PROCEEDINGS OF SCIENCE

On the R_K Theory Error

Saad Nabeebaccus[a,b,*] **and Roman Zwicky**[c]

[a] *Department of Physics, Universität Zürich, Winterthurerstrasse 190, CH-8057 Zürich, Switzerland*

[b] *IJCLab, CNRS/IN2P3, Université Paris-Saclay, Orsay 91898, France*

[c] *Higgs Centre for Theoretical Physics, School of Physics and Astronomy, University of Edinburgh, Edinburgh EH9 3JZ, Scotland*

E-mail: saad.nabeebaccus@ijclab.in2p3.fr, roman.zwicky@ed.ac.uk

We compute the double differential rate of $\bar{B} \to \bar{K}\ell^+\ell^-$ at $O(\alpha)$ in QED, using a mesonic effective theory framework. While soft and soft collinear logarithms cancel at the differential level in any set of kinematic variables, the cancellation of hard collinear logarithms $\ln m_\ell$ depends on the choice of differential variables even in the photon-inclusive case. Crucially, using gauge invariance, we show that structure-dependent QED corrections do not lead to additional hard collinear logs. Using photon energy cuts, emulating the LHCb procedure, we report a correction to R_K of $\approx 1.7\%$ due to QED. This correction is accidentally small due to the cuts and is expected to be largely captured by PHOTOS. Finally, the effect of migration of radiation is discussed, whereby QED corrections at larger q^2 (lepton pair momentum squared) can "leak" into lower values. This effect could be relevant in view of charmonium resonances and is deserving of further attention.

11th International Workshop on the CKM Unitarity Triangle (CKM2021)
22-26 November 2021
The University of Melbourne, Australia

*Speaker

PoS(CKM2021)071

1. Introduction

In the Standard Model (SM), lepton flavours couple to gauge bosons with the same coupling strength, giving rise to the concept of *lepton flavour universality* (LFU). Thus, an important test of the SM is to consider *LFU ratios*, and compare theory with experiment. An example is R_K, defined as the ratio of branching fractions of $B \to K\mu^+\mu^-$ and $B \to Ke^+e^-$,

$$R_K\left[q^2_{\rm min}, q^2_{\rm max}\right] = \frac{\int_{q^2_{\rm min}}^{q^2_{\rm max}} dq^2 \frac{d\Gamma(B\to K\mu^+\mu^-)}{dq^2}}{\int_{q^2_{\rm min}}^{q^2_{\rm max}} dq^2 \frac{d\Gamma(B\to Ke^+e^-)}{dq^2}} \,, \tag{1}$$

in bins of $q^2 = (\ell^+ + \ell^-)^2$ (the lepton pair momentum squared). The advantage of such ratios is that (non-perturbative) QCD corrections cancel as they are independent of the lepton flavour.

However, since 2014, the LHCb collaboration has been reporting discrepancies,

$$R_K\left[1.1\text{GeV}^2, 6\text{GeV}^2\right] = 0.846^{+0.042+0.013}_{-0.039-0.012} \,, \tag{2}$$

with the latest measurment [1], whereas in theory, one expects

$$R_K\left[1.1\text{GeV}^2, 6\text{GeV}^2\right] \approx 1 + \Delta_{\rm QED} \,, \tag{3}$$

where the matter of $\Delta_{\rm QED}$ correction is the subject of this proceeding. Ignoring the latter, Eq. (2) implies a 3.1 σ deviation and, together with other deviations in B-physics, has created quite some excitement.

Let us turn to QED. The crucial point is that the lepton masses do break LFU and that their scales are so different from the b mass scale that they can give rise to significant corrections. Concretely, the fine structure constant $\frac{\alpha}{\pi} \approx 2 \cdot 10^{-3}$ is enhanced by collinear logs by an order of magnitude $\frac{\alpha}{\pi} \ln \frac{m_e}{m_B} \gtrsim$ 2-3 % and with order one coefficient can amount up to 10% [2, 3]. In practice the situation is more subtle as the experiment aims to subtract $\Delta_{\rm QED}$ from the result reported in (2) by using the `PHOTOS` Monte Carlo tool. Thus, the title of this proceeding is slightly misleading and the crucial question is how large the corrections beyond `PHOTOS` are, as the latter treats the mesons as point particles. In computing the QED corrections to $B \to K\ell^+\ell^-$ at the double differential level (in q^2 and the lepton angle), we are able to give a largely positive answer, namely that R_K is a rather safe observable.

