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A Simple Approach to the Charmonium Spectrum

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In this work we derive an analytical mass formula for the charmonium states. Starting from the two-body Hamiltonian with the Cornell potential, we use a version of the uncertainty relation (which depends on the n and l quantum numbers) and write the momentum in terms of the relative coordinate. Minimizing the obtained Hamiltonian with respect to this coordinate we determine the equilibrium separation and calculate the energy of the state, which depends on the charm mass, on the coupling constant and on the string tension. We include the spin interactions. These parameters are fixed through a fit of the existing data. We obtain a good fit of the data. This mass formula can be useful to make a first estimate of new states and can also be easily adapted to multiquark systems such as the X,Y and Z states.

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1. Introduction

Since the discovery of charm, the chamonium $(c - \bar{c})$ system has been one of the best tools to understand the color force. The charmonium spectrum has been measured experimentally with great accuracy [1]. The lowest lying states have been calculated with lattice QCD [2] and good agreement with data was found. After reproducing the spectrum, the next step in understanding the QCD interaction is to decompose the obtained masses into the individual contributions of the short range one-gluon exhcange, the long range confining potential, the spin contribution, ... etc. This study has not yet been done in lattice QCD and here is where models can be useful. In the case of heavy quarkonium states (charmonium and bottomonium), modelling is simpler because the system is, to a good approximation, non-relativistic.

In this work we study the charmonium spectrum using a simple analytical model. We treat the system non-relativistically and use the Cornell quark-antiquark potential [3]. However, instead of solving the Schrödinger equation, we simply use the uncertainty relation to substitute the momentum by the associate spatial coordinate in the expression of the Hamiltonian. Then, we minimize the Hamiltonian with respect to this coordinate to find the equilibrium radius of the system. This procedure, sometimes called variational approach or variational principle (VP), is widely used in textbooks of quantum mechanics to develop the intuition of the students [4]. In the case of the hydrogen atom, it is surprisingly succesful. Recently, the VP was introduced in the study the heavy quarkonium spectrum in Ref. [5]. In the present work, we improve the formalism of [5] combining the variational principle and the spin formalism as implemented in [6] in an analytical way. With this improvement, we can calculate, for example, the $\eta_c - J/\psi$ mass difference.

2. Formalism

The Cornell quark-antiquark potencial is given by [3]:

$$V(r) = -\frac{4}{3}\frac{\alpha_s}{r} + br \tag{1}$$

where $\alpha_s (= g_s^2/4\pi)$ is QCD fine structure constant and b is the string tension. The Hamiltonian can be written as:

$$H = m_c + m_{\bar{c}} + \frac{p_r^2}{2\mu} + \frac{l(l+1)}{2\mu r^2} + V(r) \tag{2}$$

where $m_c (= m_{\bar{c}})$ is the charm mass, μ is the reduced mass and l is the angular momentum. In order to apply the uncertainty relation, we follow [5] and define the symmetrical point, $u = r/\sqrt{3}$, and the symmetrical momentum, $\beta_u = p_r/\sqrt{3}$. They are the radial position and momentum with equal components in cartesian coordinates. The uncertainty relation can then be written as:

$$u \cdot \beta_u = \frac{\delta}{3} \tag{3}$$

with [5]

$$\delta = (\sqrt{2(n-1)+l+3})\delta_{n,(1-l)} + \left(2(n-1)+l+\frac{3}{2}\right)(1-\delta_{n,(1-l)}) \tag{4}$$

In the above equation $\delta_{n,(1-l)}$ is the Kronecker delta. Inserting (4) into (3) and the latter into (2) we find:

$$H_{VP} = m_c + m_{\bar{c}} + \frac{\delta^2 + l(l+1)}{6\mu}\frac{1}{u^2} + V(\sqrt{3}u) \tag{5}$$

Minimizing this new Hamiltonian with respect to u, we find the equilibrium value $u = u_{min}$ and the corresponding energy for each n and l combination, obtaining thus the full spectrum.

