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Reconstructing dS_3 with Wilson Lines

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In the Chern-Simons formulation of three-dimensional gravity, Wilson lines probe gravitational physics. Wilson lines for anti- de Sitter spacetime have been shown to reconstruct known gravitational observables such as geodesic lengths, Green's functions and the quasinormal mode spectrum. In these proceedings, we will review recent work generalizing this to de Sitter gravity. We will explain how Wilson lines can reproduce Green's functions for massive scalar fields in Euclidean de Sitter spacetime, and will give Lorentzian versions of these results by performing an analytic continuation. There is a crucial role played by endpoint states which satisfy a singlet condition. The existence of exact results for $su(2)$ Wilson loops from non-abelian localization suggests that the quantization of the Wilson line may be more tractable in de Sitter compared to anti- de Sitter spacetime.

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1. Introduction

Even classical gravity in de Sitter spacetime can be subtle. Quantum gravity in de Sitter spacetime is even more confusing for many reasons. For instance, the finite de Sitter entropy suggests that the quantum description of de Sitter should be encoded by a Hilbert space with only a finite number of states. In the known realization of holography in terms of a higher-spin theory [1], the boundary theory also lives at an instance of *time* instead of at a spatial boundary. The extent to which this should be viewed as a full quantum theory, rather than some statistical description, remains puzzling. This is not to even mention puzzles of the late-time behavior of de Sitter vacua, which generically undergo eternal inflation, leading to a measure problem [2]. Since de Sitter spacetime approximates the current phase of our universe, it is essential that we adapt our toolkit of holography and quantum gravity to this case.

The Chern-Simons formulation of three-dimensional gravity has long been an avenue for tackling questions of quantization of gravity [3, 4]. A natural observable from the Cherns-Simons perspective is a Wilson loop, or if one does not insist on gauge-invariance, a Wilson line. Moving beyond the semi-classical description of a Wilson line in a fixed background, we can also consider the quantum generalization where the metric is allowed to fluctuate. Interestingly, whereas de Sitter gravity has often turned out to be more subtle than its anti- de Sitter counterpart, this is an avenue where it is possible to gain comparatively *more* traction in de Sitter. For instance, EdS$_3$ has isometry algebra $so(4) \simeq su(2) \times su(2)$. In the Chern-Simons formulation, this is the gauge algebra. $su(2)$ is a compact algebra, and there are more mathematical results to draw on than for non-compact algebras. For instance, there exist exact results from non-abelian localization for the Chern-Simons path integral and also the Wilson loop expectation value [5–7]. This can potentially give us a handle on fully exact results and their gravitational duals. In contrast, in the anti- de Sitter case progress in computing the full Wilson line has only been made perturbatively in $1/c$, with c the central charge of the dual conformal field theory [8].

The bulk of these proceedings consists of a pedagogical review of [9]. Many details of the construction can be found in that paper; here the aim is to avoid technicalities in favor of giving a straightforward overview of applications of Wilson lines to de Sitter gravity. This includes shortcuts for readability, along with some additional background. There will also be some allusions to further developments as well as connections with other aspects of de Sitter gravity treated at the workshop. In the spirit of the workshop, it is also interesting to learn what the representations used to build Wilson lines have to tell us about standard representations of $so(d+1,1)$ that can be used to construct fields in de Sitter spacetime (see [10] for a recent review). This, along with the study of the quantum case and its gravitational description, is the subject of work in progress [11].

2. Review of Chern-Simons theory and Wilson lines for AdS$_3$

We will begin with a very brief review of Chern-Simons theory and the role of Wilson lines as gravitational probes for anti- de Sitter (AdS) spacetime. There is a classical equivalence between 3d gravity in AdS and Chern-Simons theory with gauge algebra $so(2,2) \simeq sl(2) \times sl(2)$ [4]. Specifically, consider two copies of the $sl(2)$ Chern-Simons action

$$S = S_{CS}[A] - S_{CS}[\bar{A}] \tag{1}$$

with

$$S_{CS}[A] = \frac{k}{4\pi} \int_{\mathcal{M}} \mathrm{Tr}\left(A \wedge \mathrm{d}A + \frac{2}{3} A \wedge A \wedge A\right) , \tag{2}$$

where $A \in sl(2)_L, \bar{A} \in sl(2)_R$. This is equal to the three-dimensional Einstein-Hilbert gravitational action for AdS with curvature radius ℓ and gravitational constant G_3, given a Chern-Simons coupling k related to these parameters as

