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Heisenberg Parabolic Subgroup of SO*(8) and Invariant Differential Operators

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In the present paper we continue the project of systematic construction of invariant differential operators on the example of the non-compact algebra $so^*(8) \cong so(6,2)$. We use the maximal Heisenberg parabolic subalgebra $\mathscr{P} = \mathscr{M} \oplus \mathscr{A} \oplus \mathscr{N}$ with $\mathscr{M} = so^*(4) \oplus so(3)$. We give the main multiplets of indecomposable elementary representations. This includes the explicit parametrization of the intertwining differential operators between the ERS.
Due to the recently established parabolic relations the multiplet classification results are valid also for the two algebras $so(p,q)$ (for $(p,q) = (5,3),(4,4)$) with maximal Heisenberg parabolic subalgebra: $\mathscr{P}' = \mathscr{M}' \oplus \mathscr{A}' \oplus \mathscr{N}'$, $\mathscr{M}' = so(p-2,q-2) \oplus sl(2,\mathbb{R})$, $\mathscr{M}'^{\mathbb{C}} \cong \mathscr{M}^{\mathbb{C}}$.

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1. Introduction

Invariant differential operators play very important role in the description of physical symmetries. In recent papers [1], [2] (to which we refer for extensive list of literature on the subject) we started the systematic explicit construction of invariant differential operators. We gave an explicit description of the building blocks, namely, the parabolic subgroups and subalgebras from which the necessary representations are induced. Thus we have set the stage for study of different non-compact groups.

In the present paper we focus on the algebra $so^*(8)$. The algebras $so^*(2n)$ (for $n \geq 4$) form a class of Lie algebras which have maximal Heisenberg parabolic subalgebras. The latter are given as: $\mathscr{P} = \mathscr{M} \oplus \mathscr{A} \oplus \mathscr{N}$, where $\mathscr{M} = so^*(2n-4) \oplus so(3)$.

We note that there are low rank level coincidences: $so^*(4) \cong so(3) \oplus so(2,1)$, $so^*(6) \cong su(3,1)$, which are well studied or with different parabolics, cf. e.g., [3]. Thus, the first non-trivial case of this class is $so^*(8)$.

2. Preliminaries

Let G be a semisimple non-compact Lie group, and K a maximal compact subgroup of G. Then we have an Iwasawa decomposition $G = KA_0N_0$, where A_0 is abelian simply connected vector subgroup of G, N_0 is a nilpotent simply connected subgroup of G preserved by the action of A_0. Further, let M_0 be the centralizer of A_0 in K. Then the subgroup $P_0 = M_0A_0N_0$ is a minimal parabolic subgroup of G. A parabolic subgroup $P = MAN$ is any subgroup of G which contains a minimal parabolic subgroup.

The importance of the parabolic subgroups comes from the fact that the representations induced from them generate all (admissible) irreducible representations of G [4–6].

Let ν be a (non-unitary) character of A, $\nu \in \mathscr{A}^*$, let μ fix an irreducible representation D^μ of M on a vector space V_μ.

We call the induced representation $\chi = \mathrm{Ind}_P^G(\mu \otimes \nu \otimes 1)$ an *elementary representation* of G [7]. Their spaces of functions are:

$$\mathscr{C}_\chi = \{\mathscr{F} \in C^\infty(G, V_\mu) \mid \mathscr{F}(gman) = e^{-\nu(H)} \cdot D^\mu(m^{-1})\,\mathscr{F}(g)\} \tag{2.1}$$

where $a = \exp(H) \in A$, $H \in \mathscr{A}$, $m \in M$, $n \in N$. The representation action is the $left$ regular action:

$$(\mathscr{T}^\chi(g)\mathscr{F})(g') = \mathscr{F}(g^{-1}g'), \quad g, g' \in G. \tag{2.2}$$

For our purposes we need to restrict to *maximal* parabolic subgroups P, so that $\mathrm{rank}A = 1$. Thus, for our representations the character ν is parameterized by a real number d, called the conformal weight or energy.

An important ingredient in our considerations are the *highest/lowest weight representations* of $\mathscr{G}$. These can be realized as (factor-modules of) Verma modules V^Λ over $\mathscr{G}^\mathbb{C}$, where $\Lambda \in (\mathscr{H}^\mathbb{C})^*$, $\mathscr{H}^\mathbb{C}$ is a Cartan subalgebra of $\mathscr{G}^\mathbb{C}$, weight $\Lambda = \Lambda(\chi)$ is determined uniquely from χ [8, 9].

Actually, since our ERs will be induced from finite-dimensional representations of $\mathscr{M}$ (or their limits) the Verma modules are always reducible. Thus, it is more convenient to use *generalized Verma modules* $\tilde{V}^\Lambda$ such that the role of the highest/lowest weight vector v_0 is taken by the space $V_\mu v_0$. For the generalized Verma modules (GVMs) the reducibility is controlled only by the value of the conformal weight d. Relatedly, for the intertwining differential operators only the reducibility w.r.t. non-compact roots is essential.

One main ingredient of our approach is as follows. We group the (reducible) ERs with the same Casimirs in sets called *multiplets* [9, 10]. The multiplet corresponding to fixed values of the Casimirs may be depicted as a connected graph, the vertices of which correspond to the reducible ERs and the lines between the vertices correspond to intertwining operators. The explicit parametrization of the multiplets and of their ERs is important for understanding of the situation.

In fact, the multiplets contain explicitly all the data necessary to construct the intertwining differential operators. Actually, the data for each intertwining differential operator consists of the pair (β, m), where β is a (non-compact) positive root of $\mathcal{G}^{\mathbb{C}}$, $m \in I\!N$, such that the BGG [11] Verma module reducibility condition (for highest weight modules) is fulfilled:

$$(\Lambda+\rho, \beta^\vee) \;=\; m\,, \quad \beta^\vee \equiv 2\beta/(\beta,\beta)\,. \tag{2.3}$$

When (2.3) holds then the Verma module with shifted weight $V^{\Lambda-m\beta}$ (or $\tilde{V}^{\Lambda-m\beta}$ for GVM and β non-compact) is embedded in the Verma module V^Λ (or $\tilde{V}^\Lambda$). This embedding is realized by a singular vector v_s determined by a polynomial $\mathcal{P}_{m,\beta}(\mathcal{G}^-)$ in the universal enveloping algebra $(U(\mathcal{G}_-))\, v_0$, $\mathcal{G}^-$ is the subalgebra of $\mathcal{G}^{\mathbb{C}}$ generated by the negative root generators [12]. More explicitly, [9], $v^s_{m,\beta} = \mathcal{P}^m_\beta\, v_0$ (or $v^s_{m,\beta} = \mathcal{P}^m_\beta\, V_\mu\, v_0$ for GVMs). Then there exists [9] an intertwining differential operator

$$\mathcal{D}^m_\beta \;:\; \mathcal{C}_{\chi(\Lambda)} \;\longrightarrow\; \mathcal{C}_{\chi(\Lambda-m\beta)} \tag{2.4}$$

given explicitly by:

$$\mathcal{D}^m_\beta \;=\; \mathcal{P}^m_\beta(\widehat{\mathcal{G}^-}) \tag{2.5}$$

where $\widehat{\mathcal{G}^-}$ denotes the *right* action on the functions $\mathcal{F}$, cf. (2.1).

