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Spin in two-dimensional fermion motion with circular symmetry

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We study two-dimensional fermion motion with circular symmetry using both 3+1 and 2+1 Dirac equations with a general Lorentz structure. Using a different approach than usual, we fully develop the formalism for these equations using cylindrical coordinates and discuss the quantum numbers, spinors and differential equations in both cases when there is circular symmetry. Although there is no spin quantum number in the 2+1 case, we find that, as remarked already by other authors, in this case the spin projection s in the direction perpendicular to the plane of motion can be emulated by a parameter preserving the anti-commutation relations between the Dirac matrices. The formalism developed allowed us to recognize an equivalence between a pure vector potential and a pure tensor potential under circular symmetry, if the former is multiplied by s, for any functional form of these potentials. We apply the formalism, both in the 3+1 and 2+1 cases, to the problem of a uniform magnetic field perpendicular to the plane of motion. We fully discuss its solutions, their properties, including the energy spectra, compare them to the relativistic Landau problem and obtain the non-relativistic limit as well. This calculation enabled us to clarify the physical meaning of the s parameter, representing the spin quantum number in the 3+1 case and just a parameter in the Hamiltonian in the 2+1 case.

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1. Introduction

The study of two-dimensional motion of spin 1/2 fermions has gained recently a considerable interest due to the existence of several relativistic quantum systems in which fermions are confined to move in a plane, the most notable of which is graphene in condensed matter physics.

One common but limited way of describing this motion is through $2+1$ Dirac equation, which involves 2×2 matrices, because there is only the sign of the energy as degree of freedom, i.e., one has only positive and negative energy states, but spin is not included [1, 2]. However, the anti-commuting algebra $\{\alpha_i,\alpha_j\} = 2\delta_{ij}$, $\{\alpha_i,\beta\} = 0$ of the only three 2×2 Hermitian anti-commuting matrices, α_1, α_2, β, usually Pauli matrices, allows for the introduction of an additional parameter in the $2+1$ Dirac Hamiltonian. This algebra is satisfied as well if one multiplies one of these matrices (say $\alpha_2 \equiv \sigma_y$) by a parameter $\mathsf{s} = \pm 1$. Hagen proposed that this parameter represents the spin projection in a direction perpendicular to the motion plane [3]. However, changing s changes the $2+1$ Dirac Hamiltonian in a non-equivalent way, i.e., is not a change of representation by an unitary transformation [4]. The parameter s can label the solution but it is not a quantum number.

In this work we use the $2+1$ and $3+1$ Dirac Hamiltonians to study two-dimensional motion of spin 1/2 fermions when there is circular symmetry, i.e., the interactions only depend on the polar coordinate. We develop the full formalism for this motion in the $2+1$ and $3+1$ cases, including the quantum numbers. We uncover a quite general equivalence between a pure vector and a pure tensor potential, provided a certain relationship exists between them.

We discuss the physical role of the parameter s by applying the formalism to the problem of the motion of a spin-1/2 fermion in the xy plane subject to an uniform magnetic field along the z axis. The details of this study were published in Ref. [5].

2. 3+1 and 2+1 Dirac equations with circular symmetry

2.1 Dirac Hamiltonian in cylindrical coordinates

The time-independent Dirac equation in 3+1 dimensions, including four-vector, tensor and scalar interactions is written as (**bold** denotes vectors)

$$H\Psi = \boldsymbol{\alpha}\cdot(\boldsymbol{p}-\boldsymbol{A}) + A_0 + \beta\,(m+S) + i\beta\boldsymbol{\alpha}\cdot\boldsymbol{U} = E\Psi \tag{1}$$

where $\boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}$, $\beta = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}$ and σ_i are the Pauli matrices and I_2 is the 2×2 unity matrix. In cylindrical coordinates (ρ,φ,z) one has

$$\boldsymbol{\alpha}\cdot\boldsymbol{p} = -\mathrm{i}(\alpha_\rho\nabla_\rho + \alpha_\varphi\nabla_\varphi + \alpha_z\nabla_z) = -\mathrm{i}\left(\alpha_\rho\frac{\partial}{\partial\rho} + \alpha_\varphi\frac{1}{\rho}\frac{\partial}{\partial\varphi} + \alpha_z\frac{\partial}{\partial z}\right) \tag{2}$$