$$k = \frac{\ell}{4G_3} . \tag{3}$$

The gauge theory has Wilson loop observables

$$W_{\mathscr{R}}(C) = \mathrm{Tr}_{\mathscr{R}}\left(\mathscr{P}e^{-\oint_C A} \mathscr{P}e^{-\oint \bar{A}}\right) , \tag{4}$$

where C is a closed loop in AdS. The representations $\mathscr{R}$ we consider for gravitational applications are highest-weight $sl(2)$ representations. Given $sl(2)$ generators $J_0, J_\pm$ in the raising and lowering basis, these are constructed by starting from a highest weight state $|h,0\rangle$ satisfying

$$J_+ |h,0\rangle = 0 , \qquad J_0 |h,0\rangle = h |h,0\rangle , \tag{5}$$

with the remaining states in the representation generated through successive action of the lowering operator, $|h,k\rangle \propto (J_-)^k |h,0\rangle$. These representations have a Casimir $c_2 = h(h-1)$.

The Wilson loop can be viewed as the partition function for an auxiliary quantum system associated to the curve C [12]. The Hilbert space for this system is identified with the representation $\mathscr{R}$, and the trace in Eq. (4) is just the usual quantum mechanical trace over the Hilbert space. This partition function can be evaluated semiclassically through a version of the orbit method [6, 7]

$$W_{\mathscr{R}}(x_i, x_f) = \int \mathcal{D}U e^{-S(U,A,\bar{A})_C} . \tag{6}$$

Here U is an auxiliary field associated to the curve C, coupled to the background connections $A, \bar{A}$. The explicit construction of the action for $S(U, A, \bar{A})_C$ for AdS can be found in [13].

It will be useful for us to consider Wilson lines rather than loops. These are not observables in the same sense, as introducing endpoints means they are no longer gauge invariant. Let $|U_i\rangle, |U_f\rangle \in \mathscr{R}$ be endpoint states. Then the Wilson line can be expressed as

$$W_{\mathscr{R}}(x_i, x_f) = \langle U_f | \mathscr{P} e^{-\int_\gamma A} \mathscr{P} e^{-\int_\gamma \bar{A}} |U_i\rangle , \tag{7}$$

where γ is a bulk curve with endpoints x_i, x_f.

The Wilson line has a bulk gravitational interpretation. It probes the physics of a point particle moving from x_i to x_f with

$$m^2 = c_2 + \bar{c}_2 , \qquad s = h - \bar{h} , \qquad c_2 = h(h-1) . \tag{8}$$

On-shell with $U = 1$, the action in Eq. (6) computes the geodesic distance between x_i, x_f [13].

It also has a boundary interpretation, when we take x_i, x_f to the boundary of AdS. Then the Wilson line computes the correlator

$$W_{\mathscr{R}}(x_i, x_f) \underset{x_i, x_f \to \partial \text{AdS}}{=} \langle \psi | O(x_i) O(x_f) | \psi \rangle . \tag{9}$$

Here ψ is a heavy state dual to the gauge fields $A, \bar{A}$ that describe the bulk geometry, and $O(x)$ is a light operator with scaling dimension $(h, \bar{h})$ coming from the representation.

3. Ishibashi states

When considering Wilson lines as opposed to loops, the boundary states $|U_i\rangle, |U_f\rangle$ are an additional input. What is a natural choice for these states? Although there are many possibilities, we consider a choice of states suitable if it reproduces gravitational physics. An appropriate choice is to consider *Ishibashi states* [14–16] (see also [17] for another application to bulk physics). More specifically, we consider two possibilities, the Ishibashi state, $|\Sigma_{\text{Ish}}\rangle$, and the crosscap state, $|\Sigma_{\text{cross}}\rangle$. These are defined as satisfying the condition for $J_a \in sl(2)_L, \bar{J}_a \in sl(2)_R$ and for $a = 0, \pm 1$,

$$J_a - \bar{J}_{-a} |\Sigma_{\text{Ish}}\rangle = 0 , \tag{10}$$

$$J_a - (-1)^a \bar{J}_{-a} |\Sigma_{\text{cross}}\rangle = 0 . \tag{11}$$

Through these conditions, the singlet states can be seen as a way to tie together the barred and unbarred representations. As long as one considers expectation values in these singlet states, barred generators can be substituted for suitable unbarred generators (possibly up to a phase) and vice versa.

