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Supersymmetric extensions of oscillator- and Coulomb-like systems

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We suggest the $su(1,N|M)$-superconformal mechanics formulated in terms of phase superspace given by the non-compact analog of complex projective superspace $\mathbb{CP}^{N|M}$. We parameterized this phase space by the specific coordinates allowing us to interpret it as a higher-dimensional super-analog of the Lobachevsky plane parameterized by lower half-plane (Klein model). Then we transited to the canonical coordinates corresponding to the known separation of the "radial" and "angular" parts of (super)conformal mechanics. Relating the "angular" coordinates with action-angle variables we demonstrated that the proposed scheme allows constructing the $su(1,N|M)$ superconformal extensions of a wide class of superintegrable systems. We also proposed the superintegrable oscillator-like system with a $su(1,N|M)$ dynamical superalgebra, and found that it admits deformed $\mathcal{N}=2M$ Poincaré supersymmetry.

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1. Introduction

Kähler manifolds are Hermitian manifolds which possesses the symplectic structure obeying specific compatibility condition with Riemann (and/or complex) structure [1]. Being highly common objects in almost all areas of theoretical physics, these manifolds usually appear as configuration spaces of the particles and fields, thought they could be considered as a phase spaces of Hamiltonian systems.

On the other hand, there were some indications that Kähler phase spaces can be useful for the study of conventional Hamiltonian systems, i.e. for the systems formulated on cotangent bundle of Riemann manifolds. A very simple example of such system is one-dimensional conformal mechanics formulated in terms of Lobachevsky plane ("noncompact complex projective plane") played the role of phase space [4]. Such description, besides elegance, allows to immediately construct its $\mathcal{N} = 2M$ superconformal extension associated with $su(1.1|M)$ superalgebra. In the recent paper a similar formulation of some higher-dimensional systems was given [6] in terms of $su(1.N)$-symmetric Kähler phase space which can be considered as a non-compact version of complex projective space. Relating the angular coordinates and momenta with the action-angle variables of the angular part of the integrable conformal mechanics, we describe all symmetries of the generic superintegrable conformal-mechanical systems in terms of the powers of the $su(1.N)$ isometry generators. Then we consider the maximally superintegrable generalizations of the Euclidean oscillator/Coulomb systems and expressed all the symmetries of these superintegrable systems via $su(1.N)$ isometry generators as well. However, the supersymmetrization aspects of that system was not considered there at all. While we had strong suspect that replacing concompact projective space by its supergeneralization we can construct their $\mathcal{N}$-extended superconformal extensions of the systems considered there, as it was done in [4] for one-dimensional case. Present paper is based on [5] and [6]. We construct the superanalogs of the maximally superintegrable generalizations of the Euclidian oscillator/Coulomb systems considered in [6] as follows: we preserve the form of Hamiltonian expressed via generators of $su(1.1)$ subalgebra but extend the phase space $\widetilde{\mathbb{CP}}^N$ to phase superspace $\widetilde{\mathbb{CP}}^{N|M}$. As a result, we find that these superextensions reserves all symmetries of the initial bosonic Hamiltonians and get maximal set of functionally-independent fermionic integrals,i.e. they remains superintegrable in the sence of super-Liouville's theorem. We also find, that the constructed oscillator-like systems (in contrast with Coulomb-like ones) possess deformed $\mathcal{N} = 2M$ Poincaré supersymmetry, and express all the symmetries of these superintegrable systems via $su(N.1)$ isometry generators as well.