3. The non-compact Lie algebra $so^*(8)$

3.1 The general case of $so^*(2n)$

The group $G = SO^*(2n)$ $(n \geq 2)$ consists of all matrices in $SO(2n,\mathbb{C})$ which commute with a real skew-symmetric matrix times the complex conjugation operator C :

$$SO^*(2n) \doteq \{\, g \in SO(2n,\mathbb{C}) \mid J_n C g = g J_n C \} \tag{3.1}$$

The Lie algebra $\mathcal{G} = so^*(2n)$ is given by:

$$\begin{aligned} so^*(2n) \;&\doteq\; \{\, X \in so(2n,\mathbb{C}) \mid J_n C X = X J_n C \} = \\ &= \{\, X = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \mid\; a,b \in gl(n,\mathbb{C}),\;\; {}^t a = -a,\;\; b^\dagger = b \,\}\,. \end{aligned} \tag{3.2}$$

$\dim_R \mathcal{G} = n(2n-1)$, rank$\,\mathcal{G} = n$.

The Cartan involution is given by: $\Theta X = -X^\dagger$. Thus, $\mathcal{K} \cong u(n)$:

$$\mathcal{K} = \{\, X = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mid\; a,b \in gl(n,\mathbb{C}),\;\; {}^t a = -a = -\bar{a},\;\; b^\dagger = b = \bar{b} \,\}\,. \tag{3.3}$$

The complimentary space $\mathscr{P}$ is given by:

$$\mathscr{P} = \{ X = \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \mid a,b \in gl(n,\mathbb{C}) \ , \ {}^t a = -a = \bar{a}, \ b^\dagger = b = -\bar{b} \ \} \ . \tag{3.4}$$

$\dim_R \mathscr{P} = n(n-1)$.

We need also the root system of $\mathscr{G}^{\mathbb{C}} = so(2n,\mathbb{C})$. The positive roots are given standardly as:

$$\alpha_{ij} \ = \ \varepsilon_i - \varepsilon_j \ , \quad 1 \leq i < j \leq n \ , \tag{3.5a}$$

$$\beta_{ij} \ = \ \varepsilon_i + \varepsilon_j \ , \quad 1 \leq i < j \leq n \tag{3.5b}$$

where ε_i are standard orthonormal basis: $\langle \varepsilon_i, \varepsilon_j \rangle = \delta_{ij}$. We shall need the scalar products of the roots:

$$\langle \alpha_{ij}, \alpha_{k\ell} \rangle \ = \ \delta_{ik} - \delta_{i\ell} - \delta_{jk} + \delta_{j\ell} \tag{3.6a}$$

$$\langle \alpha_{ij}, \beta_{k\ell} \rangle \ = \ \delta_{ik} + \delta_{i\ell} - \delta_{jk} - \delta_{j\ell} \tag{3.6b}$$

$$\langle \beta_{ij}, \beta_{k\ell} \rangle \ = \ \delta_{ik} + \delta_{i\ell} + \delta_{jk} + \delta_{j\ell} \tag{3.6c}$$

Note that the highest root is β_{12}.

The simple roots are:

$$\pi \ = \ \{ \gamma_i = \alpha_{i,i+1}, \ 1 \leq i \leq n-1, \quad \gamma_n = \beta_{n-1,n} \} \tag{3.7}$$

The compact roots w.r.t. the real form $SO^*(2n)$ are α_{ij} - they form (by restriction) the root system of the semisimple part of $\mathscr{K}^{\mathbb{C}}$, namely, $\mathscr{K}_s^{\mathbb{C}} \cong su(n)^{\mathbb{C}} \cong sl(n,\mathbb{C})$, while the roots β_{ij} are $\mathscr{K}$-noncompact.

The split rank is $r \equiv [n/2]$. The minimal parabolics of $SO^*(2n)$ depend on whether n is even or odd and are:

$$\mathscr{M}_0 \ = \ so(3) \oplus \cdots \oplus so(3), \quad r \text{ factors, for } n = 2r \tag{3.8a}$$

$$= \ so(2) \oplus so(3) \oplus \cdots \oplus so(3), \quad r \text{ factors, for } n = 2r+1 \tag{3.8b}$$

The subalgebras $\mathscr{N}_0^{\pm}$ which form the root spaces of the root system $(\mathscr{G}, \mathscr{A}_0)$ are of real dimension $n(n-1) - [n/2]$.

The maximal parabolic subalgebras have $\mathscr{M}$-factors as follows [1]:

$$\mathscr{M}_j^{\max} \ = \ so^*(2n-4j) \oplus su^*(2j) \ , \quad j = 1, \ldots, r \ . \tag{3.9}$$

The $\mathscr{N}^{\pm}$ factors in the maximal parabolic subalgebras have dimensions: $\dim (\mathscr{N}_j^{\pm})^{\max} = j(4n - 6j - 1)$.

The case $j = 1$ is special. In this case we have a maximal Heisenberg parabolic with $\mathscr{M}$-factor:

$$\mathscr{M}^{\max}_{\text{Heisenberg}} \ = \ so^*(2n-4) \oplus so(3) \tag{3.10a}$$

$$\text{rank}\, \mathscr{M}^{\max}_{\text{Heisenberg}} \ = \ n-1 \tag{3.10b}$$

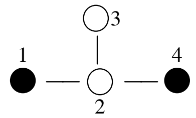
which we use in this paper.

3.2 The case $so^*(8)$

Further we restrict to our case of study $\mathcal{G} = so^*(8) \cong so(6,2)$ with minimal parabolic:

$$\mathcal{M}_0 = so(3) \oplus so(3) \tag{3.11}$$

The Satake-Dynkin diagram of $\mathcal{G}$ is:



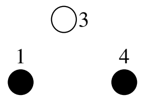
(3.12)

where by standard convention the black dots represent the $so(3)$ subalgebras of $\mathcal{M}_0$.

