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Approaching small- and large-box regimes in field theory

Erich Cavalcanti*

Centro Brasileiro de Pesquisas Físicas/MCTI,
22290-180 Rio de Janeiro, RJ, Brazil

E-mail: erichcavalcanti@gmail.com

We propose to investigate in detail how field theory behaves in restricted spaces. To do so, we consider a parametric representation of Feynman amplitudes, which allows us to discuss the behavior up to all orders in a perturbative expansion and extract consequences valid in a global setting. As a first step, we employ periodically compactified spaces and consider a scalar field theory. We show two valid and equivalent representations: a large-box representation (best suited near the bulk limit) and a small-box representation (best suited near the limit of dimensional reduction). In the small-box regime, we discuss the approach to dimensional reduction and show how it differs from a static-mode approximation.

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*Speaker

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1. Introduction

In recent years, some authors have considered field theories, like QCD, with circle compactification and how it might give new clues regarding the renormalon problem (See references in Ref. [2]). However, there is a lack of investigation in the literature regarding the topic of dimensional reduction of Feynman amplitudes and the general behavior of spatial compactifications up to all loops. Of course, the discussion about only one dimension is well-known, as one can immediately recall from field theory at finite temperature. There, we know how Feynman amplitudes behave both in the limit of high- and low- temperatures. However, as far as we know the general scenario was first discussed by the author and some collaborators years ago, see Refs. [3, 4], where the dimensional reduction was investigated both for a scalar and a fermionic field, and also taking into account the dependence with four different boundary conditions: periodic, antiperiodic, Dirichlet and Neumann. However, these works were restricted just to one-loop contributions.

When we proceed to extend the discussion up to all loops in perturbation theory, we find another small hole in the literature. Although the parametric representation for field theory is well-established and textbook content, there were very few attempts to extend it to the scenario with compactified dimensions. Perhaps the justification for this is the fact that finite-size effects do not contribute if one is interested in the study of renormalizability. Anyway, this does not justify this lack of understanding. To fill this gap we can extend the Schwinger parametric representation to a compactified scenario. This is the theme of the present research, for further details we refer to the full article, see Ref. [2].

2. Parametric representation : Standard

The scenario under investigation is a scalar field theory without derivative couplings. The field lives initially in a D dimensional Euclidean space. The amplitude corresponding to some Feynman graph G is a composition of the internal propagators (I is the number of internal propagators), the vertices (V is the number of vertices), and a constant related to the coupling constants and symmetry factors of the graph.

$$\widetilde{\mathcal{I}}_G = C_G \prod_{i=1}^{I} \left[\int \frac{d^D K_i}{(2\pi)^D} \frac{1}{K_i^2 + m_i^2} \right] \prod_{v=1}^{V} \left[(2\pi)^D \delta^D \left(P_v - \sum_i \varepsilon_{vi} K_i \right) \right], \tag{1}$$

Here ϵ_{vi} is the incidence matrix. We build it by assigning $+1$ when the line starts at vertex and -1 when it ends at the vertex.

What follows is a standard procedure. Firstly one considers some parametrization, such as the Schwinger parametrization, then we deal with both the conservation Dirac deltas and the integrals over the internal momenta. We go on until we obtain the final form of the parametric representation in terms of the so-called Symanzik polynomials $\mathcal{U}$ and $\mathcal{V}$.

$$\mathcal{I}_G = C_G \left[\prod_i \int_0^\infty du_i \right] \frac{e^{-\sum_i u_i m_i^2}}{(4\pi)^{\frac{D}{2}L}} \frac{e^{-\frac{\mathcal{V}(p)}{\mathcal{U}}}}{\mathcal{U}^{\frac{D}{2}}} \tag{2}$$

Perhaps the most significant thing here is that $\mathcal{U}$ and $\mathcal{V}$ can be obtained directly by the topology of the original diagrams. That is, in principle, it became a bit easier to obtain the amplitude of some

Feynman diagram of any order. The Symanzik polynomials are given by the 1-tree contributions ($\mathcal{U}$) and the 2-tree contributions ($\mathcal{V}$). For the interested reader, we refer to e Refs. [5, 8] for a further discussion of the topic.

3. Parametric representation : compactified spaces

There are many prescriptions to consider quantum field theory in restricted dimensions. Here we employ the perspective where some spatial directions are periodically compactified. You can picture it as a restriction on the field itself, that satisfies some periodic identification. Or also as deformation of the space itself, that becomes something like a hypertorus. This is illustrated in Fig. 1. We can say that our field lives in D dimensional space where d of them have a finite length, like a box. In the large box limit we say that the system approaches the bulk, that is, the scenario without any compactification. In the small box limit, we say the system approaches the dimensional reduction, that is, the length scale of some dimensions is so small that can be ignored and the system seems to live in a space with fewer dimensions.

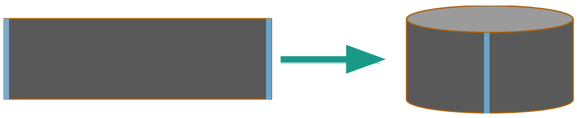

Figure 1: Representation of the periodic compactification employed.

