delta}}{\delta A_\mu} P_\mu^\nu \{ \gamma_\nu \bar{\lambda} \lambda \gamma^{\rho\sigma} A_\rho (A - 2\partial \square^{-1} \partial \cdot A)_\sigma \}_{\alpha\alpha} + \int \text{tr} \frac{\overleftarrow{\delta}}{\delta A_\mu} A_\mu \square^{-1} \partial \cdot A + O(\mathcal{G}) \quad (40)$$

with the non-Abelian transversal projector

$$P_\mu^\nu = \delta_\mu^\nu \mathbb{1} - D_\mu c \bar{c} \frac{\partial \mathcal{G}}{\partial A_\nu} \quad \Rightarrow \quad \frac{\partial \mathcal{G}}{\partial A_\mu} P_\mu^\nu = 0 = P_\mu^\nu D_\nu \quad (41)$$

forcing the flow onto the gauge surface: $R_g \mathcal{G} \sim \mathcal{G}$. For the Landau gauge, $\mathcal{G} = \partial \cdot A$, all expressions simplify considerably. We have reversed the direction of the derivatives since acting towards the left is more convenient for the graphical representation.

So the upshot of both versions A and B is that our explicit construction formula (15) carries over to gauge theory, for $D \leq 4$ in any gauge and for $D=6$ and 10 in the Landau gauge,

$$T_g A = \mathcal{P} \exp \left\{ - \int_0^g dh R_h[A] \right\} A = \sum_n g^n c_n r_{n_s}[A] \dots r_{n_2}[A] r_{n_1}[A] A \quad (42)$$

from a decomposition into homogeneous pieces

$$R_g[A] = r_1[A] + g r_2[A] + g^2 r_3[A] + \dots \quad \text{with} \quad \int A \frac{\delta}{\delta A} r_k[A] = k r_k[A]. \quad (43)$$

Let us finally look at the diagrammatics in the Landau gauge [9]. With the solid line representing the free fermion propagator $(i\partial)^{-1}$ and the dashed line standing for the free ghost propagator \square^{-1} , we obtain the tree expansion

$$\begin{aligned} \overleftarrow{R}_g &= \left(\text{solid line} \right) + g \left(\text{solid line with wavy line} \right) + g^2 \left(\text{solid line with two wavy lines} + \text{solid line with dashed line and wavy line} \right) \\ &+ g^3 \left(\text{solid line with three wavy lines} + \text{solid line with dashed line and two wavy lines} + \text{solid line with dashed line and wavy line and wavy line} \right) + O(g^4) \end{aligned}$$

Iterating this in the universal formula (42) produces (with rules analogous to the scalar case)

$$\begin{aligned} T_g A &= \text{solid line} - g \left(\text{solid line with wavy line} \right) - \frac{1}{2} g^2 \left(\text{solid line with two wavy lines} - \text{solid line with dashed line and wavy line} \right) \\ &- \frac{1}{6} g^3 \left(2 \text{solid line with three wavy lines} - \text{solid line with dashed line and two wavy lines} - 2 \text{solid line with dashed line and wavy line and wavy line} \right. \\ &\quad \left. - \text{solid line with wavy line and wavy line} + \text{solid line with dashed line and wavy line and wavy line} + 2 \text{solid line with dashed line and wavy line} \right) + O(g^4) \end{aligned}$$

where the color structure follows the graphical one, and we have suppressed the Lorentz and spinor indices. In fact, performing the spin traces creates various contractions of Lorentz indices on the gauge-field legs and on the propagators due to $(i\not{\partial})^{-1} = i\gamma^\mu \partial_\mu \square^{-1}$, so that the number of terms at $O(g^n)$ grows rapidly with n , namely 1, 3, 34, 344, Nevertheless, the expansion is algorithmic and may be implemented on a computer. Explicit computations were performed to order g^3 in [10] and to order g^4 in [13]. For first evaluations of correlators, see [6, 15].