2. Non-compact complex projective superspace

Our goal is to study the systems on the Kähler phase space with $su(N.1|M)$ isometry superalgebra. Let us parametrize the complex projective superspace with w, z^α bosonic and η^A fermionic coordinates obeying the Poisson bracket relations

$$\{w, \bar{w}\} = -A(w - \bar{w}), \quad \{z^\alpha, \bar{z}^\beta\} = \imath A\delta^{\alpha\bar{\beta}}, \quad \{\theta^A, \bar{\theta}^B\} = A\delta^{A\bar{B}},$$
$$\{w, \bar{z}^\alpha\} = A\bar{z}^\alpha, \quad \{w, \bar{\theta}^A\} = A\bar{\theta}^A. \tag{1}$$

where

$$A := \frac{1}{g}\left(\imath(w-\bar{w}) - \sum_{\gamma=1}^{N-1} z^\gamma \bar{z}^\gamma + \imath \sum_{C=1}^{M} \theta^C \bar{\theta}^C\right), \tag{2}$$

These Poisson brackets are associated with the supersymplectic structure

$$\begin{aligned}\Omega = \frac{\imath}{g}\Bigg[&\frac{1}{A^2}dw\wedge d\bar{w} - \frac{\imath z^\alpha}{A^2}dw\wedge d\bar{z}^\alpha - \frac{\theta^A}{A^2}dw\wedge d\bar{\theta}^A \\ &+ \frac{\imath\bar{z}^\alpha}{A^2}dz^\alpha\wedge d\bar{w} + \left(\frac{g\delta_{\alpha\bar{\beta}}}{A} + \frac{\bar{z}^\alpha z^\beta}{A^2}\right)dz^\alpha\wedge d\bar{z}^\beta - \frac{\imath\bar{z}^\alpha\theta^A}{A^2}dz^\alpha\wedge d\bar{\theta}^A \\ &- \frac{\bar{\theta}^A}{A^2}d\theta^A\wedge d\bar{w} + \frac{\imath\bar{\theta}^A z^\alpha}{A^2}d\theta^A\wedge d\bar{z}^\alpha - \left(\frac{\imath g\delta_{A\bar{B}}}{A} + \frac{\bar{\theta}^A\theta^B}{A^2}\right)d\theta^A\wedge d\bar{\theta}^B\Bigg].\end{aligned} \tag{3}$$

It is defined by the Kähler potential

$$\mathcal{K} = -g\log(\imath(w-\bar{w}) - z^\alpha\bar{z}^\alpha + \imath\theta^A\bar{\theta}^A). \tag{4}$$

In what follows we will call this space " noncompact projective superspace $\widetilde{\mathbb{CP}}^{N|M}$ ". The isometry algebra of this space is $su(N.1|M)$.It is defined by the following Killing potentials

$$H := v^N\bar{v}^N|_{J=g} = \frac{w\bar{w}}{A}, \quad K := v^0\bar{v}^0|_{J=g} = \frac{1}{A}, \quad D := (v^N\bar{v}^0 + v^0\bar{v}^N)|_{J=g} = \frac{w+\bar{w}}{A}, \tag{5}$$

$$H_\alpha := \bar{v}^\alpha v^N|_{J=g} = \frac{\bar{z}^\alpha w}{A}, \quad K_\alpha := \bar{v}^\alpha v^0|_{J=g} = \frac{\bar{z}^\alpha}{A}, \quad h_{\alpha\bar{\beta}} := \bar{v}^\alpha v^\beta|_{J=g} = \frac{\bar{z}^\alpha z^\beta}{A}, \tag{6}$$

$$Q_A := \bar{\eta}^A v^N|_{J=g} = \frac{\bar{\theta}^A w}{A}, \quad S_A := \bar{\eta}^A v^0|_{J=g} = \frac{\bar{\theta}^A}{A}, \quad \Theta_{A\bar{\alpha}} := \bar{\eta}^A v^\alpha|_{J=g} = \frac{\bar{\theta}^A z^\alpha}{A}, \tag{7}$$

$$R_{A\bar{B}} := \imath\bar{\eta}^A\eta^B|_{J=g} = \imath\frac{\bar{\theta}^A\theta^B}{A}. \tag{8}$$

Constructed super-Kähler structure can be viewed as a higher dimensional analog of the Klein model of Lobachevsky space, where the latter is parameterized by the lower half-plane. The generators (5) define conformal subalgebra $su(1.1)$ and are separated from the rest $su(N.1)$ generators. Thus they can be interpreted as the Hamiltonian of conformal mechanics, the generator of conformal boosts and the generator of dilatation.