2. Framework

2.1 Effective meson theory

We start from an *effective meson theory*, where the $\bar{B} \to \bar{K}\ell_1\bar{\ell}_2$ decay is mediated by

$$\mathcal{L}^{\rm EFT}_{\rm int} = g_{\rm eff}\, L^\mu V^{\rm EFT}_\mu + \text{h.c.} \,, \tag{4}$$

a lepton-hadron current interaction

$$L_\mu \equiv \bar{\ell}_1\gamma^\mu(C_V + C_A\gamma_5)\ell_2, \quad V^{\rm EFT}_\mu = \sum_{n\geq 0} \frac{f_\pm^{(n)}(0)}{n!}(-D^2)^n\left[(D_\mu B^\dagger)K \mp B^\dagger(D_\mu K)\right] \,, \tag{5}$$

where the leptons $\ell_{1,2}$ are kept generic, $g_{\rm eff} \equiv 2\frac{G_F}{\sqrt{2}}\lambda_{\rm CKM}$, $C_{V(A)} \equiv -\frac{\alpha}{2\pi}C_{9(10)}$ are Wilson coefficients, $f_\pm^{(n)}(0)$ represent the $n^{\rm th}$ derivative of the form factor $f_\pm(q^2)$, evaluated at $q^2 = 0$, and D_μ is the covariant derivative, used to enforce gauge invariance in minimal form. This framework goes *beyond* scalar QED (treating the mesons as point particles) in expanding the form factor. The matching condition for the expansion is the reproduction of the leading oder (LO) matrix element

$$\langle \bar{K}|V_\mu|\bar{B}\rangle = f_+(q^2)(p_B+p_K)_\mu + f_-(q^2)(p_B-p_K)_\mu = \langle \bar{K}|V_\mu^{\rm EFT}|\bar{B}\rangle + O(e) \ , \tag{6}$$

where $V_\mu \equiv \bar{s}\gamma_\mu(1-\gamma_5)b$. Other than that, mesons are treated as pointlike particles. That is to say the itself photon does not resolve the mesons. An important aspect to clarify are the kinematics as they control the IR-safety.

2.2 Kinematics

We consider two sets of variables for the differential distribution of $\bar{B}(p_B) \to \bar{K}(p_K)\ell^+(\ell^+)\ell^-(\ell^-)\gamma(k)$ process, assuming that radiation is not detected:

$$\{q_a^2, c_a\} = \begin{cases} q_\ell^2 = (\ell_1+\ell_2)^2, & c_\ell = -\left(\frac{\vec{\ell}_1\cdot\vec{p}_K}{|\vec{\ell}_1||\vec{p}_K|}\right)_{q-{\rm RF}} & [\text{“Hadron collider” variables}] \ , \\ q_0^2 = (p_B-p_K)^2 \ , & c_0 = -\left(\frac{\vec{\ell}_1\cdot\vec{p}_K}{|\vec{\ell}_1||\vec{p}_K|}\right)_{q_0-{\rm RF}} & [\text{“B-factory” variables}] \ , \end{cases} \tag{7}$$

where $q-$RF and q_0-RF denotes the rest frames (RF) of

$$q \equiv \ell^+ + \ell^- \ , \quad q_0 \equiv p_B - p_K = q + k \ , \tag{8}$$

and $c_\ell \equiv \cos\theta_\ell$, $c_0 \equiv \cos\theta_0$. This is illustrated in Fig. 1.

For the real radiation, one needs to integrate over the photon momentum, and for this purpose, we define a cut-off for the photon energy (related to detector resolution) in a Lorentz invariant way

$$\bar{p}_B^2 \equiv (p_B-k)^2 \ \geq \ m_B^2\,(1-\delta_{\rm ex}) \ , \tag{9}$$

using the experimentally reconstructed mass of the B meson.

3. Computations and IR-safe Differential Variables $\{q_0^2, c_0\}$

In computing the real and virtual parts, we employ phase space slicing to separate the IR sensitive terms into integrals that can be computed analytically. This leads to numerically stable cancellation of the IR divergences.