3. Application to the supermembrane

In the last part of this talk I would like to describe a recent application [16] of the Nicolai map towards a quantization of the maximal supersymmetric membrane, an outstanding unsolved problem.⁴ The $D=11$ supermembrane [17] can be obtained as an $N \rightarrow \infty$ limit of a maximally supersymmetric (so-called BFSS) matrix model [18–22]. More concretely, in a Minkowski background in the light-cone gauge, the supermembrane can be viewed as a one-dimensional gauge theory of area-preserving diffeomorphisms (APD), which is regularized by the $SU(N)$ BFSS matrix model. This matrix model arises also in two other ways. Firstly, it can be seen as the worldline theory of a large number of $D0$ -branes in type IIA string theory (the double dimensional reduction of the supermembrane). Secondly, it appears as the Kaluza–Klein compactification of super Yang–Mills theory from 1+9 to 1+0 dimensions. The Yang–Mills, matrix-model and APD coupling g can be seen to be proportional to the membrane tension T , which combines the two key parameters of string theory via $T = g_s^{-2/3}(\alpha')^{-1}$. Hence, a perturbative quantization of the BFSS matrix model (in powers of g) can serve as a low- T expansion of the quantum supermembrane. Here, we attempt to set this up via the Nicolai map, by dimensionally reducing its $D=10$ $SU(N)$ super Yang–Mills version to a map for the matrix quantum mechanics and finally (in the $N \rightarrow \infty$ limit) for the APD gauge theory.

The Nicolai map for super Yang–Mills theory was described in the previous section. Let us allow for $D = 3, 4, 6$ or 10 . The dimensional reduction from $\mathbb{R}^{1,D-1}$ to $\mathbb{R}^{1,0}$ effects

$$\partial_\mu \rightarrow (\partial_t, 0), \quad A_\mu^A \rightarrow (\omega^A, X_a^A), \quad \lambda_a^A \rightarrow \theta_a^A, \quad D_\mu \rightarrow (D_t = \partial_t + g\omega \times, gX_a^A \times) \quad (44)$$

where $\mu = (0, a) = (0, 1, \dots, D-1)$, $\alpha = 1, \dots, r$ and $A = 1, \dots, N^2-1$. We use the \times symbol to hide the $SU(N)$ structure constants, as in $(\omega \times)^{AB} \equiv f^{ACB} \omega^C$. The spinor index notation is a bit sloppy here: while λ^A is an $SO(D)$ Majorana spinor, the $SO(D-1)$ Majorana θ^A has only half as many components (the other half gets projected out). The non-dynamical Lagrange multiplier ω^A enforces the Gauß constraint. The Lorenz gauge simplifies to

$$\mathcal{G}(A) = \partial \cdot A \quad \longrightarrow \quad \mathcal{G}(\omega) = \dot{\omega} \equiv \partial_t \omega = D_t \omega, \quad (45)$$

thus $\mathcal{G}=0$ forces ω to be constant in time. Interestingly, the reduced temporal gauge $\omega=0$ implies the reduced Lorenz gauge $\dot{\omega}=0$. Hiding color, Lorentz and spin indices, and integrating out the auxiliary D field, the Yang–Mills lagrangian reduces as follows,

$$\begin{aligned} \mathcal{L}_{\text{YM}} &= -\frac{1}{4}F^2 - \frac{1}{2\xi}\mathcal{G}(A)^2 - \frac{i}{2}\bar{\lambda}\not{D}\lambda + \bar{c}\frac{\partial\mathcal{G}}{\partial A}Dc \quad \longrightarrow \\ \mathcal{L}_{\text{MM}} &= \frac{1}{2}(D_t X)^2 - \frac{1}{4}g^2(X \times X)^2 - \frac{i}{2}\theta(D_t + g\gamma \cdot X \times)\theta - \frac{1}{2\xi}\dot{\omega}^2 + \bar{c}\partial_t D_t c. \end{aligned} \quad (46)$$

⁴ See also N. Nicolai’s talk at the Humboldt Kolleg on Quantum Gravity and Fundamental Interactions, which was part of the same Corfu Summer Institute 2021.

To construct the coupling flow operator for the matrix model, we may either employ version A of the previous section or directly dimensionally reduce the Yang–Mills flow operator already given there. Either way, one arrives at

$$\overleftarrow{R}_g = -\frac{1}{r} \iiint \overleftarrow{\frac{\delta}{\delta X_a}} \left[\underbrace{(\gamma_a \mathbb{1} - g X_a \times D_t^{-1})}_{\text{from } P_\mu^\nu} \theta \theta \underbrace{\left(\frac{1}{2} \gamma^{cd} X_c \times X_d + \gamma^d \omega \times X_d \right)}_{\text{from } \gamma^{\rho\sigma} A_\rho A_\sigma} \right]_{\alpha\alpha} \quad (47)$$

where the Euclidean indices $a, b, \dots = 1, \dots, D-1$ and the spin trace $[\dots]_{\alpha\alpha}$ have been exhibited but color and the temporal argument in $X_a^A(t)$ are suppressed. This operator is to be iterated on X to yield $(T_g X)_a^A(t)$. Since no $\frac{\delta}{\delta \omega}$ appears, $R_g \omega = 0$, and hence $T_g \omega = \omega$ respects the gauge slice. For simplicity, we pass to the temporal subgauge $\omega \equiv 0$. Then, only odd powers of g show up in the perturbative expansion of R_g .⁵

