PROCEEDINGS OF SCIENCE

Symmetries at Null Boundaries: 3-dimensional Einstein gravity

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Gauge transformations are usually viewed as redundancies in the description of gauge theories and the physical observables must be gauge invariant. This should be revisited in presence of boundaries where a part of gauge transformations to which there are non vanishing surface charges associated, can become physical "non-proper" gauge transformations. One can use these surface charges to label different points of the solution phase space. Here we consider the Einstein gravity in presence of a given null boundary. We construct the maximal solution-phase space, find its symmetries and calculate the associated surface charges. Surface charges and their algebra depend on the slicing in solution phase space. We discuss the implications of the change of slicing in different aspects of solution phase space, from integrability to algebra of surface charges.

The talk based on [1, 2] and presented by V. Taghiloo in RDP online PhD school and workshop "Aspects of Symmetry", Tbilisi, Georgia,

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1. Introduction

Gauge theories enjoy local symmetries and we usually treat them as redundancies in description of theory. But in presence of boundaries a part of these transformations, large gauge transformations/diffeomorphisms, can become physical. They are large in the sense that they act non trivially at the boundary. From the first and second Noether theorems, one can associate non vanishing charges to these transformations. Then, these charges can be used to label different points of the solution phase space.

In presence of boundaries in addition to bulk degrees of freedom (d.o.f) we need to account for boundary d.o.f. In other words, we need to enlarge the Hilbert space of the theory in such a way it includes these new boundary d.o.f. The surface charges associated with large diffeomorphisms provide a natural way to describe these boundary d.o.f. In this talk, we focus on formulating the Einstein gravity in three dimensions in presence of a null boundary. The main property of 3-dimensional Einstein gravity is that it does not have any bulk d.o.f. So, its dynamics only arises from the boundary d.o.f.

To calculate the surface charges associated with large diffeomorphisms, we use the covariant phase space method [3–6] which gives the charge variation over the solution phase space. In general, this charge variation could be non-integrable. In this regard, we provide a specific statement that in absence of bulk (hard) modes passing through the boundary, there are specific slicings in the solution phase space of theory which yield integrable expressions for the charge variation, *integrable slicing* [1, 2, 7, 8]. To clarify these notions we will give an explicit example in the context of the three-dimensional Einstein gravity.

2. Null Boundary Symmetry (NBS) Algebra, *3d* Gravity Case

We consider the three-dimensional Einstein gravity in presence of the cosmological constant, with the following action

$$S = \frac{1}{16\pi G}\int \mathrm{d}^3x\,\sqrt{-g}\,(R-2\Lambda)\ . \tag{1}$$

Depending on Λ, $\Lambda < 0, \Lambda = 0, \Lambda > 0$ we respectively have AdS_3, flat or dS_3 gravities. The stationary-action principle yields

$$\mathcal{E}_{\mu\nu} := R_{\mu\nu} - 2\Lambda g_{\mu\nu} = 0\,. \tag{2}$$

To construct the solution phase space of this theory, we take the following ansatz for the line element [1, 2, 8]

$$ds^2 = -V dv^2 + 2\eta dv dr + g\,(d\phi + U dv)^2 \tag{3}$$

here we assume $r = 0$ to be a null surface, $V(r = 0) = 0$, and denote this null hypersurface by $\mathcal{N}$. We also assume the Taylor expandability of the line element and write it as

$$\begin{aligned} V &= 2\left(\eta\kappa - \partial_v\eta + \mathcal{U}\partial_\phi\eta\right) r + O(r^2)\,,\\ U &= \mathcal{U} - \frac{\eta\Upsilon}{\Omega^2} r + O(r^2)\,,\\ g &= \Omega - 2\eta\lambda\, r + O(r^2)\,. \end{aligned} \tag{4}$$

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All functions which appear in these expansions, are generic functions of v and ϕ. They have geometrical meanings, specially we can think about κ, $\mathcal{U}$, Υ and Ω as surface gravity, velocity aspect of the boundary, angular momentum aspect and area density respectively. Einstein's equations at leading order expansion of r lead to

$$\mathcal{D}_v\Theta_l - \kappa\Theta_l + \frac{1}{D-2}\Theta_l^2 = 0\,, \tag{5a}$$

$$\mathcal{D}_v(\Omega\mathcal{H}) - \Omega\partial_\phi\kappa = 0\,, \tag{5b}$$

$$\mathcal{D}_v\Theta_n + \kappa\Theta_n + \Theta_l\Theta_n - \left(\bar{\nabla}_C\mathcal{H}^C + \mathcal{H}^C\mathcal{H}_C\right) - \Lambda = 0\,. \tag{5c}$$