The differential rate is parameterised as follows

$$\begin{aligned} d^2\Gamma_{\bar{B}\to\bar{K}\ell_1\bar{\ell}_2}(\delta_{\rm ex}) &= d^2\Gamma^{\rm LO} + \frac{\alpha}{\pi}\sum_{i,j}\hat{Q}_i\hat{Q}_j\left(\mathcal{H}_{ij} + \mathcal{F}_{ij}^{(a)}(\delta_{\rm ex})\right)\, dq_a^2 dc_a + O(\alpha^2) \\ &= d^2\Gamma^{\rm LO}\left[1 + \Delta^{(a)}(q_a^2, c_a; \delta_{\rm ex})\right]\, dq_a^2 dc_a + O(\alpha^2) \ , \end{aligned} \tag{10}$$

where $d^2\Gamma^{\rm LO}$ corresponds to the LO differential rate, the indices i, j run over all charged particles in the decay, and $\mathcal{H}$ and $\mathcal{F}$ stand for the virtual and real contributions respectively. The charges $\hat{Q}_i$ are defined such that $\sum_i \hat{Q}_i = 0$, cf. [3].

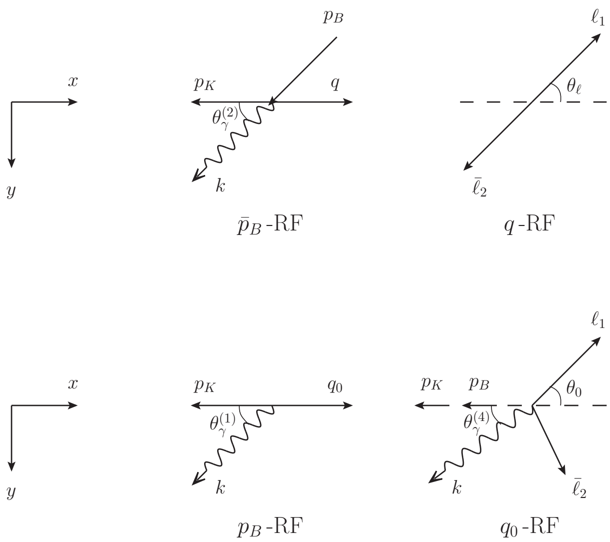

Figure 1: Decay kinematics for the different RFs of interest. The dashed line corresponds to the decay axis, defined by the direction of the outgoing kaon.

In order to separate IR sensitive regions of the real integration, the *two cutoff phase space slicing* procedure [4] is employed. It requires the introduction of two small unphysical parameters $\{\omega_s, \omega_c\}$ that allows the separation of the real part as follows

$$\mathcal{F}_{ij}^{(a)}(\delta_{\text{ex}}) = \frac{d^2\Gamma^{\text{LO}}}{dq^2 dc_\ell}\tilde{\mathcal{F}}_{ij}^{(s)}(\omega_s) + \tilde{\mathcal{F}}_{ij}^{(hc)(a)}(\underline{\delta}) + \Delta\mathcal{F}_{ij}^{(a)}(\underline{\delta}) \,, \tag{11}$$

where $\underline{\delta} = \{\omega_s, \omega_c, \delta_{\text{ex}}\}$, and s and hc stand for 'soft' and 'hard-collinear'. By *hard collinear divergences*, we mean any photon emission with energy *above* the soft cut-off ω_s, collinear to either lepton in the final state.

This procedure allows us to compute the IR sensitive real integrals $\tilde{\mathcal{F}}_{ij}^{(s)}$ and $\tilde{\mathcal{F}}_{ij}^{(hc)(a)}$ *analytically* up to terms of $O(\omega_{c,s})$ which are negligible. The relevant dependence on the unphysical cut-offs is logarithmic and cancel in the sum of the three terms. The *numerical* integration in $\Delta\mathcal{F}_{ij}^{(a)}$ is performed in the region where

$$\bar{p}_B^2 \leq m_B^2\,(1-\omega_s)\,, \quad k\cdot\ell_{1,2} \geq \omega_c m_B^2 \,. \tag{12}$$