With a solid line now depicting the one-dimensional propagator $\partial_t^{-1} = \frac{1}{2} \text{sgn}(t) =: \epsilon(t)$ up to a constant (and a linear term in case of a zero mode on a circle), the diagrammatical expansion of the flow operator reads

$$\overleftarrow{R}_g = g \left[\text{diagram 1} \right] + g^3 \left\{ \text{diagram 2} + \text{diagram 3} + \text{diagram 4} + \text{diagram 5} + \text{diagram 6} + \text{diagram 7} + \text{diagram 8} \right\} + O(g^5)$$

giving rise to the branched-tree expansion

$$T_g X_a = X_a - \frac{1}{2} g^2 \left[\text{diagram 1} \right] + \frac{1}{8} g^4 \left\{ 2 \left[\text{diagram 2} \right] + 2 \left[\text{diagram 3} \right] + 2 \left[\text{diagram 4} \right] + \left[\text{diagram 5} \right] + \left[\text{diagram 6} \right] - \left[\text{diagram 7} \right] + 2 \left[\text{diagram 8} \right] - 2 \left[\text{diagram 9} \right] \right\} + \dots$$

Remarkably, this expression passes all tests. The free-action condition is met for any value of D ,

$$\frac{1}{2} (T_g X)^{\cdot 2} = \frac{1}{2} \dot{X}^{\cdot 2} + \sum_{2 \text{ trees}} g^\# \left[\text{diagram 1} \right] \stackrel{!}{=} \frac{1}{2} \dot{X}^{\cdot 2} - \frac{g^2}{4} \left[\text{diagram 2} \right]$$

where # stands for the various g -powers in the sum. It is nontrivial that all but one term cancel in the infinite sum over double trees. The determinant matching, in contrast, works only for $D \in \{3, 4, 6, 10\}$,

$$\sum_{\text{trees}} \sum_{\text{contractions}} g^\# \left[\text{diagram 1} \right] = \sum_{\text{loop+trees}} g^\# \left[\text{diagram 2} \right] \stackrel{!}{=} \sum_{\text{loop}} g^\# \left[\text{diagram 3} \right]$$

Here, it is amazing that (with the help of the Jacobi identity) all loops with trees attached cancel out, leaving only the standard one-loop graphs.

⁵ At least up to eighth order, where a nonzero contribution $\sim (\gamma_{a_1} \dots \gamma_{a_9})_{\alpha\alpha} \sim \epsilon_{a_1 \dots a_9}$ seems possible.

The $N \rightarrow \infty$ limit leads to the area-preserving-diffeomorphism (APD) gauge theory,

$$X_a^A(t) \rightarrow X_a(\vec{\sigma}, t) \quad \text{and} \quad f^{ABC} \rightarrow \int d^2\sigma \sqrt{w(\vec{\sigma})} Y^A(\vec{\sigma}) \{Y^B(\vec{\sigma}), Y^C(\vec{\sigma})\}, \quad (48)$$

with membrane coordinates $\vec{\sigma} = (\sigma^1, \sigma^2)$, a complete orthonormal basis $\{Y^A(\vec{\sigma})\}$ of functions on the membrane, and an irrelevant reference density $w(\vec{\sigma})$, which cancels when inserting the APD bracket

$$\{A(\vec{\sigma}), B(\vec{\sigma})\} = \frac{1}{\sqrt{w(\vec{\sigma})}} (\partial_{\sigma^1} A(\vec{\sigma}) \partial_{\sigma^2} B(\vec{\sigma}) - \partial_{\sigma^2} A(\vec{\sigma}) \partial_{\sigma^1} B(\vec{\sigma})). \quad (49)$$

Using the Y basis, the $N \rightarrow \infty$ limit of the color summation is converted into an integral over $\vec{\sigma}$, and the matrix interaction gets encoded in the APD bracket, e.g.