While these equations cannot be solved by finite sums of states in the representation, it is possible to take infinite sums of states. In terms of states $|h, k, \bar{k}\rangle = |h, k, \rangle \otimes |h, \bar{k}\rangle$ for the full representation combining both barred and unbarred copies, the solutions are

$$|\Sigma_{\mathrm{Ish}}\rangle = \sum_{k=0}^{\infty} |h, k, k\rangle \ , \qquad |\Sigma_{\mathrm{cross}}\rangle = \sum_{k=0}^{\infty} (-1)^k |h, k, k\rangle \ . \tag{12}$$

With this solution for the Ishibashi state, one can compute the AdS_3 Wilson line explicitly [16]. Schematically, the result is

$$\begin{aligned} W_{\mathscr{R}}(x_i, x_f) &= \text{character associated to the representation } \mathscr{R} \\ &= \text{Green's function between } x_i, x_f \ . \end{aligned} \tag{13}$$

These are two fairly universal relations that we would like to highlight in these proceedings. As we will see, similar statements can be made for Wilson lines in de Sitter, but with slight differences and subtleties. We would also like to understand exactly *how* the Green's function can be reproduced for de Sitter, specifically in terms of which representations and endpoint states are necessary to use in the construction.

4. Why de Sitter?

It may seem like a pedagogical exercise to repeat this analysis for de Sitter, but in fact there is something to learn by using Wilson lines as tools to probe bulk physics in de Sitter spacetime. Can we reproduce the subtleties of de Sitter gravity? Such subtleties exist even classically, for instance the Green's function and smearing functions for dS_3 do not result simply from analytic continuation of their AdS_3 counterparts [18–20]. Is there a way to reproduce Green's functions and smearing functions in de Sitter? As we will see, our Wilson line analysis allows us to see clearly how these differences appear in the Chern-Simons language, in terms of duplicate contributing representations.

The advantage of studying de Sitter gravity using Wilson lines also extends beyond the classical analysis. There exist results in AdS_3 for quantizing a Wilson line order by order in $1/c$ [8]. But in fact, in the Chern-Simons approach de Sitter spacetime is a better starting point for quantizing a Wilson line. This is because there exist exact results from nonabelian localization and other methods for $su(2)$ Chern-Simons theory, which is the gauge algebra relevant for EdS_3 [5–7]. This is analogous to the past use of exact results in Chern-Simons theory to match the 1-loop Euclidean gravitational partition function [21]. When considering Wilson lines, the main subtlety comes from the need to use non-unitary representations for de Sitter, as we will see in Section 5.1. The application of exact results to the quantization of de Sitter Wilson lines will be treated in further detail in [11].

Of course, there are also numerous subtleties that come up in any treatment of de Sitter gravity. For instance, in the Lorenztian dS_3 case it is natural to attach the boundaries of Wilson lines to future timelike infinity, but unlike for AdS there is no CFT there to set boundary conditions.

5. Euclidean EdS$_3$

To probe gravity in de Sitter using Wilson lines, we start by considering three-dimensional Euclidean de Sitter (EdS$_3$), which is just a three-sphere, S^3. EdS$_3$ has isometry algebra $so(4) \simeq su(2) \times su(2)$, thus we consider $su(2)$ Chern-Simons theory.

The description of EdS$_3$ gravity using Chern-Simons theory proceeds similarly to the negative cosmological constant case, but with a different gauge algebra. Consider two copies of the $su(2)$ Chern-Simons action

$$S = S_{CS}[A] - S_{CS}[\bar{A}] \tag{14}$$

with

$$S_{CS}[A] = -\frac{k}{4\pi}\int_{\mathcal{M}} \mathrm{Tr}\left(A \wedge \mathrm{d}A + \frac{2}{3}A \wedge A \wedge A\right) , \tag{15}$$

where $A \in su(2)_L, \bar{A} \in su(2)_R$. One can check that this likewise reduces to the three-dimensional EdS$_3$ Einstein-Hilbert gravitational action with curvature radius ℓ and gravitational constant G_3, given a Chern-Simons coupling taken to be

$$k = \frac{\ell}{4G_3} . \tag{16}$$

Now let us consider the geometry of EdS$_3$. The metric on the three-sphere can be written in Hopf coordinates as

$$\frac{ds^2}{\ell^2} = dr^2 + \cos^2 r d\tau^2 + \sin^2 r d\phi^2 . \tag{17}$$

We can relate this to the Chern-Simons connections $A, \bar{A}$ through the relation

$$g_{\mu\nu} = -\frac{\ell^2}{2}\mathrm{Tr}\left[(A_\mu - \bar{A}_\mu)(A_\nu - \bar{A}_\nu)\right] . \tag{18}$$