with

$$\alpha_\rho = \boldsymbol{\alpha}\cdot\widehat{\boldsymbol{\rho}} = \begin{pmatrix} 0 & \sigma_\rho \\ \sigma_\rho & 0 \end{pmatrix},\ \alpha_\varphi = \boldsymbol{\alpha}\cdot\widehat{\boldsymbol{\varphi}} = \begin{pmatrix} 0 & \sigma_\varphi \\ \sigma_\varphi & 0 \end{pmatrix},\ \ \alpha_z = \boldsymbol{\alpha}\cdot\widehat{\boldsymbol{z}} = \begin{pmatrix} 0 & \sigma_z \\ \sigma_z & 0 \end{pmatrix}$$

$$\sigma_\rho = \begin{pmatrix} 0 & e^{-i\varphi} \\ e^{+i\varphi} & 0 \end{pmatrix},\quad \sigma_\varphi = i\sigma_\rho\sigma_z = -\mathrm{i}\begin{pmatrix} 0 & e^{-i\varphi} \\ -e^{+i\varphi} & 0 \end{pmatrix} = \frac{\partial\sigma_\rho}{\partial\varphi} \tag{3}$$

The matrices α_i and σ_i, $i = \rho, \varphi, z$ obey the same commutation and anti-commutation relations as the corresponding Cartesian components, with the cylindrical indexes ordered as ρ, φ, z. For example, one has $\alpha_i\alpha_j = I\delta_{ij} + \mathrm{i}\epsilon_{ijk}\Sigma_k$ with $\Sigma_i = \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix}$, $i, j, k = \rho, \varphi, z$, $\epsilon_{\rho\,\varphi\,z} = 1$.

Using the relations $\alpha_\varphi = \mathrm{i}\alpha_\rho\Sigma_z$ and $L_z = -\mathrm{i}\frac{\partial}{\partial\varphi}$, the Hamiltonian in (1) reads now

$$
\begin{aligned}
H =& - \mathrm{i}\alpha_\rho \left(\frac{\partial}{\partial\rho} - \mathrm{i}A_\rho + \beta U_\rho - \frac{1}{\rho} L_z\Sigma_z + \Sigma_z A_\varphi + \mathrm{i}\Sigma_z\beta U_\varphi \right) \\
& - \mathrm{i}\alpha_z \left(\frac{\partial}{\partial z} - A_z + \beta U_z \right) + A_0 + \beta\,(m+S) \ . \qquad (4)
\end{aligned}
$$

For the time-independent $2+1$ Dirac equation one has

$$
H\Psi = \left[\boldsymbol{\sigma} \cdot (\boldsymbol{p} - \boldsymbol{A}) + A_0 + \sigma_z\,(m+S) \right] \Psi = E\Psi \ . \qquad (5)
$$

Here boldface denotes two-dimensional vectors in the plane xy. In the spinor space, one defines $\boldsymbol{\sigma} = (\sigma_x, \mathsf{s}\sigma_y)$, with $\mathsf{s} = \pm 1$ to allow for different non-equivalent realizations of the algebra of 2×2 matrices s_i, $i = x, y, z$ as mentioned in the Introduction. In cylindrical (polar) coordinates, using the notation as before, one gets $\boldsymbol{\sigma} \cdot \boldsymbol{p} = -\mathrm{i}\left(\sigma_\rho \frac{\partial}{\partial\rho} + \sigma_\varphi \frac{1}{\rho}\frac{\partial}{\partial\varphi} \right)$, with

$$
\sigma_\rho = \sigma_x \cos\varphi + \mathsf{s}\,\sigma_y \sin\varphi = \begin{pmatrix} 0 & e^{-i\mathsf{s}\,\varphi} \\ e^{+i\mathsf{s}\,\varphi} & 0 \end{pmatrix} \qquad (6)
$$

$$
\sigma_\varphi = -\sigma_x \sin\varphi + \mathsf{s}\,\sigma_y \cos\varphi = -\mathrm{i}\mathsf{s} \begin{pmatrix} 0 & e^{-i\mathsf{s}\,\varphi} \\ -e^{+i\mathsf{s}\,\varphi} & 0 \end{pmatrix} = -\mathrm{i}\mathsf{s}\,\sigma_z\sigma_\rho = \frac{\partial\sigma_\rho}{\partial\varphi} \ . \qquad (7)
$$