In the next section we will analize in details superconformal mechanics and its dynamical superalgebra, which is the isometry algebra defined by the generators (5),(6),(7),(8).

3. $su(1, N|M)$ superconformal algebra

The generators (Killing potentials) (5),(6),(7),(8) form $su(1, N|M)$ superalgebra ([5]). Its explicit expression with separated $su(1, 1)$ subalgebra is represented below. For the convenience it is divided into three sectors: "bosonic", "fermionic" and "mixed" ones.

"Bosonic" sector: $su(1, N) \times u(M)$ algebra

The bosonic sector is the direct product of the $su(1, N)$ algebra defined by the generators (5),(6), and the $u(M)$ algebra defined by the R-symmetry generators (8). Explicitly, the $su(1, N)$

algebra is given by the relations

$$\{H,K\} = -D, \quad \{H,D\} = -2H, \quad \{K,D\} = 2K, \tag{9}$$
$$\{H,K_\alpha\} = -H_\alpha, \quad \{H,H_\alpha\} = \{H,h_{\alpha\bar\beta}\} = 0, \tag{10}$$
$$\{K,H_\alpha\} = K_\alpha, \quad \{K,K_\alpha\} = \{K,h_{\alpha\bar\beta}\} = 0, \tag{11}$$
$$\{D,K_\alpha\} = -K_\alpha, \quad \{D,H_\alpha\} = H_\alpha, \quad \{D,h_{\alpha\bar\beta}\} = 0, \tag{12}$$
$$\{K_\alpha,K_\beta\} = \{H_\alpha,H_\beta\} = \{K_\alpha,H_\beta\} = 0, \tag{13}$$
$$\{K_\alpha,\overline{K}_\beta\} = -\imath K\delta_{\alpha\bar\beta}, \quad \{H_\alpha,\overline{H}_\beta\} = -\imath H\delta_{\alpha\bar\beta}, \quad \{h_{\alpha\bar\beta},h_{\gamma\bar\delta}\} = \imath(h_{\alpha\bar\delta}\delta_{\gamma\bar\beta} - h_{\gamma\bar\beta}\delta_{\alpha\bar\delta}), \tag{14}$$
$$\{K_\alpha,h_{\beta\bar\gamma}\} = -\imath K_\beta\delta_{\alpha\bar\gamma}, \quad \{H_\alpha,h_{\beta\bar\gamma}\} = -\imath H_\beta\delta_{\alpha\bar\gamma}, \quad \{K_\alpha,\overline{H}_\beta\} = h_{\alpha\bar\beta} + \frac{1}{2}(I - \imath D)\,\delta_{\alpha\bar\beta}, \tag{15}$$

where

$$I := g + \sum_{\gamma=1}^{N-1} h_{\gamma\bar\gamma} + \sum_{C=1}^{M} R_{C\bar C} \tag{16}$$

The R-symmetry generators form $u(M)$ algebra and commutes with all generators of $su(1,N)$:

$$\{R_{A\bar B},R_{C\bar D}\} = \imath(R_{A\bar D}\delta_{C\bar B} - R_{C\bar B}\delta_{A\bar D}), \quad \{R_{A\bar B},(H;K;D;K_\alpha;H_\alpha;h_{\alpha\bar\beta})\} = 0. \tag{17}$$

It is clear that the generators H, D, K form conformal algebra $su(1,1)$, the generators $h_{\alpha\bar\beta}$ form the algebra $u(N-1)$, and all together - the $su(1,1)\times u(N-1)$ algebra. Notice, that the generator I in (16) defines the Casimir of conformal algebra $su(1,1)$:

$$\mathcal{I} := \frac{1}{2}I^2 = \frac{1}{2}D^2 - 2HK. \tag{18}$$

Hence, choosing H as a Hamiltonian, we get that H_α, $h_{\alpha\bar\beta}, R_{A\bar B}$ define its constant of motion. Similarly, choosing the generator K as a Hamiltonian, we get that it has constants of motion $K_\alpha, h_{\alpha\bar\beta}, R_{A\bar B}$.