The connections can be related to group elements

$$A = g_L \mathrm{d} g_L^{-1} , \qquad \bar{A} = \tilde{g}_R^{-1}\mathrm{d}\tilde{g}_R . \tag{19}$$

We can in particular make the choice

$$g_L = e^{-irL_2}e^{-i(\phi+\tau)L_3} , \qquad \tilde{g}_R = e^{i(\phi-\tau)L_3}e^{-irL_2} . \tag{20}$$

These define connections that reproduce the metric for the three-sphere in the coordinates given by Eq. (17).

We can now evaluate de Sitter Wilson lines in three steps.

5.1 Step 1: Build representations

To probe gravitational physics in EdS$_3$ using Wilson lines, we must construct infinite dimensional $su(2)$ representations. Since unitary $su(2)$ representations are finite dimensional, this means we must construct non-unitary representations.

The $su(2)$ algebra is

$$[L_i, L_j] = i\epsilon_{ijk}L_k . \tag{21}$$

Expressed in terms of raising and lowering operators $L_\pm = L_1 \pm iL_2$, $L_0 = L_3$, this is

$$[L_0, L_\pm] = \pm L_\pm \ , \qquad [L_+, L_-] = 2L_0 \ . \tag{22}$$

The simplest way to get a non-unitary representation, maintaining consistency with the commutation relations, is to take L_3 to be Hermitian and $L_{1,2}$ anti-Hermitian, in other words

$$L_\pm^\dagger = -L_\mp \ , \qquad L_0^\dagger = L_0 \ . \tag{23}$$

We can now construct non-unitary highest weight representations. We start with the highest weight state defined through

$$L_+ |l,0\rangle = 0 \ , \tag{24}$$

$$L_0 |l,0\rangle = l\, |l,0\rangle \ . \tag{25}$$

Then we successively build the remaining states by acting with lowering operators $|l,p\rangle \propto (L_-)^p |l,0\rangle$. The proportionality constant can be determined by relating the overlap of states, subject to the unit normalization of the highest weight state.

The Casimir for these representations can be determined by squaring the condition, Eq. (43), requiring that the highest weight state is annihilated by the raising operator. It is given by

$$L^2 |l,p\rangle = c_2 |l,p\rangle \ , \qquad c_2 = l(l+1) \ . \tag{26}$$

One can also check that with the choices above, all states in the representation have a positive norm.

The representations have an associated character

$$\mathrm{ch}(\mathscr{R}) = \sum_{p=0}^{\infty} \langle l,p| e^{i\alpha L_0} |l,p\rangle = \frac{e^{i\alpha(l+1)}}{e^{i\alpha}-1} \ . \tag{27}$$

This will be useful to compare to once we compute the Wilson line.

From the gravitational perspective, there are two defining properties of these representations:

Negative Casimir: Imposing that the norm of all states is positive implies $l < 0$. We have a negative Casimir in the window $-1 < l < 0$, which corresponds to

$$-\frac{1}{2} < c_2 < 0 \ . \tag{28}$$

It will turn out that in order to match an appropriate Green's function, the Casimir should be related to the mass as $c_2 = -m^2\ell^2/4$. Thus, to reproduce gravitational physics, it is necessary to use representations such as these that have a negative Casimir in some window.

Two representations with fixed Casimir: Inverting the Casimir Eq. (26) for l, we have

$$l_\pm = -\frac{1 \pm \sqrt{1+4c_2}}{2} \ . \tag{29}$$

There are in fact two solutions in the window Eq. (28). We define the two distinct representations that contribute within this window to be

$$\mathscr{R}_+ : \qquad -1 < l_+ < -\frac{1}{2}\,, \tag{30}$$

$$\mathscr{R}_- : \qquad -\frac{1}{2} < l_- < 0\,. \tag{31}$$

With the choice $c_2 = -m^2\ell^2/4$, one can already start to recognize $l_\pm$ in Eq. (29) as parameters that appear in the EdS_3 Green's function [20].