These correspond to 3+1 definitions if $\mathsf{s} = 1$. With $L_z = -\mathrm{i}\partial/\partial\varphi$, the Dirac Hamiltonian is

$$
H = -\mathrm{i}\sigma_\rho \left(\frac{\partial}{\partial\rho} - \mathrm{i}A_\rho - \mathsf{s}\frac{1}{\rho} L_z\sigma_z + \mathsf{s}\sigma_z A_\varphi \right) + A_0 + \sigma_z\,(m+S) \ . \qquad (8)
$$

2.2 Planar motion with circular symmetry

Constraining the motion to the plane xy with circular symmetry means that: i) there would be no momenta nor forces along the z axis; ii) all potentials would be a function of the polar coordinate ρ only. After gauging away the potential A_ρ (see [5]) the reduced Hamiltonian reads

$$
h = -\mathrm{i}\alpha_\rho \left(\frac{\partial}{\partial\rho} + \beta U_\rho - \frac{1}{\rho} L_z\Sigma_z + \Sigma_z A_\varphi \right) + A_0 + \beta\,(m+S) \ . \qquad (9)
$$

where all potentials depend only on ρ. The corresponding $2+1$ hamiltonian reads

$$
h = -\mathrm{i}\sigma_\rho \left(\frac{\partial}{\partial\rho} - \mathsf{s}\frac{1}{\rho} L_z\sigma_z + \mathsf{s}\sigma_z A_\varphi \right) + A_0 + \sigma_z\,(m+S) \ . \qquad (10)
$$

2.3 Quantum numbers and spinor wave function

In the 3 + 1 problem one can show that $J_z = L_z + S_z$ and $K = \beta\left(\frac{I}{2} + L_z\Sigma_z\right)$ are constants of motion, together with βS_z [5]. However, only two of those are independent, so only two quantum numbers from two of these operators can be used to label a quantum state. From [5]), one has

$$\psi_{k\,m_j} = \frac{1}{\sqrt{\rho}}\begin{pmatrix} ig_{k\,m_j}(\rho)h_{k\,m_j}(\varphi) \\ f_{k\,m_j}(\rho)h_{-k\,m_j}(\varphi)\end{pmatrix} \qquad \chi_s = \begin{pmatrix}\delta_{s\,,1} \\ \delta_{s\,,-1}\end{pmatrix} \qquad s = -1, 1\ , \tag{11}$$

where $h_{k\,m_j}(\varphi) = \Phi_m(\varphi)\chi_s = \frac{1}{\sqrt{2\pi}}\,e^{\mathrm{i}m\varphi}\chi_s$, $J_z\psi_{k\,m_j} = m_j\psi_{k\,m_j}$, $K\psi_{k\,m_j} = k\psi_{k\,m_j}$, $\sigma_z\chi_s = s\chi_s$. For the 2 + 1 case, $j_z = L_z + (\mathsf{s}/2)\sigma_z$ is a constant of motion, and the two-component spinor is [5]

$$\psi_{m_j} = \frac{1}{\sqrt{\rho}}\left[\,\mathrm{i}\,g^{+}_{m_j}(\rho)h^{+1}_{m_j}(\varphi) + g^{-}_{m_j}(\rho)h^{-1}_{m_j}(\varphi)\right]\ , \tag{12}$$

where $h^{e}_{m_j}(\varphi) = \frac{1}{\sqrt{2\pi}}\,e^{\mathrm{i}m\varphi}\begin{pmatrix}\delta_{e\,,1} \\ \delta_{e\,,-1}\,e^{\mathrm{i}\mathsf{s}\varphi}\end{pmatrix}$ with $e = \pm 1$.