We shall use the Heisenberg maximal parabolic with $\mathcal{M}$-subalgebra (3.10):

$$\mathcal{M} = so^*(4) \oplus so(3) \cong so(2,1) \oplus so(3) \oplus so(3) \tag{3.13}$$

The Satake-Dynkin diagram of $\mathcal{M}$ is a subdiagram of (3.12):



(3.14)

Note the symmetry

$$\mathcal{R} \,:\, 1 \longleftrightarrow 4 \tag{3.15}$$

of the above diagrams. It will play a role in the representation theory.

From the above follows that the $\mathcal{M}$-compact roots of $\mathcal{G}^{\mathbb{C}}$ are (given in terms of the simple roots):

$$\alpha_{12} = \gamma_1,\ \alpha_{34} = \gamma_3,\ \beta_{34} = \gamma_4 \tag{3.16}$$

By definition the above are the positive roots of $\mathcal{M}^{\mathbb{C}}$.

The positive $\mathcal{M}$-noncompact roots of $\mathcal{G}^{\mathbb{C}}$ in terms of the simple roots are:

$$\gamma_{12} = \gamma_1 + \gamma_2,\ \gamma_{13} = \gamma_1 + \gamma_2 + \gamma_3,\ \gamma_2,\ \gamma_{23} = \gamma_2 + \gamma_3, \tag{3.17a}$$

$$\begin{aligned} &\beta_{12} = \gamma_1 + 2\gamma_2 + \gamma_3 + \gamma_4,\ \beta_{13} = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4, \\ &\beta_{14} = \gamma_1 + \gamma_2 + \gamma_4,\ \beta_{23} = \gamma_2 + \gamma_3 + \gamma_4,\ \beta_{24} = \gamma_2 + \gamma_4 \end{aligned} \tag{3.17b}$$

where for convenience we use the notation $\gamma_{ij} \equiv \alpha_{i,j+1}$ The above set is invariant under the symmetry $\mathcal{R}$: $\gamma_{12} \longleftrightarrow \beta_{24}$, $\gamma_{13} \longleftrightarrow \beta_{23}$, the rest are invariant themselves.

To characterize the Verma modules we shall use first the Dynkin labels:

$$m_i \equiv (\Lambda + \rho, \gamma_i^\vee), \quad i = 1, \ldots, 4, \tag{3.18}$$

where ρ is half the sum of the positive roots of $\mathcal{G}^{\mathbb{C}}$. Thus, we shall use :

$$\chi_\Lambda = \{m_1, m_2, m_3, m_4\} \tag{3.19}$$

Note that when all $m_i \in I\!\!N$ then χ_Λ characterizes the finite-dimensional irreps of $\mathcal{G}^{\mathbb{C}}$ and its real forms, in particular, $so^*(8)$. Furthermore, $m_1, m_3, m_4 \in I\!\!N$ characterizes the finite-dimensional irreps of the $\mathcal{M}$ subalgebra.

The eigenvalue of 2nd order Casimir is given in terms of the Dynkin labels as:

$$\mathscr{C}_2 \;=\; \tfrac{1}{2}(m_1^2+2m_1m_2+m_1m_3+m_1m_4+2m_2^2+2m_2m_3+2m_2m_4+m_3^2+m_3m_4+m_4^2)-7 \quad (3.20)$$

where the normalization is chosen so that the Casimir is zero for the trivial one-dimensional irrep when all $m_i = 1$.

We shall use also the Harish-Chandra parameters:

$$m_{ij} \;=\; (\Lambda+\rho, \gamma_{ij}^{\vee})\,, \quad (3.21a)$$
$$\hat{m}_{ij} \;=\; (\Lambda+\rho, \beta_{ij}^{\vee}) \quad (3.21b)$$

For the $\mathscr{M}$-noncompact roots of $\mathscr{G}^{\mathbb{C}}$ the HC parameters in terms of the Dynkin labels are (compare (3.17)):

$$\begin{aligned} \text{NHCP} \;=\; \{\; & m_{12} = m_1+m_2,\; m_{13} = m_1+m_2+m_3, \\ & m_2,\; m_{23} = m_2+m_3, \\ & \hat{m}_{12} = m_1+2m_2+m_3+m_4, \\ & \hat{m}_{13} = m_1+m_2+m_3+m_4, \\ & \hat{m}_{14} = m_1+m_2+m_4, \\ & \hat{m}_{23} = m_2+m_3+m_4,\; \hat{m}_{24} = m_2+m_4 \;\} \end{aligned} \quad (3.22)$$

The above set is also invariant under $\mathscr{R} \;:\; m_1 \longleftrightarrow m_4$.

4. Main multiplets of SO*(8)

The main multiplets are in 1-to-1 correspondence with the finite-dimensional irreps of $so^*(8)$, i.e., they are labelled by the four positive Dynkin labels $m_i \in I\!N$.

We take $\chi_0 = \chi_\Lambda$, cf. (3.19). It has one embedded Verma module with HW $\Lambda_a = \Lambda_0 - m_2\gamma_2$. The number of ERs/GVMs in a main multiplet is 24. We give the whole multiplet as follows:

$$\begin{aligned}
\chi_0 &= \{m_1, m_2, m_3, m_4\} \\
\chi_a &= \{m_{12}, -m_2, m_{23}, m_{2,4}\}, \quad \Lambda_a = \Lambda_0 - m_2\gamma_2 \\
\chi_b &= \{m_2, -m_{12}, m_{13}, m_{12,4}\}, \quad \Lambda_b = \Lambda_a - m_1\gamma_{12} \\
\chi_c &= \{m_{13}, -m_{23}, m_2, m_{24}\}, \quad \Lambda_c = \Lambda_a - m_3\gamma_{23} \\
\chi_d &= \{m_{12,4}, -m_{2,4}, m_{24}, m_2\}, \quad \Lambda_d = \Lambda_a - m_4\beta_{24} \\
\chi_e &= \{m_{23}, -m_{13}, m_{12}, m_{14}\}, \quad \Lambda_e = \Lambda_c - m_1\gamma_{12} \\
\chi_f &= \{m_{2,4}, -m_{12,4}, m_{14}, m_{12}\}, \quad \Lambda_f = \Lambda_b - m_4\beta_{24} \\
\chi_g &= \{m_{14}, -m_{24}, m_{2,4}, m_{23}\}, \quad \Lambda_g = \Lambda_c - m_4\beta_{24} = \Lambda_d - m_3\gamma_{23} \\
\chi_h &= \{m_{24}, -m_{14}, m_{12,4}, m_{13}\}, \quad \Lambda_h = \Lambda_e - m_4\beta_{24} \\
\chi_i &= \{m_3, -m_{13}, m_1, m_{14,2}\}, \quad \Lambda_i = \Lambda_e - m_2\gamma_{13} \\
\chi_j &= \{m_4, -m_{12,4}, m_{14,2}, m_1\}, \quad \Lambda_j = \Lambda_f - m_2\beta_{14} \\
\chi_k &= \{m_{14,2}, -m_{24}, m_4, m_3\}, \quad \Lambda_k = \Lambda_g - m_2\beta_{23}
\end{aligned} \tag{4.1}$$