In practice, we employ an extension of the so-called Matsubara formalism. The usual imaginary time Matsubara formalism is used to build models of quantum field theory at finite temperature. A context where dimensional reduction means the limit of very high temperatures. We can extend it for d spatial dimensions. The momenta are now changed to the momenta of the remaining uncompactified dimensions and frequencies related to the compactified dimensions. The integrals over the compactified dimensions are now summed over the modes. And the Dirac deltas became Kronecker deltas. Refs. [6, 7]

This simple modification turns the computation a bit more intricate. The main reason is that originally one could make any shift in the internal momenta to manipulate the integrals, as the momenta belonged to the reals. Now, we have discrete frequencies and we are not allowed to make any shift as we please. This restriction introduces the need for some caution during the computation. To see the steps we refer to Ref. [2], where we define the function G_α.

$$\mathcal{I}_G = C_G \left[\prod_i \int_0^\infty du_i \right] \frac{e^{-\sum_i u_i m_i^2}}{(4\pi)^{\frac{(D-d)}{2} L}} \frac{e^{-\frac{\mathcal{V}(p)}{\mathcal{U}}}}{\mathcal{U}^{\frac{D-d}{2}}} \prod_\alpha \left\{ \sum_{\substack{n_\alpha^{(\ell)} \in \mathbb{Z} \\ \forall \ell \in [1,L]}} \frac{e^{-\frac{4\pi^2}{L_\alpha^2} G_\alpha}}{L_\alpha^L} \right\}, \tag{3}$$

After some manipulations, we get to a point of bifurcation. We have at least two different paths to follow. We can consider the Feynman amplitude in a small-box regime, where the compactified dimensions can approach the limit of dimensional reduction. And we can consider the amplitude in a large-box regime, where the compactified dimensions can approach the bulk limit of infinite length.

It is important to point out that both paths are equivalent, in the sense that we can analytically go from one to another. Anyway, each of them is more suited to one scenario and converges faster in different scales. In principle, one can extract information about dimensional reduction using the large-box representation, and on the bulk using the small-box representation, see Ref [3].

4. Small- and large-box regimes

Firstly let us consider the small-box regime. The small-box approximation means that the length scales L_α of the d compactified dimensions are small in comparison with an arbitrary inverse mass scale Λ. That is, the value of ΛL_α is very small. The consequence of this is that each new mode n introduces an exponential suppression to our amplitude. This means that in the limit of dimensional reduction, where the product approaches zero, the surviving and dominant contribution is the one that minimizes the exponential. That is why we refer to the minimum of the function G.

$$\mathcal{I}_G \sim C_G \left[\prod_i \int_0^\infty du_i\right] \frac{e^{-\sum_i u_i m_i^2}}{(4\pi)^{\frac{(D-d)}{2}L}} \frac{e^{-\frac{\mathcal{V}(p)}{\mathcal{U}}}}{\mathcal{U}^{\frac{D-d}{2}}} \prod_\alpha \left[\frac{e^{-\frac{4\pi^2}{L_\alpha^2}\min G_\alpha}}{L_\alpha^L}\right]. \tag{4}$$

One could, perhaps, expect that in the limit of dimensional reduction the amplitude is just the original one but now with fewer dimensions, this happens in the static mode approximation of finite-temperature field theory, for example. However, when we consider the minimum of G we find the simple result that there might be some surviving information from the compactified dimensions. This surviving information is due to the external modes, as there is no justification whatsoever for them to be taken as zero. One might, perhaps, think of it as the production of a dynamic mass due to dimensional reduction.

On the other hand, the large-box approximation means that the length scales L_α of the d compactified dimensions are large in comparison with an arbitrary inverse mass scale Λ. That is, the value of ΛL_α is very large. In this regime the original expression has a very slow convergence, new modes do not produce a suppression as before. To deal with it we can use an identity of the Jacobi theta, which is a common procedure in the topic of zeta regularization. After some manipulation, we get a new expression that converges faster in the large-box regime.

$$\mathcal{I}_G = C_G \left[\prod_i \int_0^\infty du_i\right] \frac{e^{-\sum_i u_i m_i^2}}{(4\pi)^{\frac{D}{2}L}} \frac{e^{-\frac{\mathcal{V}(P;u)}{\mathcal{U}(u)}}}{\mathcal{U}^{\frac{D}{2}}(u)} \sum_{\substack{n_\alpha^{(\ell)}\in\mathbb{Z} \\ \forall\ell,\forall\alpha}} e^{-\sum_\alpha \frac{L_\alpha^2}{4\mathcal{U}(u)}\mathbf{n}_\alpha^t A(u)\mathbf{n}_\alpha} e^{\frac{2\pi i}{\mathcal{U}(u)}\sum_\alpha \mathbf{n}_\alpha^t A(u)\widetilde{B}_\alpha(u)}. \tag{5}$$

There are two main comments here. First, this expression makes it very easy to see the bulk limit. When the product of the length scale and the mass scale approaches infinity the dominant contribution is exactly the known contribution without any compactification. The second comment is that this expression proves that, up to all loops, the Feynman amplitude in compactified dimensions is separable just like at one-loop. We have a bulk contribution independent of the compactifications (that must be renormalized) and contributions of each compactified dimension. Also, if we make a small parametric transformation, the contribution of compactified dimensions can always be written as a sum over Bessel functions of the second kind.

5. Conclusions

We exhibit two different representations to deal with Feynman diagrams up to all loops in the scenario with restricted dimensions. The formulation discussed is valid for periodic boundary conditions and scenarios with scalar fields. Using this representation we obtain that some known behavior of Feynman amplitudes at 1-loop are indeed valid up to all loops, as the behavior for the small- and large-box regimes. For further details see Refs. [1, 2].

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