"Fermionic" sector

The Poisson brackets between fermionic generators (7) have the form

$$\{S_A,\overline{S}_B\} = K\delta_{A\bar B}, \quad \{Q_A,\overline{Q}_B\} = H\delta_{A\bar B}, \quad \{S_A,\overline{Q}_B\} = -\imath R_{A\bar B} + \frac{\imath}{2}(I - \imath D)\,\delta_{A\bar B}, \tag{19}$$
$$\{\Theta_{A\bar\alpha},\overline{\Theta}_{B\bar\beta}\} = R_{A\bar B}\delta_{\beta\bar\alpha} + h_{\beta\bar\alpha}\delta_{A\bar B}, \quad \{S_A,\overline{\Theta}_{B\bar\alpha}\} = K_\alpha\delta_{A\bar B}, \quad \{Q_A,\overline{\Theta}_{B\bar\alpha}\} = H_\alpha\delta_{A\bar B}, \tag{20}$$
$$\{S_A,S_B\} = \{Q_A,Q_B\} = \{\Theta_{A\bar\alpha},\Theta_{B\bar\beta}\} = \{S_A,Q_B\} = \{S_A,\Theta_{B\bar\alpha}\} = \{Q_A,\Theta_{B\bar\alpha}\} = 0. \tag{21}$$

Hence, the functions Q_A play the role of supercharges for the Hamiltonian H, and the functions S_A define the supercharges of the Hamiltonian given by the generator of conformal boosts K.

"Mixed" sector

The mixed sector is given by the relations

$$\{H, Q_A\} = \{H, \Theta_{A\bar{\alpha}}\} = 0, \quad \{H, S_A\} = -Q_A, \tag{22}$$

$$\{K, S_A\} = \{K, \Theta_{A\bar{\alpha}}\} = 0, \quad \{K, Q_A\} = S_A, \tag{23}$$

$$\{D, S_A\} = -S_A, \quad \{D, Q_A\} = Q_A, \quad \{D, \Theta_{A\bar{\alpha}}\} = 0 \tag{24}$$

$$\{Q_A, \overline{K}_\alpha\} = -\Theta_{A\bar{\alpha}}, \quad \{Q_A, H_\alpha\} = \{Q_A, \bar{H}_\alpha\} = \{Q_A, \bar{K}_\alpha\} = \{Q_A, h_{\alpha\bar{\beta}}\} = 0, \tag{25}$$

$$\{S_A, \overline{H}_\alpha\} = \Theta_{A\bar{\alpha}}, \quad \{S_A, K_\alpha\} = \{S_A, \bar{K}_\alpha\} = \{S_A, H_\alpha\} = \{S_A, h_{\alpha\bar{\beta}}\} = 0, \tag{26}$$

$$\{\Theta_{A\bar{\alpha}}, K_\beta\} = \imath S_A \delta_{\beta\bar{\alpha}}, \quad \{\Theta_{A\bar{\alpha}}, H_\beta\} = \imath Q_A \delta_{\beta\bar{\alpha}}, \quad \{\Theta_{A\bar{\alpha}}, h_{\beta\bar{\gamma}}\} = \imath \Theta_{A\bar{\gamma}} \delta_{\beta\bar{\alpha}}, \tag{27}$$

$$\{\Theta_{A\bar{\alpha}}, \bar{H}_\alpha\} = \{\Theta_{A\bar{\alpha}}, \bar{K}_\alpha\} = 0,$$

$$\{S_A, R_{B\bar{C}}\} = -\imath S_B \delta_{A\bar{C}}, \quad \{Q_A, R_{B\bar{C}}\} = -\imath Q_B \delta_{A\bar{C}}, \quad \{\Theta_{A\bar{\alpha}}, R_{B\bar{C}}\} = -\imath \Theta_{B\bar{\alpha}} \delta_{A\bar{C}}. \tag{28}$$