$$\begin{aligned}
\chi_k^+ &= \{m_{14,2}, -m_{14}, m_4, m_3\}, \quad \Lambda_k^+ = \Lambda_k - m_1\beta_{12} \\
\chi_j^+ &= \{m_4, -m_{14}, m_{14,2}, m_1\}, \quad \Lambda_j^+ = \Lambda_j - m_3\beta_{12} \\
\chi_i^+ &= \{m_3, -m_{14}, m_1, m_{14,2}\}, \quad \Lambda_i^+ = \Lambda_i - m_4\beta_{12} \\
\chi_h^+ &= \{m_{24}, -m_{14,2}, m_{12,4}, m_{13}\}, \quad \Lambda_h^+ = L_h - m_2\beta_{12} \\
\chi_e^+ &= \{m_{23}, -m_{14,2}, m_{12}, m_{14}\}, \quad \Lambda_e^+ = \Lambda_i^+ - m_2\beta_{24} \\
\chi_f^+ &= \{m_{2,4}, -m_{14,2}, m_{14}, m_{12}\}, \quad \Lambda_f^+ = \Lambda_h^+ - m_3\beta_{14} \\
\chi_g^+ &= \{m_{14}, -m_{14,2}, m_{2,4}, m_{23}\}, \quad \Lambda_g^+ = \Lambda_h^+ - m_1\beta_{23} = \Lambda_k^+ - m_2\gamma_{12} \\
\chi_d^+ &= \{m_{12,4}, -m_{14,2}, m_{24}, m_2\}, \quad \Lambda_d^+ = \Lambda_f^+ - m_1\beta_{23} \\
\chi_c^+ &= \{m_{13}, -m_{14,2}, m_2, m_{24}\}, \quad \Lambda_c^+ = \Lambda_g^+ - m_4\gamma_{13} \\
\chi_b^+ &= \{m_2, -m_{14,2}, m_{13}, m_{12,4}\}, \quad \Lambda_b^+ = \Lambda_e^+ - m_3\beta_{14} = \Lambda_f^+ - m_4\gamma_{13} \\
\chi_a^+ &= \{m_{12}, -m_{14,2}, m_{23}, m_{2,4}\}, \quad \Lambda_a^+ = \Lambda_b^+ - m_1\beta_{23} \\
\chi_0^+ &= \{m_1, -m_{14}, m_3, m_4\}, \quad \Lambda_0^+ = \Lambda_a^+ - m_2\beta_{13}
\end{aligned} \tag{4.2}$$

We shall label the signature of the ERs of $\mathcal{G}$ also as follows:

$$\chi = [n_1, n_2, n_3; c], \quad c = -\tfrac{1}{2}m_{14,2}, \quad n_1 = m_1,\ n_2 = m_3,\ n_3 = m_4, \tag{4.3}$$

where the last entry labels the characters of $\mathcal{A}$, the first three entries of χ are labels of the finite-dimensional irreps of $\mathcal{M}$ when all $n_j > 0$ or limits of the latter when some $n_j = 0$. Note that $m_{14,2} = m_1 + 2m_2 + m_3 + m_4$ is the Harish-Chandra parameter for the highest root β_{12}.

Using this labelling signatures may be given in the following pair-wise manner:

$$\begin{aligned}
\chi_0^\pm &= [m_1, m_3, m_4\,;\, \pm\tfrac{1}{2} m_{14,2}] \\
\chi_a^\pm &= [m_{12}, m_{23}, m_{2,4}\,;\, \pm\tfrac{1}{2} m_{14}], \\
\chi_b^\pm &= [m_2, m_{13}, m_{12,4}\,;\, \pm\tfrac{1}{2} m_{24}], \\
\chi_c^\pm &= [m_{13}, m_2, m_{24}\,;\, \pm\tfrac{1}{2} m_{12,4}], \\
\chi_d^\pm &= [m_{12,4}, m_{24}, m_2\,;\, \pm\tfrac{1}{2} m_{13}], \\
\chi_e^\pm &= [m_{23}, m_{12}, m_{14}\,;\, \pm\tfrac{1}{2} m_{2,4}], \\
\chi_f^\pm &= [m_{2,4}, m_{14}, m_{12}\,;\, \pm\tfrac{1}{2} m_{23}], \\
\chi_g^\pm &= [m_{14}, m_{2,4}, m_{23}\,;\, \pm\tfrac{1}{2} m_{12}], \\
\chi_h^\pm &= [m_{24}, m_{12,4}, m_{13}\,;\, \pm\tfrac{1}{2} m_2], \\
\chi_i^\pm &= [m_3, m_1, m_{14,2}\,;\, \pm\tfrac{1}{2} m_4], \\
\chi_j^\pm &= [m_4, m_{14,2}, m_1\,;\, \pm\tfrac{1}{2} m_3], \\
\chi_k^\pm &= [m_{14,2}, m_4, m_3\,;\, \pm\tfrac{1}{2} m_1]
\end{aligned} \tag{4.4}$$

We note the symmetry of the ERs under the operation $\mathscr{R}$ (3.15) which involves here the change $m_1 \longleftrightarrow m_4$ and also the interchange of the 1st and 3rd entries in the signatures:

$$\chi_b^\pm \longleftrightarrow \chi_d^\pm\ , \quad \chi_e^\pm \longleftrightarrow \chi_g^\pm\ , \quad \chi_i^\pm \longleftrightarrow \chi_k^\pm \tag{4.5}$$

the other signatures are self-conjugate.