Once the minimization and performed, u_{min} is found and $V(\sqrt{3}u) = V(\sqrt{3}u_{min})$ is computed, the energy of the spin interaction is introduced as a perturbation (as in [6]). As usual, we consider the three components of the spin interaction: spin-spin (V_{SS}), spin-orbit(V_{LS}) and tensor (V_T). They are given by:

$$V_{spin}(r) = V_{SS}(r) + V_{LS}(r) + V_T(r) = C_{SS}(r)\,\mathbf{S_c}\cdot\mathbf{S_{\bar{c}}} + C_{LS}(r)\,\mathbf{L}\cdot\mathbf{S} + C_T(r)\,\langle\mathbf{S_{c\bar{c}}}\rangle \tag{6}$$

For the Cornell potential, the coefficients are given by:

$$C_{SS}(r) = \frac{32\pi\alpha_s}{9m_c^2}\left(\frac{\sigma}{\sqrt{\pi}}\right)^3 e^{-\sigma^2 r^2} \tag{7}$$

$$C_{LS}(r) = \frac{2\alpha_s}{m_c^2}\frac{1}{r^3} - \frac{b}{2m_c^2}\frac{1}{r} \tag{8}$$

$$C_T(r) = \frac{4\alpha_s}{m_c^2}\frac{1}{r^3} \tag{9}$$

where $\sigma = 0.5859$ GeV is a smearing parameter, taken from [6]. To evaluate the scalar products in (6) we use $\mathbf{S}^2 = \mathbf{S_c^2} + \mathbf{S_{\bar{c}}^2} + 2\,\mathbf{S_c}\cdot\mathbf{S_{\bar{c}}}$ and $\mathbf{J}^2 = \mathbf{L}^2 + \mathbf{S}^2 + 2\,\mathbf{L}\cdot\mathbf{S}$. Finally, $\langle\mathbf{S_{c\bar{c}}}\rangle$ is given by:

$$\langle\mathbf{S_{c\bar{c}}}\rangle = \begin{cases} 0, & \text{if } l = 0 \text{ or } S = 0, \\ -\frac{2l}{(2l+3)}, & \text{if } J = l+1, l \neq 0 \text{ and } S = 1, \\ +2, & \text{if } J = l, l \neq 0 \text{ and } S = 1, \\ -\frac{2(l+1)}{(2l-1)}, & \text{if } J = l-1, l \neq 0 \text{ and } S = 1. \end{cases} \tag{10}$$

Substituting (7), (8), (9) and (10) in (6) and adding this total spin contribution to the energy of the system, (5), we find the masses of the states with n, l and s quantum numbers. They are given by:

$$M(c\bar{c}) = 2m_c + \frac{\delta^2 + l(l+1)}{6\mu}\frac{1}{(2u_{min})^2} + V(\sqrt{3}u_{min}) + V_{spin}(\sqrt{3}u_{min}) \tag{11}$$

As it can be seen, the charmonium spectrum calculated with the above formula, depends on three parameters, m_c, α and b, which can be determined by fitting the experimentally measured spectrum.
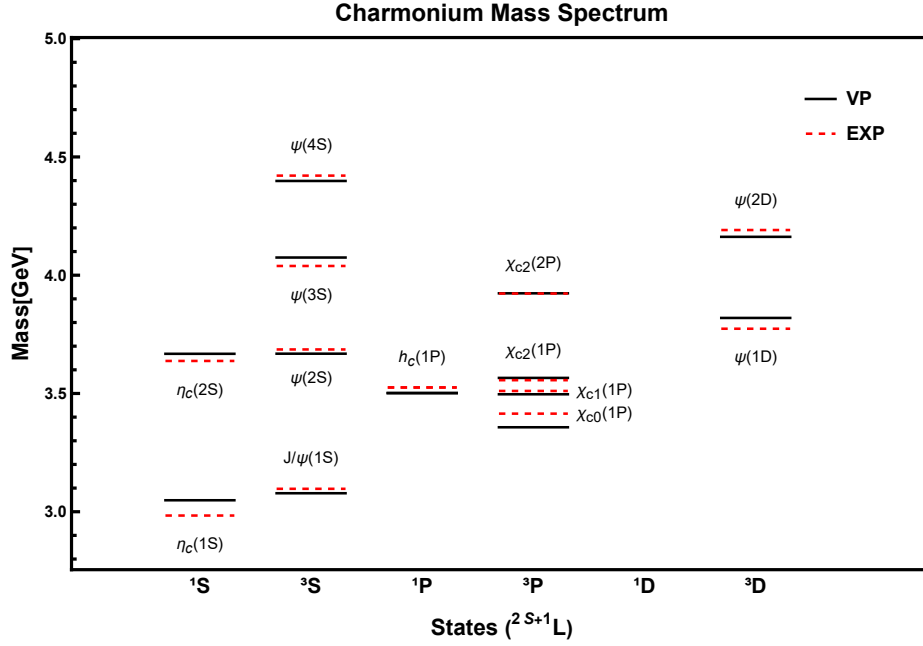
3. Results