5.2 Step 2: Construct singlet states

We now must make a suitable choice of endpoint states to use to evaluate our de Sitter Wilson lines. In analogy to the $sl(2)$ case for AdS, we consider Ishibashi and crosscap states defined by the condition for $L_a \in su(2)_L$, $\bar{L}_a \in su(2)_R$, with $a = 0, \pm 1$,

$$L_a - \bar{L}_{-a}\,|\Sigma_{\text{Ish}}\rangle = 0\,, \tag{32}$$

$$L_a - (-1)^a \bar{L}_{-a}\,|\Sigma_{\text{cross}}\rangle = 0\,. \tag{33}$$

In terms of states $|l, p, \bar{p}\rangle = |l, p, \rangle \otimes |l, \bar{p}\rangle$ for the full representation, the solutions are given by the infinite sums

$$|\Sigma_{\text{Ish}}\rangle = \sum_{p=0}^{\infty} (-1)^p\,|l, p, p\rangle\ , \tag{34}$$

$$|\Sigma_{\text{cross}}\rangle = \sum_{p=0}^{\infty} |l, p, p\rangle\ . \tag{35}$$

The result appears similar to Eq. (5.1) for the AdS_3 case. Besides the use of different representations, one difference is that the phase $(-1)^p$ now appears in the sum for the Ishibashi state, rather than the crosscap state.

5.3 Step 3: Evaluate Wilson line

It remains to evaluate the Wilson line, and we now have all the ingredients required to perform an explicit computation. We start by taking group elements, Eq. (20), and the connections, Eq. (19). The Wilson line with Ishibashi state endpoints is given by

$$W_{\mathscr{R}}(x_i, x_f) = \langle\Sigma_{\text{Ish}}|\,\mathscr{P}e^{-\int_\gamma A}\mathscr{P}e^{-\int_\gamma \bar{A}}\,|\Sigma_{\text{Ish}}\rangle\ . \tag{36}$$

We can evaluate the path ordered exponentials in terms of the group elements evaluated at the endpoints,

$$\mathscr{P}e^{-\int_\gamma A} = g_L(x_f)g_L(x_i)^{-1}\,, \qquad \mathscr{P}e^{-\int_\gamma \bar{A}} = g_R(x_f)^{-1}g_R(x_i)\,. \tag{37}$$

Using the explicit group elements for S^3 given in Eq. (20), the explicit endpoint states Eq. (34), and evaluating in Eq. (36), this gives

$$W_{\mathscr{R}}(x_i, x_f) = \frac{e^{i\alpha(l+1)}}{e^{i\alpha} - 1}\,, \tag{38}$$

where

$$\cos\left(\frac{\alpha}{2}\right) = \cos(r_f)\cos(r_i)\cos(\tau_f - \tau_i) + \sin(r_f)\sin(r_i)\cos(\phi_f - \phi_i) \tag{39}$$

is related to the invariant distance on S^3. But this is just the character, Eq. (27), associated to the non-unitary $su(2)$ representations,

$$W_{\mathscr{R}}(x_i, x_f) = \mathrm{ch}(\mathscr{R}) \ . \tag{40}$$

The result is directly analogous to the AdS$_3$ case, except that here the character is the one associated to the non-unitary $su(2)$ representations we used for the EdS$_3$ Wilson line.

Likewise, one can check that the Wilson line can also be related to the Euclidean Green's function $G(\Theta)$ for a scalar field on S^3 with mass $m^2\ell^2 = -4c_2$,

$$G(\Theta) = a_{-l} W_{\mathscr{R}_+}(x_i, x_f) + a_{-l-1} W_{\mathscr{R}_-}(x_i, x_f) \ . \tag{41}$$

Here Θ is the invariant distance between the endpoint coordinates x_i and x_f of the Wilson line. As was suggested earlier, the mass is directly related to the Casimir of the contributing representations, which must be *negative* in order to reproduce the Green's function. Unlike for the AdS$_3$ case, here for de Sitter one needs *both* representations contributing to a fixed Casimir to relate the Wilson line to the Euclidean Green's function.

6. Local fields

We can also probe gravitational physics directly from the representations we used to evaluate the Wilson line. In this section, we will give a schematic overview of the construction that involves taking some shortcuts. The details and careful treatment can be found in [9].