The ERs in the multiplet are related also by intertwining integral operators introduced in [14]. These operators are defined for any ER, the general action being:

$$\begin{aligned}
&G_{KS}\ :\ \mathscr{C}_\chi \longrightarrow \mathscr{C}_{\chi'}\ , \\
&\chi = [n_1, n_2, n_3\,;\, c]\ , \qquad \chi' = [n_1, n_2, n_3\,;\, -c].
\end{aligned} \tag{4.6}$$

The main multiplets are given explicitly in Fig. 1. We use the notation: $\Lambda^\pm = \Lambda(\chi^\pm)$. Each intertwining differential operator is represented by an arrow accompanied either by a symbol i_{jk} encoding the root γ_{jk} and the number $m_{\gamma_{jk}}$ which is involved in the BGG criterion, or a symbol $i_{\widehat{jk}}$ encoding the root β_{jk} and the number $m_{\beta_{jk}}$ from BGG.

In addition pairs $\chi^\pm$ are symmetric w.r.t. to the dashed line in the middle the figure - this represents the Weyl symmetry realized by the Knapp-Stein operators (4.6): $G_{KS}\ :\ \mathscr{C}_{\chi^\mp} \longleftrightarrow \mathscr{C}_{\chi^\pm}$.

Some comments are in order.

Matters are arranged so that in every multiplet only the ER with signature χ_0^- contains a finite-dimensional nonunitary representation in a finite-dimensional subspace $\mathscr{E}$. The latter corresponds to the finite-dimensional irrep of $so^*(8)$ with signature $[m_1, \ldots m_4]$. The subspace $\mathscr{E}$ is annihilated by the operator G^+, and is the image of the operator G^-. The subspace $\mathscr{E}$ is annihilated also by the intertwining differential operator $\mathscr{F}^{m_2}_{\gamma_2}$ acting from χ_0^- to χ_a^-. When all $m_i = 1$ then $\dim \mathscr{E} = 1$, and in that case $\mathscr{E}$ is also the trivial one-dimensional UIR of the whole algebra $\mathscr{G}$. Furthermore in that case the conformal weight is zero: $d = \frac{5}{2} + c = \frac{5}{2} - \frac{1}{2}(m_1 + 2m_2 + m_3 + m_4)_{|_{m_i=1}} = 0$.

In the conjugate ER χ_0^+ there is a unitary subrepresentation in an infinite-dimensional subspace $\mathscr{D}$ with conformal weight $d = \frac{5}{2} + c = \frac{5}{2} + \frac{1}{2}(m_1 + 2m_2 + m_3 + m_4)$. It is annihilated by the operator G^-, and is in the image of the operators G^+ acting from χ_0^- and $\mathscr{D}^{m_2}_{\widehat{13}}$ acting from χ_a^+.

Finally, we remind that according to [2] the above considerations are applicable also for the algebra $so(p,q)$ (with $p+q=8$, $p \geq q \geq 3$) with maximal Heisenberg parabolic subalgebra: $\mathscr{P}' = \mathscr{M}' \oplus \mathscr{A}' \oplus \mathscr{N}'$, $\mathscr{M}' = so(p-2,q-2) \oplus sl(2,\mathbb{R})$, $\mathscr{M}'^{\mathbb{C}} \cong \mathscr{M}^{\mathbb{C}}$.

5. Reduced multiplets

5.1 Main reduced multiplets

There are four main reduced multiplets M_k, $k = 1,2,3,4$, which may be obtained by setting the parameter $m_k = 0$.

The main reduced multiplet M_1 contains 15 GVMs (ERs), see Fig. 2a. Their signatures are given as follows:

$$\begin{aligned}
\chi_0^\pm &= [0, m_3, m_4\,; \pm\tfrac{1}{2} m_{24,2}] \\
\chi_a^\pm &= [m_2, m_{23}, m_{2,4}\,; \pm\tfrac{1}{2} m_{24}] = \chi_b^\pm, \\
\chi_e^\pm &= [m_{23}, m_2, m_{24}\,; \pm\tfrac{1}{2} m_{2,4}] = \chi_c^\pm, \\
\chi_f^\pm &= [m_{2,4}, m_{24}, m_2\,; \pm\tfrac{1}{2} m_{23}] = \chi_d^\pm, \\
\chi_h^\pm &= [m_{24}, m_{2,4}, m_{23}\,; \pm\tfrac{1}{2} m_2] = \chi_g^\pm, \\
\chi_i^\pm &= [m_3, 0, m_{24,2}\,; \pm\tfrac{1}{2} m_4], \\
\chi_j^\pm &= [m_4, m_{24,2}, 0\,; \pm\tfrac{1}{2} m_3], \\
\chi_k &= [m_{24,2}, m_4, m_3\,; 0]
\end{aligned} \tag{5.1}$$

Note that some of the inducing representations, namely, $\chi_0^\pm$, $\chi_i^\pm$, $\chi_j^\pm$, are limits of $\mathscr{M}$ representations, while the rest are finite-dimensional IRs (as in the main multiplets). We shall call the latter *relevant* ERs.

The main reduced multiplet M_2 contains 15 GVMs (ERs) with signatures given as follows, see Fig. 2b:

$$\begin{aligned}
\chi_a^\pm &= [m_1, m_3, m_4\,; \pm\tfrac{1}{2} m_{1,34}] = \chi_0^\pm, \\
\chi_b^\pm &= [0, m_{1,3}, m_{1,4}\,; \pm\tfrac{1}{2} m_{34}], \\
\chi_c^\pm &= [m_{1,3}, 0, m_{34}\,; \pm\tfrac{1}{2} m_{1,4}], \\
\chi_d^\pm &= [m_{1,4}, m_{34}, 0\,; \pm\tfrac{1}{2} m_{1,3}], \\
\chi_e^\pm &= [m_3, m_1, m_{1,34}\,; \pm\tfrac{1}{2} m_4] = \chi_i^\pm, \\
\chi_f^\pm &= [m_4, m_{1,34}, m_1\,; \pm\tfrac{1}{2} m_3] = \chi_j^\pm, \\
\chi_g^\pm &= [m_{1,34}, m_4, m_3\,; \pm\tfrac{1}{2} m_1] = \chi_k^\pm, \\
\chi_h &= [m_{34}, m_{1,4}, m_{1,3}\,; 0],
\end{aligned} \tag{5.2}$$

Note that the ER $\chi_0^+ = \chi_a^+$ is not only relevant but also contains a limit of the representation on $\mathscr{D}$ with conformal weight $d = \frac{5}{2} + \frac{1}{2}(m_1 + m_3 + m_4)$.