To find the best values of the parameters m_c, α and b, we perform a least-square fit, in which we minimize the quantity χ^2 defined as:

$$\chi^2 = \sum_i^n (M_i^{theo} - M_i^{exp})^2 \cdot w_i \tag{12}$$

where the weight $w_i = 1$ is the same for all the states. Fitting the data we find $m_c = 1.5461$ GeV, $\alpha_s = 0.7972$ and $b = 0.1787$ GeV^2. The spectrum is shown in Fig. 1. As it can be seen, the overall agreement between the VP formula (11) and the data is quite good. Looking at $\eta_c(1S)$ and $J/\psi(1S)$, we can observe that the introduction of spin interactions produced a shift in the levels in the right direction, but not strong enough. The experimental mass difference between $\eta_c(1S)$ and $J/\psi(1S)$ is 113 MeV and in our model only 30 MeV.

Another way to show the quality of the fit is through the ratio Theory/Experiment. The same

Figure 1: Charmonium spectrum. The colored dashed lines are the experimental data from [1]. The solid lines are our results with $\alpha_s = 0.7972$, $b = 0.1787$ GeV^2 and $m_c = m_{\bar{c}} = 1.5461$ GeV.

numbers shown in Fig. 1 are displayed again in Table 1. Looking at the last column we can see that the biggest discrepancy between experimental data and model predictions is of the order of 2%.

4. Conclusion

We have derived a new mass formula for the charmonium spectrum. It is based on the use of the uncertainty relation and on the variational principle. The quark-antiquark interaction is described by the Cornell potential. Spin interactions are included as perturbations. The mass formula reproduces the energy levels with 2 % accuracy.

This formula can be used to make approximate predictions of unmeasured energy levels. It can also be applied to the study of four-quark states, such as the very recently observed T_{4c} [6] and T_{cc}^+ [7] and others [8], provided that they are treated as diquark-antidiquark systems.

Acknowledgments

$n^{(2s+1)}l_j$	u_{min}	VP	Meson	Exp	VP/Exp
1^1S_0	1.2114	3.0485	$\eta_c(1S)$	2.9839	1.02
1^3S_1	1.2114	3.0782	$J/\psi(1S)$	3.0969	0.99
1^3P_0	1.9657	3.3571	$\chi_{c0}(1P)$	3.4147	0.98
1^3P_1	1.9657	3.5658	$\chi_{c1}(1P)$	3.5107	1.02
1^1P_1	1.9657	3.5017	$h_c(1P)$	3.5254	0.99
1^3P_2	1.9657	3.4966	$\chi_{c2}(1P)$	3.5562	0.98
2^1S_0	2.3188	3.6677	$\eta_c(2S)$	3.6375	1.01
2^3S_1	2.3188	3.6682	$\psi(2S)$	3.6861	1.00
1^3D_1	2.7161	3.8193	$\psi(1D)$	3.7737	1.01
1^3D_2	2.7161	3.8675	-	-	-
1^1D_2	2.7161	3.8403	-	-	-
1^3D_3	2.7161	3.8299	-	-	-
2^3P_0	2.9312	3.8935	-	-	-
2^3P_1	2.9312	3.9524	-	-	-
2^1P_1	2.9312	3.9297	-	-	-
2^3P_2	2.9312	3.9234	$\chi_{c2}(2P)$	3.9225	1.00
3^1S_0	3.2902	4.0747	-	-	-
3^3S_1	3.2902	4.0747	$\psi(3S)$	4.0390	1.01
2^3D_1	3.5176	4.1622	$\psi(2D)$	4.1910	0.99
2^3D_2	3.5176	4.1794	-	-	-
2^1D_2	3.5176	4.1645	-	-	-
2^3D_3	3.5176	4.1547	-	-	-
4^1S_0	4.1248	4.3984	-	-	-
4^3S_1	4.1248	4.3984	$\psi(4S)$	4.4210	0.99

Table 1: Masses of the $c\bar{c}$ states. All the masses are in GeV, and u_{min}, in GeV^{-1}. Experimental data are from [1].

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