The first two equations are Raychaudhuri and Damour's equations respectively. Here we have also introduced three further geometric quantities

$$\Theta_l = \frac{\mathcal{D}_v\Omega}{\Omega}, \qquad \Theta_n = \frac{\lambda}{\Omega}, \qquad \mathcal{H} = \frac{\Upsilon}{2\Omega} + \frac{\partial_\phi\eta}{2\eta}\,. \tag{6}$$

For latter convenience we introduce the differential operators $\mathcal{D}_v$ and $\mathcal{L}_{\mathcal{U}}$ which their action on a codimension one function $O_w(v,\phi)$ of weight w is defined through [2, 6]

$$\mathcal{D}_v O_w = \partial_v O_w - \mathcal{L}_{\mathcal{U}} O_w\,, \tag{7a}$$

$$\mathcal{L}_{\mathcal{U}} O_w = \mathcal{U}\partial_\phi O_w + w O_w \partial_\phi \mathcal{U}\,, \tag{7b}$$

where $\mathcal{U}$ is a function of weight -1. Weights of different functions can be found in Table 1.

$w=-1$	$\mathcal{U}\,,Y\,,\tilde{Y}$
$w=0$	$\eta\,,T\,,W\,,\Theta_l\,,\Theta_n\,,\kappa\,,\Gamma\,,\tilde{T}\,,\tilde{W}\,,\mathcal{P}\,,\partial_v$
$w=1$	$\Omega\,,\lambda\,,\mathcal{H}\,,\partial_\phi$
$w=2$	$\Upsilon\,,\mathcal{J}$

Table 1: Weight w for various quantities defined and used in this section.

Solution phase space. A careful analysis of Einstein's equations shows the solution phase space is parameterized by three codimension one functions: $\eta(v,\phi)$, $\Omega(v,\phi)$ and $\Upsilon(v,\phi)$. In other words, the determination of these boundary data yields a unique solution. We interpret them as labels for the boundary degrees of freedom. Based on these kinds of symmetry analyses, it has been shown that the boundary d.o.f on a generic null surface show a local thermodynamic description [9].

Null boundary preserving diffeomorphisms. Diffeomorphisms generated by the vector field

$$\xi = T\,\partial_v + r(\mathcal{D}_v T - W)\,\partial_r + \left(Y - r\frac{\eta}{\Omega}\partial_\phi T\right)\partial_\phi + \mathcal{O}(r^2) \tag{8}$$

keep $r=0$ as a null surface. Our null boundary preserving diffeomorphisms are specified by three symmetry generators, $T=T(v,\phi)$, $W=W(v,\phi)$ and $Y=Y(v,\phi)$. They are called supertranslation, superscaling and superrotation respectively. Since the Einstein equations are covariant, these diffeomorphisms move us in the solution space, namely

$$\delta_\xi\eta = 2\eta\mathcal{D}_v T + T\partial_v\eta - W\eta + Y\partial_\phi\eta\,, \tag{9a}$$

$$\delta_\xi\Omega = T\Omega\Theta_l + \partial_\phi[\Omega(Y+\mathcal{U}T)]\,, \tag{9b}$$

$$\delta_\xi\Upsilon = T\mathcal{D}_v\Upsilon + \mathcal{L}_{(Y+T\mathcal{U})}\Upsilon + \Omega(\partial_\phi W - \Gamma\partial_\phi T)\,. \tag{9c}$$

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Algebra of null boundary symmetries. It is well known that the large diffeomorphisms make an algebra. In our case, due to the explicit field dependence of them (8), we need to use the adjusted Lie bracket [10, 11]. If we do so, then we will have

$$[\xi(T_1, W_1, Y_1), \xi(T_2, W_2, Y_2)]_{\text{adj. bracket}} = \xi(T_{12}, W_{12}, Y_{12}) \tag{10}$$

where

$$T_{12} = \left(T_1\partial_v + Y_1\partial_\phi\right) T_2 - (1 \leftrightarrow 2), \tag{11a}$$
$$W_{12} = \left(T_1\partial_v + Y_1\partial_\phi\right) W_2 - (1 \leftrightarrow 2), \tag{11b}$$
$$Y_{12} = \left(T_1\partial_v + Y_1\partial_\phi\right) Y_2 - (1 \leftrightarrow 2). \tag{11c}$$