The main reduced multiplet M_3 contains 15 GVMs (ERs) with signatures given as follows, see Fig. 2c:

$$\begin{aligned}
\chi_0^\pm &= [m_1, 0, m_4\,;\, \pm\tfrac{1}{2}m_{12,2,4}] \\
\chi_a^\pm &= [m_{12}, m_2, m_{2,4}\,;\, \pm\tfrac{1}{2}m_{12,4}] = \chi_c^\pm\,, \\
\chi_e^\pm &= [m_2, m_{12}, m_{12,4}\,;\, \pm\tfrac{1}{2}m_{2,4}] = \chi_b^\pm\,, \\
\chi_g^\pm &= [m_{12,4}, m_{2,4}, m_2\,;\, \pm\tfrac{1}{2}m_{12}] = \chi_d^\pm\,, \\
\chi_h^\pm &= [m_{2,4}, m_{12,4}, m_{12}\,;\, \pm\tfrac{1}{2}m_2] = \chi_f^\pm\,, \\
\chi_i^\pm &= [0, m_1, m_{12,2,4}\,;\, \pm\tfrac{1}{2}m_4]\,, \\
\chi_j &= [m_4, m_{12,2,4}, m_1\,;\, 0]\,, \\
\chi_k^\pm &= [m_{12,2,4}, m_4, 0\,;\, \pm\tfrac{1}{2}m_1]
\end{aligned} \tag{5.3}$$

The main reduced multiplet M_4 contains 15 GVMs (ERs) with signatures given as follows, see Fig. 2d:

$$\begin{aligned}
\chi_0^\pm &= [m_1, m_3, 0\,;\, \pm\tfrac{1}{2}m_{13,2}] \\
\chi_a^\pm &= [m_{12}, m_{23}, m_2\,;\, \pm\tfrac{1}{2}m_{13}] = \chi_d^\pm\,, \\
\chi_f^\pm &= [m_2, m_{13}, m_{12}\,;\, \pm\tfrac{1}{2}m_{23}] = \chi_b^\pm\,, \\
\chi_g^\pm &= [m_{13}, m_2, m_{23}\,;\, \pm\tfrac{1}{2}m_{12}] = \chi_c^\pm\,, \\
\chi_h^\pm &= [m_{23}, m_{12}, m_{13}\,;\, \pm\tfrac{1}{2}m_2] = \chi_e^\pm\,, \\
\chi_i &= [m_3, m_1, m_{13,2}\,;\, 0]\,, \\
\chi_j^\pm &= [0, m_{13,2}, m_1\,;\, \pm\tfrac{1}{2}m_3]\,, \\
\chi_k^\pm &= [m_{13,2}, 0, m_3\,;\, \pm\tfrac{1}{2}m_1]
\end{aligned} \tag{5.4}$$

Note that in all four main reduced multiplets the ER χ_a^+ contains a minimal irrep with conformal weight $d_{m_i=1} = 5$. The $\mathcal{M}$ inducing representation of the minimal irreps is of dimension 1 in the case M_2, and of dimension 4 in the other three cases.

5.2 Next reduced multiplets

The reduced multiplet M_{12} contains 7 GVMs (ERs) with signatures given as follows, see Fig. 3a:

$$\begin{aligned}
\chi_b^\pm &= [0, m_3, m_4\,;\, \pm\tfrac{1}{2}m_{34}] = \chi_a^\pm = \chi_0^\pm\,, \\
\chi_e^\pm &= [m_3, 0, m_{34}\,;\, \pm\tfrac{1}{2}m_4] = \chi_c^\pm = \chi_i^\pm\,, \\
\chi_f^\pm &= [m_4, m_{34}, 0\,;\, \pm\tfrac{1}{2}m_3] = \chi_d^\pm = \chi_j^\pm\,, \\
\chi_h &= [m_{34}, m_4, m_3\,;\, 0] = \chi_g^\pm = \chi_k\,,
\end{aligned} \tag{5.5}$$

The reduced multiplet M_{13} contains 10 GVMs (ERs) with signatures given as follows, see Fig. 3b:

$$\begin{aligned}
\chi_0^\pm &= [0,0,m_4\,;\pm\tfrac{1}{2}m_{2,2,4}] \\
\chi_a^\pm &= [m_2,m_2,m_{2,4}\,;\pm\tfrac{1}{2}m_{2,4}] = \chi_c^\pm = \chi_b^\pm = \chi_e^\pm\,, \\
\chi_h^\pm &= [m_{2,4},m_{2,4},m_2\,;\pm\tfrac{1}{2}m_2] = \chi_g^\pm = \chi_f^\pm\,, \\
\chi_i^\pm &= [0,0,m_{2,2,4}\,;\pm\tfrac{1}{2}m_4]\,, \\
\chi_j &= [m_4,m_{2,2,4},0\,;0]\,, \\
\chi_k &= [m_{2,2,4},m_4,0\,;0]
\end{aligned} \tag{5.6}$$

The main reduced multiplet M_{14} contains 10 GVMs (ERs) with signatures given as follows, see Fig. 3c:

$$\begin{aligned}
\chi_0^\pm &= [0,m_3,0\,;\pm\tfrac{1}{2}m_{23,2}] \\
\chi_a^\pm &= [m_2,m_{23},m_2\,;\pm\tfrac{1}{2}m_{23}] = \chi_b^\pm = \chi_d^\pm = \chi_f^\pm\,, \\
\chi_h^\pm &= [m_{23},m_2,m_{23}\,;\pm\tfrac{1}{2}m_2] = \chi_e^\pm = \chi_g^\pm\,, \\
\chi_i &= [m_3,0,m_{23,2}\,;0]\,, \\
\chi_j^\pm &= [0,m_{23,2},0\,;\pm\tfrac{1}{2}m_3]\,, \\
\chi_k &= [m_{23,2},0,m_3\,;0]
\end{aligned} \tag{5.7}$$

The reduced multiplet M_{23} contains 7 GVMs (ERs) with signatures given as follows, see Fig. 3d:

$$\begin{aligned}
\chi_c^\pm &= [m_1,0,m_4\,;\pm\tfrac{1}{2}m_{1,4}] = \chi_0^\pm = \chi_a^\pm\,, \\
\chi_e^\pm &= [0,m_1,m_{1,4}\,;\pm\tfrac{1}{2}m_4] = \chi_i^\pm = \chi_b^\pm\,, \\
\chi_h &= [m_4,m_{1,4},m_1\,;0] = \chi_j^\pm = \chi_f\,, \\
\chi_g^\pm &= [m_{1,4},m_4,0\,;\pm\tfrac{1}{2}m_1] = \chi_k^\pm = \chi_d^\pm\,,
\end{aligned} \tag{5.8}$$