3. Motion under an uniform magnetic field perpendicular to the plane of motion

The polar first-order coupled differential equations for the 3 + 1 and 2 + 1 cases read [5]

$$\frac{df_{k\,m_j}}{d\rho} + \frac{k}{\rho}\,f_{k\,m_j} - U_\rho\,f_{k\,m_j} - s\,A_\varphi\,f_{k\,m_j} = -\left[E - (A_0 + m + S)\right]g_{k\,m_j} \tag{13}$$

$$\frac{dg_{k\,m_j}}{d\rho} - \frac{k}{\rho}\,g_{k\,m_j} + U_\rho\,g_{k\,m_j} + s\,A_\varphi\,g_{k\,m_j} = (E - A_0 + m + S)\,f_{k\,m_j}\ , \tag{14}$$

$$\frac{dg^{+}_{m_j}}{d\rho} - \frac{m_j\mathsf{s}}{\rho}\,g^{+}_{m_j} + \mathsf{s}\,A_\varphi\,g^{+}_{m_j} = +\left[E - A_0 + m + S\right]g^{-}_{m_j} \tag{15}$$

$$\frac{dg^{-}_{m_j}}{d\rho} + \frac{m_j\mathsf{s}}{\rho}\,g^{-}_{m_j} - \mathsf{s}\,A_\varphi\,g^{-}_{m_j} = -\left[E - (A_0 + m + S)\right]g^{+}_{m_j}\ . \tag{16}$$

One notices that in eqs. (14) and (13) only the combination $U_\rho + s\,A_\varphi$ enters, meaning that the tensor potential U_ρ and sA_φ are equivalent as far as planar motion with circular symmetry is concerned: pure tensor potentials and pure sA_φ potentials (no other potentials present) with the same functional ρ dependence give exactly the **same** energy spectrum (see [6]).

Motion under an uniform magnetic field $\boldsymbol{B} = B\hat{z}$ implies $A_\varphi = \frac{1}{2}B\rho$ (symmetric gauge) and $U_\rho = A_0 = S = 0$ ($A_0 = S = 0$ in the 2 + 1 case). The second-order differential equations for $g_{k\,m_j}$ and $g^{+}_{m_j}$, are, respetively,

$$\frac{d^2 g_{k\,m_j}}{d\rho^2} - \frac{k(k-1)}{\rho^2}\,g_{k\,m_j} + sB\left(k + \frac{1}{2}\right)g_{k\,m_j} - \frac{1}{4}B^2\rho^2\,g_{k\,m_j} = -(E^2 - m^2)\,g_{k\,m_j}\ , \tag{17}$$

$$\frac{d^2 g^{+}_{m_j}}{d\rho^2} - \frac{m_j(m_j - \mathsf{s})}{\rho^2} + \mathsf{s}\,B\left(\frac{1}{2} + \mathsf{s}\,m_j\right) - \frac{1}{4}\,B^2\rho^2\,g^{+}_{m_j} = -(E^2 - m^2)\,g^{+}_{m_j}\ . \tag{18}$$

From these two equations and the fact that $k = m_j s$ [5], one sees that if one makes to correspondence $\mathsf{s} \to s$, one gets the same solutions for the polar functions $g_{k\,m_j}$ and $g^{+}_{m_j}$.

4. Conclusions

In this work we studied the planar motion of spin 1/2 relativistic fermions with circular symmetry using both $3+1$ and $2+1$ Dirac equations. We developed the full formalism in both cases, with general Lorentz potentials. As an application, we solved the problem of fermion motion in a constant magnetic field perpendicular to the plane of motion.

We found that for planar motion with circular symmetry, the tensor potential and the only non-trivial component of the electromagnetic vector potential $\boldsymbol{A}$ are equivalent if the latter is multiplied by the spin quantum number s. We also found that the parameter s in a generalized $2+1$ Dirac Hamiltonian provides the same solutions as in the $3+1$ case provided a correspondence between s and s is made. However, these numbers are very different in nature. One represents a spin quantum number $(3+1)$ and the other just a parameter in a Hamiltonian which does not describe spin $(2+1)$.

In future works, it would be interesting to apply these results to two-dimensional relativistic fermion systems like graphene. The parameter in the effective graphene $2+1$ Dirac equation should describe other, yet unknown, feature of graphene, and a correspondence could be made with the equivalent quantum number in a 3+1 Dirac equation for graphene, as in [14].

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