Represent the $su(2)_L, su(2)_R$ generators as Killing vectors $\mathcal{L}_a, \bar{\mathcal{L}}_a$ of the three-sphere

$$L_a, \bar{L}_a \to \mathcal{L}_a, \bar{\mathcal{L}}_a \ . \tag{42}$$

Likewise, we can promote the states in the representation constructed in Section 5.1 to fields,

$$|\ell; p, \bar{p}\rangle \to \Phi_{p,\bar{p}} \ . \tag{43}$$

By combining the action of the raising and lowering operators on fields, the Casimir equation gives

$$(\nabla^2 + \bar{\nabla}^2)\Phi_{p,\bar{p}}(x) = 2l(l+1)\Phi_{p,\bar{p}}(x) \ , \tag{44}$$

where $\nabla^2 = \delta^{ab}\mathcal{L}_a\mathcal{L}_b$ and likewise for the barred copy. Thus the $\Phi_{p,\bar{p}}$ are just local fields[1] on the three-sphere.

These local fields can also be solved for explicitly. The highest weight conditions become

$$\mathcal{L}_+\Phi_{0,0} = 0 \ , \qquad \mathcal{L}_0\Phi_{0,0} = l\Phi_{0,0} \ , \tag{45}$$

and the full set of fields can be built from $\Phi_{0,0}$ by acting using lowering operators $\mathcal{L}_-, \bar{\mathcal{L}}_-$ on this highest weight state. One can solve Eq. (43), and then explicitly act by lowering operators

[1]More precisely, we call these "pseudo-fields" since both representations are necessary to form a complete basis.

in differential operator form. The solutions $\Phi_{p,\bar{p}}$ are simply the quasinormal modes on the three-sphere.

We can define a state $|U(x)\rangle$ expanded over the representation,

$$|U(x)\rangle = g_L^{-1}(x) g_R(x) |\Sigma_{\rm Ish}\rangle \tag{46}$$

$$= \sum_{p,\bar{p}=0}^{\infty} \Phi^*_{p,\bar{p}}(x) |l, p, \bar{p}\rangle \ . \tag{47}$$

One can check that the Wilson line is simply the overlap

$$W_{\mathscr{R}}(x_i, x_f) = \left\langle U(\tau_f, r_f, \phi_f + \pi) \right| U(\tau_i, r_i, \phi_i)\rangle \ . \tag{48}$$

The extra shift in π in the angular direction is necessary as a result of the use of non-unitary representations in the de Sitter case, and is not present in the AdS case.

By comparing Eqs. (47) and (34) and using the explicit expression for quasinormal modes on the three-sphere, we can represent the Ishibashi state as

$$|\Sigma_{\rm Ish}\rangle = \sum_{p,\bar{p}} \Phi^*_{p,\bar{p}}(\tau = 0, r = 0) |l, p, \bar{p}\rangle = |U(\tau = 0, r = 0)\rangle \ , \tag{49}$$

and thus it describes a state situated on $(r = 0, \tau = 0)$ of the three-sphere. Of course, this result is gauge ambiguous and depends on our choice of coordinates.

7. Analytic continuation to Lorentzian dS_3

Ultimately we wish to use Wilson lines to probe gravity in Lorentzian dS_3 spacetime. To reproduce local bulk fields in this case, we now perform an analytic continuation from the three-sphere implemented by the Wick rotation

$$\tau \to \frac{it}{\ell} \ . \tag{50}$$

One can also implement this Wick rotation on the Killing vectors and fields $\Phi_{p,\bar{p}}$, and work out how the representation transforms. This is summarized in Table 1. It turns out that the non-unitary $su(2)_L \times su(2)_R$ representation, $\mathscr{R}$, we constructed to obtain gravitational physics from Wilson lines on the three-sphere nicely transforms under this Wick rotation to a unitary $sl(2)_L \times sl(2)_R$ representation, $\tilde{\mathscr{R}}$, with generators $\mathcal{H}_0, \mathcal{H}_\pm$ and a barred copy that are entirely Hermitian.

After analytic continuation, the fields lie in highest weight $sl(2)$ representations with Casimir $h(h-1)$, where $h = -l$ (analogously, $h = l + 1$ for $\tilde{\mathscr{R}}_-$). They solve the dS_3 wave equation

$$(\nabla^2 + \bar{\nabla}^2)\Phi_{p,\bar{p}} = 2h(h-1)\Phi_{p,\bar{p}} \ , \tag{51}$$

where $\nabla^2 = -\eta^{ab}\mathcal{H}_a\mathcal{H}_b$, and likewise for the barred copy, are related to the de Sitter Laplacian as $\nabla^2 + \bar{\nabla}^2 = -\frac{1}{2}\nabla^2_{dS_3}$. Like for the Euclidean case, the fields can be constructed explicitly starting with the highest weight state which solves

$$\mathcal{H}_+\Phi_{0,0} = 0 \ , \qquad \mathcal{H}_0\Phi_{0,0} = h\Phi_{0,0} \ , \tag{52}$$

EdS$_3$	dS$_3$
$SO(4) \simeq SU(2) \times SU(2)$ Chern-Simons	$SL(2,\mathbb{C})$ Chern-Simons
Non-unitary highest weight $su(2)$ representations	Unitary highest weight $sl(2)$ representations

Table 1: The analytic continuation from EdS$_3$ to dS$_3$.

and likewise for the barred operators, with the additional modes constructed by successive action of lowering operators $\mathcal{H}_-, \bar{\mathcal{H}}_-$. The solutions are just quasinormal modes for Lorentzian dS$_3$.