The reduced multiplet M_{24} contains 7 GVMs (ERs) with signatures given as follows, see Fig. 3e:

$$\begin{aligned}
\chi_d^\pm &= [m_1,m_3,0\,;\pm\tfrac{1}{2}m_{1,3}] = \chi_a^\pm = \chi_0^\pm\,, \\
\chi_f^\pm &= [0,m_{1,3},m_1\,;\pm\tfrac{1}{2}m_3] = \chi_b^\pm = \chi_j^\pm\,, \\
\chi_g^\pm &= [m_{1,3},0,m_3\,;\pm\tfrac{1}{2}m_1] = \chi_c^\pm = \chi_k^\pm\,, \\
\chi_h &= [m_3,m_1,m_{1,3}\,;0] = \chi_e^\pm = \chi_i\,,
\end{aligned} \tag{5.9}$$

The reduced multiplet M_{34} contains 10 GVMs (ERs) with signatures given as follows, see Fig.

3f:

$$\begin{aligned}
\chi_0^\pm &= [m_1,0,0\,;\pm\tfrac{1}{2}m_{12,2}] \qquad (5.10)\\
\chi_a^\pm &= [m_{12},m_2,m_2\,;\pm\tfrac{1}{2}m_{12}] = \chi_c^\pm = \chi_d^\pm = \chi_g^\pm,\\
\chi_h^\pm &= [m_2,m_{12},m_{12}\,;\pm\tfrac{1}{2}m_2] = \chi_e^\pm = \chi_f^\pm,\\
\chi_i &= [0,m_1,m_{12,2}\,;0],\\
\chi_j &= [0,m_{12,2},m_1\,;0],\\
\chi_k^\pm &= [m_{12,2},0,0\,;\pm\tfrac{1}{2}m_1]
\end{aligned}$$

The minimal irreps with $m_i = 1$ in the singlets χ_h in the multiplets M_{12}, M_{23}, M_{24} have conformal weight $d = \frac{5}{2}$.

The minimal irreps with $m_i = 1$ in the ERs χ_a^+ in the multiplets M_{13}, M_{14}, M_{34} have conformal weight $d = \frac{7}{2}$.

The $\mathscr{M}$ inducing representation of the minimal irreps is of dimension 2 in all cases.

5.3 Third level reduction of multiplets

The reduced multiplet M_{123} contains 3 GVMs (ERs) with signatures given as follows (Fig. 4a):

$$\begin{aligned}
\chi_e^\pm &= [0,0,m_4\,;\pm\tfrac{1}{2}m_4] = \chi_0^\pm = \chi_c^\pm = \chi_i^\pm = \chi_b^\pm, \qquad (5.11)\\
\chi_f &= [m_4,m_4,0\,;0] = \chi_d^\pm = \chi_j^\pm = \chi_h,
\end{aligned}$$

The reduced multiplet M_{124} contains 3 GVMs (ERs) with signatures given as follows (Fig. 4b):

$$\begin{aligned}
\chi_e &= [m_3,0,m_3\,;0] = \chi_0^\pm = \chi_c^\pm = \chi_i^\pm = \chi_h, \qquad (5.12)\\
\chi_f^\pm &= [0,m_3,0\,;\pm\tfrac{1}{2}m_3] = \chi_d^\pm = \chi_j^\pm = \chi_b^\pm,
\end{aligned}$$

The reduced multiplet M_{134} contains 7 GVMs (ERs) with signatures given as follows (Fig. 4c):

$$\begin{aligned}
\chi_0^\pm &= [0,0,0\,;\pm\tfrac{1}{2}m_{2,2}] \qquad (5.13)\\
\chi_a^\pm &= [m_2,m_2,m_2\,;\pm\tfrac{1}{2}m_2] = \chi_c^\pm = \chi_b^\pm = \chi_h^\pm = \chi_e^\pm,\\
\chi_i &= [0,0,m_{2,2}\,;0],\\
\chi_j &= [0,m_{2,2},0\,;0],\\
\chi_k &= [m_{2,2},0,0\,;0]
\end{aligned}$$

Note that all 13 $\mathscr{M}$-noncompact roots are represented via the corresponding intertwining differential operators.
The minimal irrep with $m_2 = 1$ in the ER χ_a^+ has conformal weight $d = 3$.
The $\mathscr{M}$ inducing representation of the minimal irrep is of dimension 1.

The reduced multiplet M_{234} contains 3 GVMs (ERs) with signatures given as follows (Fig. 4d):

$$\begin{aligned}
\chi_c^\pm &= [m_1,0,0\,;\pm\tfrac{1}{2}m_1] = \chi_0^\pm = \chi_a^\pm = \chi_g^\pm, \qquad (5.14)\\
\chi_e &= [0,m_1,m_1\,;0] = \chi_i^\pm = \chi_b^\pm = \chi_h,
\end{aligned}$$

Acknowledgments.

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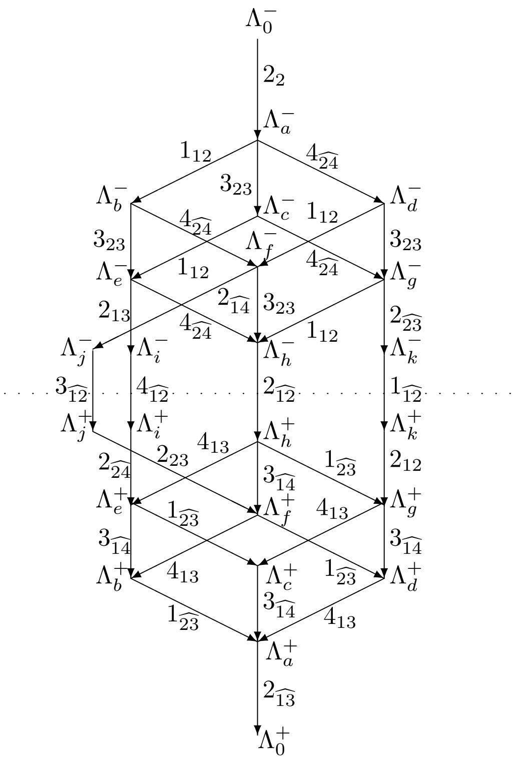

Fig. 1. Main multiplets for $SO^*(8)$
using induction from maximal Heisenberg parabolic