We find that the soft and soft-collinear divergences cancel at the double differential level, independent of the choice of differential variables and photon energy cut-off δ_{ex}. This is expected, by virtue of the KLN theorem. The hard-collinear logarithms ($\ln m_\ell$) are more interesting as the KLN theorem guarantees their cancellation in the fully inclusive case (i.e. photon inclusive and integration over differential variables (7)). Thus, the question is: Does the cancellation survive in any of the two sets of differential variables used? It turns out that $\{q_0^2, c_0\}$ are the *collinear-safe* variables, whereas for $\{q^2, c_\ell\}$, the hard-collinear logs do not cancel. This effect is at the heart of the 10% QED corrections quoted in the introduction. Intuitively, the $\{q_0^2, c_0\}$ variables are collinear-safe since for those, the B and K mesons can be thought of as one particle (with 4-momentum $p_B - p_K$) and then its decay is analogous to a $Z \to \ell^+\ell^-$ decay which is non-differential and IR finite by the KLN theorem.

For the outlook on R_K, a central result of our work is that we were able to show, using *gauge invariance* and the lepton equation of motion, that structure-dependent corrections (absent in our calculation, since we treated photon interactions with the mesons as *scalar* interactions) do **not** contribute to hard-collinear logarithms $\ln m_\ell$ (cf. section 3.4 of [3]). If this was not the case then it would be very hard to justify a solid

4. Results in terms of Plots

We present the results as plots of QED corrections normalised with the LO differential rate, ie.

$$\Delta^{(a)}(q_a^2;\delta_{\rm ex}) = \left(\frac{d\Gamma^{\rm LO}}{dq_a^2}\right)^{-1} \frac{d\Gamma(\delta_{\rm ex})}{dq_a^2}\Big|_\alpha \,, \tag{13}$$

with the numerator and denominator separately integrated over the angular variable c_a (defined in (7)). Our main plots are given in Figs. 2 and 3, cf. the main paper for further plots [3].

Fig. 2 is divided into the q_0^2- and q^2-variable on the left and right and bottom and top correspond to neutral and charged mesons respectively. For q_0^2 in the photon inclusive case (dashed lines and $\delta_{\rm ex} = \delta_{\rm ex}^{\rm inc}$), the $\ln m_\ell$ terms cancel, as previously stated, and thus radiative corrections are small ($O(\frac{\alpha}{\pi})$). In the charged case, the "hard-collinear" logs of the kaon mass, $\ln m_K$, (bottom left) do seem to give rise to a sizeable physical effect (up to $\sim 2\%$). On the other hand, in the variable q^2, the hard-collinear logarithms do not cancel at the differential level, and this explains why none of the lines remain close to zero. Of course, when one integrates over q^2 in the fully inclusive case, the hard-collinear logs have to cancel, which we have checked analytically. This can be seen from the dashed lines on plots on the RHS in Fig. 2 going from positive to negative values with increasing q^2. For a photon energy cut-off corresponding to $\delta_{\rm ex} = 0.1$, which is in between the values used for electrons and muons by LHCb, the QED effects are sizeable and are of course more pronounced for electrons as can be seen from the plots.

Estimating the QED correction to R_K based on our short distance analysis, using photon cuts to emulate the LHCb procedure [1], we find

$$\Delta_{\rm QED} R_K \approx \frac{\Delta\Gamma_{K\mu\mu}}{\Gamma_{K\mu\mu}}\Bigg|_{q_0^2\in[1,6]\,{\rm GeV}^2}^{m_B^{\rm rec}=5.175\,{\rm GeV}} - \frac{\Delta\Gamma_{Kee}}{\Gamma_{Kee}}\Bigg|_{q_0^2\in[1,6]\,{\rm GeV}^2}^{m_B^{\rm rec}=4.88\,{\rm GeV}} \approx +1.7\% \,, \tag{14}$$

which is accidentally small due to the cuts in use. In [2], a correction of $\Delta_{\rm QED} R_K \approx 3\%$ was reported, where a tight angle cut was applied, in addition to photon energy cuts. This highlights the importance of building a dedicated Monte Carlo based directly on matrix elements in order to verify `PHOTOS` [5], which is used to simulate QED corrections in the experimental analysis of LHCb. This will be discussed in [6].