The above algebra is Diff($\mathcal{N}$) $\Subset$ Weyl($\mathcal{N}$) [1, 7], where Diff($\mathcal{N}$) is generated by T, Y and Weyl($\mathcal{N}$) which denotes the Weyl scaling on $\mathcal{N}$, is generated by W. We refer to it as null boundary symmetry algebra.

Surface charges. So far we have constructed the solution phase space and also studied its symmetries. Now the natural question is what are the surface charges associated with these symmetries. To answer this question we use the covariant phase space method (CPSM) [3–5], which yields the following expression for the charge variation

$$\not{\delta} Q_\xi = \frac{1}{16\pi G}\int_{\mathcal{N}_v} d\phi \left(W\delta\Omega + Y\delta\Upsilon + T\not{\delta}\mathcal{A}\right) , \tag{12}$$

with

$$\not{\delta}\mathcal{A} = -2\Omega\delta\Theta + \Omega\Theta\frac{\delta\eta}{\eta} - \Gamma\delta\Omega + \mathcal{U}\delta\Upsilon \, . \tag{13}$$

Here we have introduced a new quantity

$$\Gamma := -2\kappa + 2\Theta + \frac{\partial_v\eta}{\eta} - \frac{\mathcal{U}\partial_\phi\eta}{\eta} \, . \tag{14}$$

and our integral is taken over a v-constant cross section of the null boundary $\mathcal{N}$ which we denote it by $\mathcal{N}_v$. As it is clear from the surface charge expression (12), the charge variation, $\not{\delta} Q_\xi$, is not integrable. So, we need to separate the charge variation into the integrable and non-integrable (flux) parts. To do so, we need to adopt some physical criteria. In this regard, we use the representation theorem [3, 4] in the covariant phase space method to split the charge variation. This theorem states the charge algebra is the same as the algebra of the symmetry generators up to a central extension term. We separate the charge variation in such a way it leads to a field independent central extension term. By using this criterion, we get

$$Q^{\text{I}}_\xi = \frac{1}{16\pi G}\int_{\mathcal{N}_v} d\phi \left\{W\,\Omega + Y\,\Upsilon + T\,\left(-\Gamma\Omega + \mathcal{U}\Upsilon\right)\right\} , \tag{15}$$

and

$$F_\xi(\delta g; g) = \frac{1}{16\pi G}\int_{\mathcal{N}_v} d\phi\, T\left[-2\Omega\delta\Theta + \Omega\Theta\frac{\delta\eta}{\eta} + \Omega\delta\Gamma - \Upsilon\delta\mathcal{U}\right] . \tag{16}$$

By using the modified bracket [10], we get the surface charge algebra is the same as null boundary symmetry algebra without any central extension term.

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Change of slicing. In this part, we introduce the notion of change of slicing on solution phase space [1, 2, 6–8]. To do so, we explain it in an example. Let us consider the following field dependent combinations of the symmetry generators

$$
\begin{aligned}
\tilde{W} &= W - \Gamma T - (Y + T\mathcal{U})\,\partial_\phi \mathcal{P}\,,\\
\tilde{T} &= \Omega\Theta T + \partial_\phi[\Omega(Y + T\mathcal{U})]\,,\\
\tilde{Y} &= Y + T\mathcal{U}\,.
\end{aligned}
$$

Now, we rewrite the charge variation in terms of these generators

$$
\delta Q = \frac{1}{16\pi G}\int_{\mathcal{N}_v} d\phi\,\left(\tilde{W}\delta\Omega + \tilde{Y}\delta\mathcal{J} + \tilde{T}\delta\mathcal{P}\right)
$$

where

$$
\mathcal{J} := \Upsilon + \Omega\partial_\phi\mathcal{P}, \qquad \mathcal{P} := \ln\frac{\eta}{\Theta^2}\,.
$$