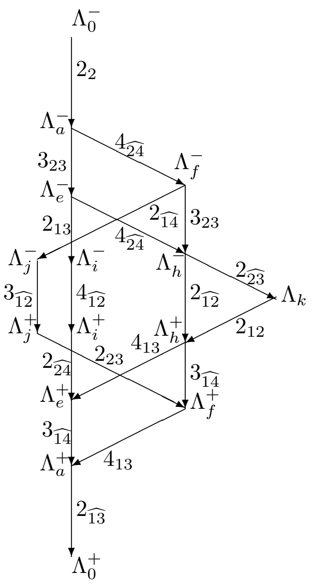

Fig. 2a. Main reduced multiplets M_1 for $SO^*(8)$ using induction from maximal Heisenberg parabolic

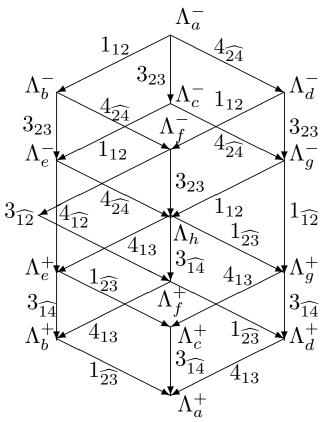

Fig. 2b. Main reduced multiplets M_2 for $SO^*(8)$ using induction from maximal Heisenberg parabolic

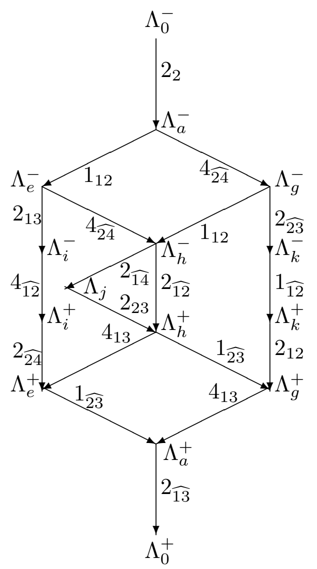

Fig. 2c. Main reduced multiplets M_3 for $SO^*(8)$ using induction from maximal Heisenberg parabolic

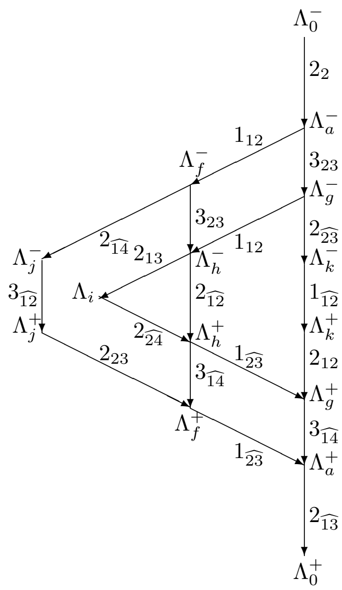

Fig. 2d. Main reduced multiplets M_4 for $SO^*(8)$ using induction from maximal Heisenberg parabolic

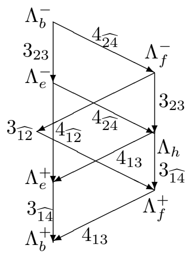

Fig. 3a. Reduced multiplets M_{12} for $SO^*(8)$ using induction from maximal Heisenberg parabolic

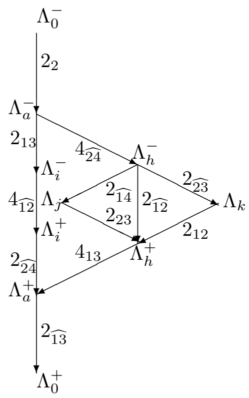

Fig. 3b. Reduced multiplets M_{13} for $SO^*(8)$ using induction from maximal Heisenberg parabolic

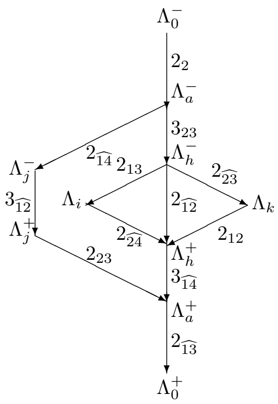

Fig. 3c. Reduced multiplets M_{14} for $SO^*(8)$ using induction from maximal Heisenberg parabolic

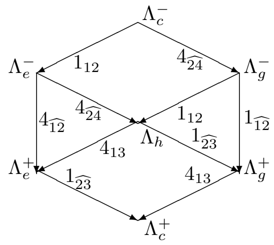

Fig. 3d. Reduced multiplets M_{23} for $SO^*(8)$ using induction from maximal Heisenberg parabolic

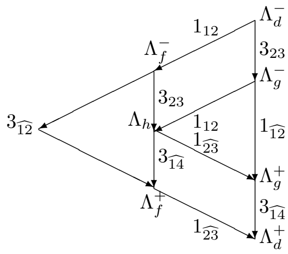

Fig. 3e. Reduced multiplets M_{24} for $SO^*(8)$ using induction from maximal Heisenberg parabolic

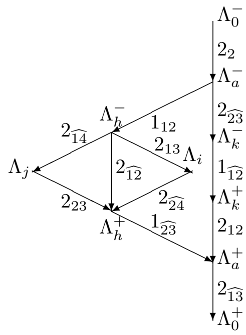

Fig. 3f. Reduced multiplets M_{34} for $SO^*(8)$ using induction from maximal Heisenberg parabolic

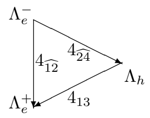

Fig. 4a. Reduced multiplets M_{123} for $SO^*(8)$ using induction from maximal Heisenberg parabolic

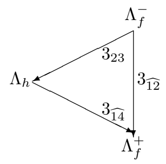

Fig. 4b. Reduced multiplets M_{124} for $SO^*(8)$ using induction from maximal Heisenberg parabolic

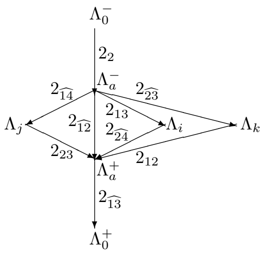

Fig. 4c. Reduced multiplets M_{134} for $SO^*(8)$ using induction from maximal Heisenberg parabolic

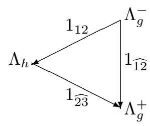

Fig. 4d. Reduced multiplets M_{243} for $SO^*(8)$ using induction from maximal Heisenberg parabolic