In the Chern-Simons description of three-dimensional gravity, whether for AdS$_3$, EdS$_3$ or dS$_3$, it is necessary for the isometry algebra to be separable into two copies of a gauge algebra, over which the unbarred and barred gauge fields can be expanded. In the Euclidean dS$_3$ case it is possible to take purely real linear combinations of the generators of the isometry algebra, the Euclidean Poincaré algebra, to form two copies of the $su(2)$ algebra. For Lorentzian dS$_3$, the isometry algebra is the Lorentzian Poincaré algebra, and this does not separate into two copies of $sl(2)$ unless one takes complex linear combinations of the generators of the Poincaré algebra. Thus, to describe dS$_3$, it is necessary to consider gauge fields $\mathcal{A}, \bar{\mathcal{A}}$ that are elements of the complex gauge algebra $sl(2,\mathbb{C})$. It is typical to additionally specify a condition relating the gauge field $\mathcal{A}$ to the complex conjugate of $\bar{\mathcal{A}}$. Then two copies of the $sl(2,\mathbb{C})$ Chern-Simons action with purely imaginary Chern-Simons couplings is can be shown to reduce to the Lorentzian dS$_3$ gravitational action with de Sitter radius ℓ and gravitational constant G_3 given the matching $s = \ell/(4G_3)$ [22].

It is now possible to repeat the analysis of Sections 5.2-5.3, but using the analytically continued representations and $SL(2,\mathbb{C})$ Chern-Simons theory. The Ishibashi states satisfying the analogue of Eq. (32) for the Lorentzian representations are

$$|\Sigma_{\text{Ish}}\rangle = \sum_{p=0}^{\infty} |h,p,p\rangle \ , \tag{53}$$

$$|\Sigma_{\text{cross}}\rangle = \sum_{p=0}^{\infty} (-1)^p \, |h,p,p\rangle \ . \tag{54}$$

For these unitary $sl(2)$ representations, the phase $(-1)^p$ now appears for the crosscap state rather than the Ishibashi state, thus more closely resembling the original AdS$_3$ case compared to the EdS$_3$ case with non-unitary $su(2)$ representations.

The Wilson line can also be computed explicitly. In [9], this was done for the inflationary patch of de Sitter. Using the endpoint Ishibashi state, Eq. (53), and $SL(2,\mathbb{C})$ Chern-Simons connections that describe the metric in these coordinates, the Wilson line again computes a character, but now for the analytically continued representations,

$$W_{\tilde{\mathcal{R}}}(x_i, x_f) = \text{ch}(\tilde{\mathcal{R}}) \ . \tag{55}$$

Likewise, the Wilson line can be matched to the dS$_3$ Euclidean Green's function $G(\Theta)$ for a scalar of mass $m^2\ell^2 = -4c_2$, where Θ is the invariant distance between endpoints x_i, x_f of the Wilson line:

$$G(\Theta) = a_h W_{\tilde{\mathcal{R}}_+}(x_i, x_f) + a_{1-h} W_{\tilde{\mathcal{R}}_-}(x_i, x_f) \ . \tag{56}$$

Notably, as for the EdS_3 case, both representations in the window with a fixed Casimir $h(h-1)$ contribute to the Green's function, whereas only a single representation appears for AdS_3.

Finally, we touch briefly on bulk reconstruction for de Sitter. By solving the wave equation in various asymptotically AdS spacetimes, the usual HKLL construction for AdS [23, 24] allows for the classical reconstruction of bulk fields $\Phi(x)$ in terms of a smeared integral of boundary operators $O(y)$ that are spacelike separated from the bulk point x,

$$\Phi(x) = \int dy' K(x; y') O(y') \,. \tag{57}$$

The kernel $K(x; y)$ here is known as a smearing function, and it depends on the specifics of the bulk geometry. While it is known in many cases, there also exist certain coordinate systems such as Rindler-AdS for which there are obstructions to the construction [25].