$$f^{ABC} X_b^B(t) X_c^C(t) \rightarrow \{X_b(\vec{\sigma}, t), X_c(\vec{\sigma}, t)\}. \quad (50)$$

This limit carries some subtleties. In particular, the APD bracket produces derivative (in σ) interactions, which may require a point-splitting regularization. In contrast, the absence of σ derivatives in the quadratic part of the APD action renders the latter ultralocal. This leads to singular $\delta(\vec{\sigma} - \vec{\sigma}')$ factors in the fermion determinant which, however, cancel against like factors in the Jacobian of the Nicolai map. Therefore, the map remains well-defined in the large- N limit because supersymmetry reigns!⁶ More annoyingly, when the Nicolai map is employed in the perturbative computation of APD correlators (e.g. for membrane vertex operators), the ultralocal free propagator will lead to singularities $\sim \delta(\vec{0})^{-1}$.⁷ This suggests that a partial resummation is needed to pass from a worldline propagator to a membrane world-volume propagator, in analogy with a geometric sum over mass insertions to shift from a massless propagator to a massive one. In APD language and suppressing the common $\vec{\sigma}$ arguments, the $N \rightarrow \infty$ limit of the above Nicolai map takes the form

$$\begin{aligned} T_g X_a(t) = & X_a(t) - \frac{1}{2} g^2 \iint ds du \varepsilon(t-s) \varepsilon(s-u) \{X_b(s), \{X_b(u), X_a(u)\}\} \\ & + \frac{1}{8} g^4 \iiint ds du dv dw \varepsilon(t-s) \varepsilon(s-u) \varepsilon(u-v) \varepsilon(v-w) \\ & \times \left[6 \left\{ X_b(s), \left\{ X_c(u), \left\{ X_{[a}(v), \{X_b(w), X_{c]}(w)\} \right\} \right\} \right\} \right. \\ & + 2 \left\{ X_b(s), \left\{ X_{[b}(u), \left\{ X_{|c|}(v), \{X_{a]}(w), X_c(w)\} \right\} \right\} \right\} \\ & \left. + 2 \left\{ X_a(s) - X_a(t), \left\{ X_b(u), \left\{ X_c(v), \{X_b(w), X_c(w)\} \right\} \right\} \right\} \right] \\ & + \frac{1}{8} g^4 \iiint ds du dv dw \varepsilon(t-s) \varepsilon(s-u) \varepsilon(s-v) \varepsilon(v-w) \\ & \times \left\{ \left\{ X_a(u), X_b(u) \right\}, \left\{ X_c(v), \left\{ X_b(w), X_c(w) \right\} \right\} \right\} + O(g^6). \end{aligned} \quad (51)$$

By computer, this expression can easily be continued to any desired order in the coupling.

⁶ Conversely, it explains why this limit does not exist for the purely bosonic matrix model, and why the bosonic membrane is ‘non-renormalizable’.

⁷ We thank J. Plefka for this remark.

4. Outlook

We have proposed a new angle of attack on the supermembrane, based on the Nicolai map for the APD gauge theory. The perturbative small-tension expansion offers a path to quantization. A distant goal is to establish quantum target-space Lorentz invariance for the supermembrane. Closer in reach appears a computation of physically relevant correlation functions, e.g. of graviton-emission vertex operators [23]

$$V_h[X, \theta; k] = h^{ab} \left[D_t X_a D_t X_b - \{X_a, X_c\} \{X_b, X^c\} - i \bar{\theta} \gamma_a \{X_b, \theta\} - \frac{1}{2} D_t X_a \bar{\theta} \gamma_{bc} \theta k^c - \frac{1}{2} \{X_a, X^c\} \bar{\theta} \gamma_{bcd} \theta k^d + \frac{1}{2} \bar{\theta} \gamma_{ac} \theta \bar{\theta} \gamma_{bd} \theta k^c k^d \right] e^{-i \vec{k} \cdot \vec{X} + i k^- t} \quad (52)$$

with graviton polarization h_{ab} . Another perspective is a control over the convergence of the perturbation series with the help of the universal formula (15) for the map. Puzzling is the special rôle of the Landau gauge for spacetime dimensions beyond four. Finally, it would be marvellous to detect traces of “integrability” for maximally supersymmetric Yang–Mills theory in four dimensions.

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