If we assume these new generators are field independent, $\delta\tilde{T} = \delta\tilde{Y} = \delta\tilde{W} = 0$, we get an integrable expression for the charge variation. Here Ω, $\mathcal{J}$ and $\mathcal{P}$ are entropy aspect charge, angular momentum aspect charge and expansion aspect charge respectively. In this integrable slicing, transformation laws take the following diagonal form

$$
\begin{aligned}
\delta_\xi\Omega &= \tilde{T}\,,\\
\delta_\xi\mathcal{P} &\approx -\tilde{W}\,,\\
\delta_\xi\mathcal{J} &\approx 2\mathcal{J}\partial_\phi\tilde{Y} + \tilde{Y}\partial_\phi\mathcal{J}\,.
\end{aligned}
$$

Because we changed the symmetry generators and subsequently their conjugate charges, the charge algebra also changes. So, in this slicing, it yields

$$
\begin{aligned}
\{\Omega(v,\phi),\Omega(v,\phi')\} &= \{\mathcal{P}(v,\phi),\mathcal{P}(v,\phi')\} = 0,\\
\{\Omega(v,\phi),\mathcal{P}(v,\phi')\} &= 16\pi G\delta\,(\phi-\phi')\,,\\
\{\mathcal{J}(v,\phi),\Omega(v,\phi')\} &= \{\mathcal{J}(v,\phi),\mathcal{P}(v,\phi')\} = 0,\\
\{\mathcal{J}(v,\phi),\mathcal{J}(v,\phi')\} &= 16\pi G\left(\mathcal{J}(v,\phi')\partial_\phi - \mathcal{J}(v,\phi)\partial'_\phi\right)\delta\,(\phi-\phi')\,.
\end{aligned}
$$

this is the Heisenberg $\oplus$ Witt algebra [1, 2]. It is worth to emphasis that integrable slicings are not unique. The main question is about the existence of these kinds of integrable slicings. In this regard, we suggest the following *integrability conjecture* [1, 2]:

In the absence of genuine flux passing through the boundary, there are specific slicings, integrable slicings, such that the charge variation becomes integrable.

This conjecture clarifies why we got integrable slicing in three-dimensional Einstein gravity. Einstein gravity in this dimension does not involve any bulk propagating mode. So, according to the integrability conjecture, we expect to find integrable slicing.

In order to check this conjecture, we studied topologically massive gravity (TMG) in three-dimensions [7]. Unlike the Einstein gravity in three dimensions, this theory has a chiral massive

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propagating mode. So, we expect in presence of this mode, we can not obtain integrable slicing. This expectation is exactly what we observed. Another natural test for our conjecture is the Einstein gravity in higher dimensions [2]. In this case, we also have bulk propagating gravitons and by repeating the same analyses, we observed there is no integrable slicing due to the passage of the gravitons through the boundary.

3. Outlook

We studied boundary symmetries near a generic null boundary in three dimensional Einstein gravity. This work is motivated by questions regarding black holes. In this case, we constructed the solution phase space of the theory and we have also studied the symmetries of the solution phase space and by using the covariant phase space method we computed the surface charges associated with these symmetries. We found our solution phase space is parameterized by three codimension one functions, entropy aspect charge, angular momentum aspect charge and expansion aspect charge. Corresponding to these boundary data we got three symmetry generators, supertranslation, superscaling and superrotation.

As our other important result, we established that in the three-dimensional Einstein gravity, there exists a basis on solution phase space in which the charge variation becomes integrable. In other words, the non-integrability of charges in three-dimensional gravity may be removed by working on a particular state/field-dependent basis. We discussed that the integrable basis is not unique. Finally, we observed the charge algebra is a slicing dependent concept. For example in an special slicing we get the Heisenberg $\oplus$ Witt algebra. The quantization of this algebra would be helpful to understand the quantum nature of the gravity.

Acknowledgement. We would like to especially thank Daniel Grumiller and Celine Zwikel for long term collaborations and many fruitful discussions which were crucial in developing the ideas and analysis discussed here. We thank Glenn Barnich for correspondence and Mohammad Hassan Vahidinia for the useful discussions. MMShJ acknowledges SarAmadan grant No. ISEF/M/400122. The work of VT is partially supported by IPM funds. The work of HA was supported by the NSFC Grant No. 12150410311.

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