However, the *migration of radiation* effect needs to be properly assessed, in view of the charmonium resonances which are currently not included in the results above and are difficult to handle in experiment. The effect is illustrated in Fig. 3 by choosing different shapes of q^2-dependence for the form factors. Specifically, we do so by plotting the constant part of the form factors (dashed lines) versus the q^2-expanded form factor (solid line) to linear order (cf. Eq. (5)). The pale pink colour corresponds to a photon cut-off of $\delta_{\rm ex} = 0.1$, while the dark red colour

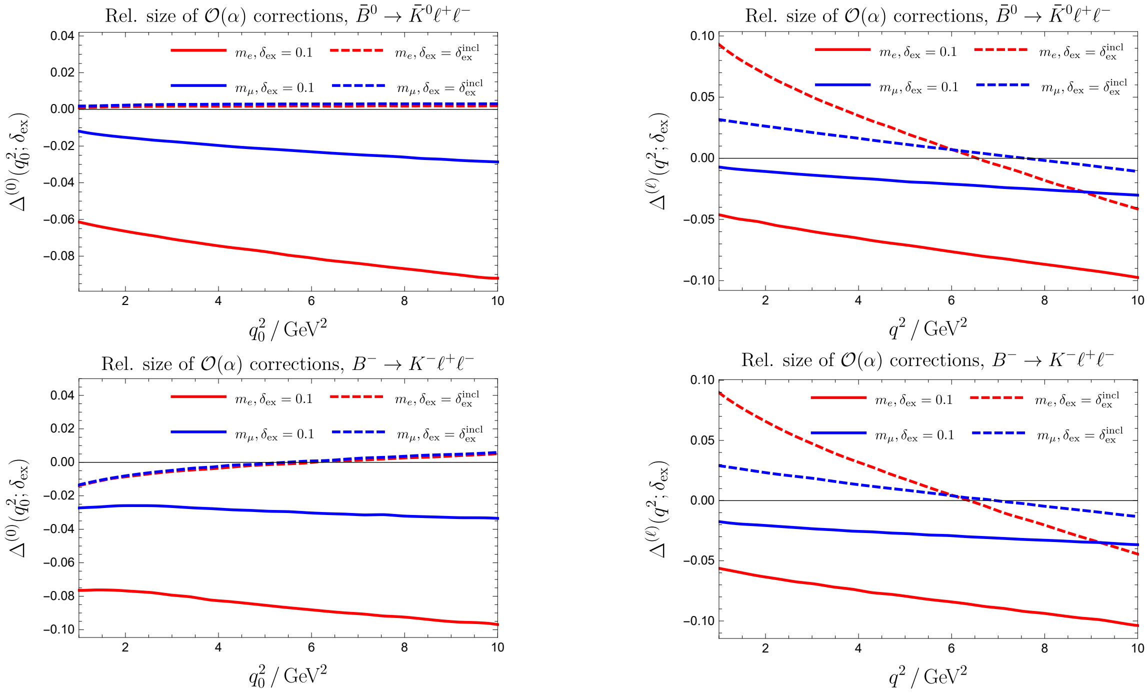

Figure 2: The relative sizes of QED corrections are shown as a function of q_a^2. The top and bottom plots represent the neutral and charged meson cases respectively. The left and right plots correspond to results differential in q_0^2 and q^2 respectively.

corresponds to the fully inclusive case. It is found that the effect of the form factors is small when differential in q_0^2. However, for q^2 the size of the relative corrections are *significantly* affected by the form factors (cf. Fig. 3). The effects are larger when one is more photon inclusive. This is due to the fact that, for the q^2 distribution, a fixed value of q^2 probes *higher* values of q_0^2, and the looser the photon energy cut-off, the wider is the range of q_0^2 that is probed (in fact, in the fully-inclusive case, the *entire* spectrum is probed).

While this effect is sizeable, it is significantly exacerbated when one considers charmonium resonances. Thus, one needs to make sure that the experimental analysis properly takes these effects into account. We will discuss them in a forthcoming publication [6].