There are subtleties when considering smearing functions for de Sitter spacetime. For the inflationary patch of dS_3, one might imagine a similar construction relating bulk fields to integrals smeared over operators at future timelike infinity. In this case, in order to reproduce the Euclidean Green's function from the two-point function of the bulk scalar [20], Ref. [18] (see also [19, 26, 27]) in fact argued that there must be two contributions to this smearing. The field $\Phi(x)$ at a bulk point x can be schematically reconstructed as

$$\Phi(x) = \int dy' K_+(x; y') O_+(y') + \int dy' K_-(x; y') O_-(y') \,, \tag{58}$$

where

$$\Delta_\pm = 1 \pm \sqrt{1 - m^2 \ell^2} \,, \tag{59}$$

and $O_\pm(y)$ are the boundary limits of the positive and negative frequency bulk modes with the divergent factor stripped off. There are contributions not only from the normalizable part O_+ dual to Φ, but also from the shadow operator with scaling dimension $\Delta_- = 2 - \Delta_+$. The appearance of both falloffs is reminiscent of our result, Eq. (56). There, unlike for AdS, we needed to consider Wilson lines associated to two different representations with fixed Casimir in order to correctly reconstruct the Green's function.

In AdS_3, there is an alternate approach to deriving the smearing function that uses Wilson lines [28]. In this method, one starts by noticing that the Ishibashi state lives at a localized bulk point, thus it describes a local bulk field in terms of sums over states in the representation. By applying the state-operator correspondence, one can relate these states to operators inserted at the origin in the CFT, $|h, p, \bar{p}\rangle = O(0,0)\,|0\rangle$. The Ishibashi state, Eq. (53), becomes

$$|\Sigma_{\rm Ish}\rangle = \sum_{p=0}^{\infty} \frac{\Gamma(2h)}{\Gamma(p+1)\Gamma(p+2h)} \mathcal{H}_{-1}^p \bar{\mathcal{H}}_{-1}^p O(0,0)\,|0\rangle \,. \tag{60}$$

Acting on this by isometries, one can translate the Ishibashi state to an arbitrary point in the bulk, thus describing any local bulk field. Some further algebraic manipulations allow one to take this to the form, Eq. (57). In de Sitter, the existence of a state-operator correspondence would rely on a dS/CFT correspondence and is far from clear, however it is highly suggestive to consider performing a similar series of steps. If one does, the need for two representations in the Ishibashi construction for de Sitter directly translates to the two contributions in Eq. (58).

8. Discussion

We have reviewed how Wilson lines evaluated in non-unitary $su(2)$ representations (and their unitary $sl(2)$ counterparts after analytic continuation) can be used to evaluate Green's functions and construct local fields in both EdS_3 and Lorentzian dS_3 spacetime. An interesting further avenue is to apply exact results for the expectation value of Wilson loops to infer quantum versions of the Wilson line, and explore its gravitational interpretation [11].

Other approaches that use Chern-Simons theory to probe gravitational physics in de Sitter include [29–31]. In our setting, we specifically found a crucial role played by non-unitary $su(2)$ representations. These representations, which describe the quasinormal modes, are useful in the context of Chern-Simons theory. However, due to a different Hermiticity choice, they differ from the usual representations that describe non-exotic scalar fields in de Sitter—the complementary and principal series of $so(d+1,1)$. They nonetheless seem to be useful to describe local bulk physics for such fields in de Sitter (analogously, the appearance of the quasinormal mode spectrum in characters related to one-loop partition functions was noted in [32–34]). It will be interesting to further understand the uses and limitations of Chern-Simons theory, which naturally involves representations that split into barred and unbarred copies, to describe observables built from the usual representations for de Sitter gravity.

This work should be extended beyond the scalar case, and there are some questions that arise in this context. For instance, de Sitter spacetime admits a rich representation theory [10, 35–37], including representations describing exotic fields such as partially massless gravitons that do not appear in flat spacetime [38, 39]. One might ask if it is possible to use Wilson lines to probe such physics.

Finally, it would also be interesting to explore further the role of the two representations, $\mathscr{R}_\pm$ (and $\tilde{\mathscr{R}}_\pm$ in the Lorentzian case). A better understanding of the quantization of the Wilson line might elucidate this issue. It would be particularly interesting to consider the role of these dual representations within a concrete microscopic Hilbert space description for de Sitter.

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