5. Conclusion

Our results show that it is important to properly take into account QED corrections, as these are enhanced by hard-collinear logarithms of the lepton mass. Soft and soft-collinear divergences are universal and cancel at the differential level. They resurface as $\ln \delta_{ex}$ effects and should be properly taken into account by the `PHOTOS` tool used in experiment.[1] Hard-collinear divergences are more interesting as they are higher in energy and could probe the mesonic structure. Moreover, they may not cancel, depending on the choice of differential variables (even in the fully photon inclusive case). However, since we were able to show, using gauge invariance, that structure-dependent corrections

[1] `PHOTOS` employs a splitting function approach which captures the leading logs of the point-like particles and resums the soft logs to all orders à la YFS. The virtual corrections are indirectly inferred from the KLN theorem which again captures the leading logs. The approach in [2] is similar in spirit to `PHOTOS`.

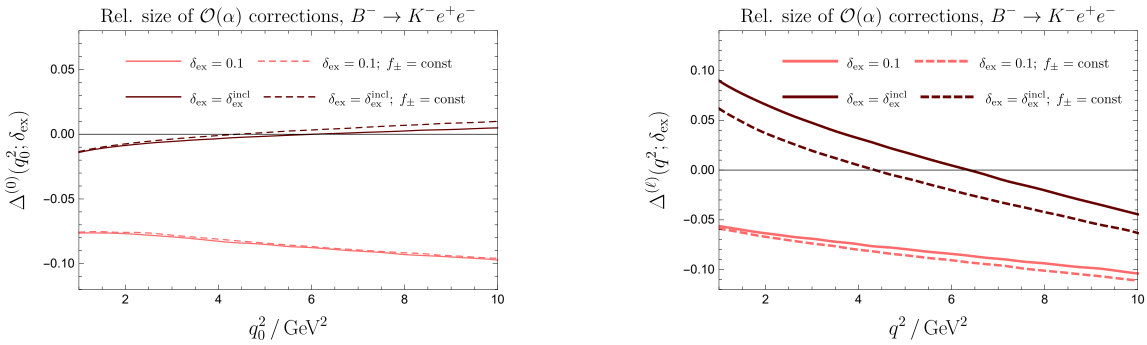

Figure 3: The effects of migration of radiation are shown. The left and right plots represent relative QED corrections in q_0^2 and q^2 respectively. Dashed lines correspond to a constant form factor, while the solid line includes the first derivative of the form factor (see Eq. (5)).

do not lead to further hard-collinear logs of the $\ln m_\ell$-type, PHOTOS can, in principle, be expected to be a reliable tool. Note that the same does not apply to the $\ln m_K$-type "collinear" logs, since the K-meson itself contains substructure.

Our results show that migration effects in the q^2-variable are sizeable for larger photon energy cut-offs. Consequently, care needs to be taken when the charmonium resonances, with their pronounced q^2-dependence, come into play. We hope to address their significance to R_K in future work [6].

Hence, the LFU observable R_K, and others of the same type such as R_{K^*} or R_D, seem robust with regard to QED corrections. That is, structure-dependent corrections cannot be expected to overthrow the main picture. Sizeable effects such as the $\ln m_K$-terms and the migration of charmonium resonances might and will become more relevant in the future. Hence, QED corrections will keep the community occupied, be it for establishing LFU or precision CKM-element extraction.

References

[1] LHCb collaboration, *Test of lepton universality in beauty-quark decays*, 2103.11769.

[2] M. Bordone, G. Isidori and A. Pattori, *On the Standard Model predictions for R_K and R_{K^*}*, *Eur. Phys. J.* **C76** (2016) 440 [1605.07633].

[3] G. Isidori, S. Nabeebaccus and R. Zwicky, *QED corrections in $\overline{B} \to \overline{K}\ell^+\ell^-$ at the double-differential level*, *JHEP* **12** (2020) 104 [2009.00929].

[4] B.W. Harris and J.F. Owens, *The Two cutoff phase space slicing method*, *Phys. Rev.* **D65** (2002) 094032 [hep-ph/0102128].

[5] N. Davidson, T. Przedzinski and Z. Was, *PHOTOS interface in C++: Technical and Physics Documentation*, *Comput. Phys. Commun.* **199** (2016) 86 [1011.0937].

[6] G. Isidori, D. Lancerini, S. Nabeebaccus and R. Zwicky, "to appear."