Looking to the all Poisson bracket relations together we conclude that

- The bosonic functions H_α, $h_{\alpha\bar{\beta}}$, and the fermionic functions Q_A, $\Theta_{A\bar{\alpha}}$ commute with the Hamiltonian H and thus, provide it by the superintegrability property [1];
- The bosonic functions K_α, $h_{\alpha\bar{\beta}}$ and the fermionic functions S_A,$\Theta_{A\bar{\alpha}}$ commute with the generator K. Hence, the Hamiltonian K defines the superintegrable system as well.
- The triples $(H, H_\alpha, Q_A,)$ and $(K, K_\alpha, S_A,)$ transform into each other under the discrete transformation

$$(w, z^\alpha, \theta^A) \to (-\frac{1}{w}, \frac{z^\alpha}{w}, \frac{\theta^A}{w}) \quad \Rightarrow D \to -D, \quad \begin{cases} (H, H_\alpha, Q_A,) & \to \quad (K, -K_\alpha, -S_A), \\ (K, K_\alpha, S_A) & \to \quad (H, H_\alpha, Q_A,) \end{cases} . \tag{29}$$

- The functions $h_{\alpha\bar{\beta}}, \Theta_{A\bar{\alpha}}$ are invariant under discrete transformation (29). Moreover, they appear to be constants of motion both for H and K. Hence, they remain to be constants of motion for any Hamiltonian being the functions of H, K.

In the next section we will express presented $su(1, N|M)$ generators in appropriate canonical coordinates and in this way we will relate presented formulae with the superextensions of conventional conformal mechanics.

4. Canonical coordinates

For the introduction of the canonical coordinates we transit from the complex coordinates to the real ones for bosonic variables and make a change of fermionic ones such that the new fermionic

[1]In accord with super-analogue of Liouville theorem [12] the system on $(2N.M)$ phase superspace is integrable iff it possess N commuting bosonic integrals (with nonvanishing and functionally independent bosonic parts) and M fermionic ones

variables will have canonical Poisson brackets. For this purpose we represent bosonic variables w, z^α as follows,

$$w = x + \imath y, \quad z^\alpha = q_\alpha \mathrm{e}^{\imath\varphi_\alpha}, \quad \text{where} \quad y < 0, q_\alpha \geq 0, \varphi_\alpha \in [0, 2\pi), \quad q^2 := \sum_{\alpha=1}^{N-1} q_\alpha^2 < -2y. \tag{30}$$

Then we write down the symplectic/Kähler one-form and identify it with the canonical one

$$\begin{aligned} \mathcal{A} &= -\frac{g}{2}\frac{dw + d\bar{w} - \imath(z^\alpha d\bar{z}^\alpha - \bar{z}^\alpha dz^\alpha) + \theta^A d\bar{\theta}^A + \bar{\theta}^A d\theta^A}{\imath(w - \bar{w}) - z^\gamma \bar{z}^\gamma + \imath\theta^C\bar{\theta}^C} \\ &:= p_x dx + \pi_\alpha d\varphi_\alpha + \frac{1}{2}\chi^A d\bar{\chi}^A + \frac{1}{2}\bar{\chi}^A d\chi^A \end{aligned} \tag{31}$$

After some calculations and canonical transformation $(p_x, x) \to (-\frac{r^2}{2}, \frac{p_r}{r})$, one can obtain

$$w = \frac{p_r}{r} - \imath\frac{I}{r^2}, \quad z^\alpha = \frac{\sqrt{2\pi_\alpha}}{r}\mathrm{e}^{\imath\varphi_\alpha}, \quad \theta^A = \frac{\sqrt{2}}{r}\chi^A, \tag{32}$$

where

$$\{r, p_r\} = 1, \quad \{\varphi_\beta, \pi_\alpha\} = \delta_{\alpha\beta}, \quad \{\chi^A, \bar{\chi}^B\} = \delta^{A\bar{B}}, \qquad \pi_a \geq 0, \quad \varphi^a \in [0, 2\pi), \quad r > 0. \tag{33}$$

Hence, $r, p_r, \varphi^\alpha, \pi_\alpha, \chi^A, \bar{\chi}^A$ define canonical coordinates. They expresses via initial ones as follows

$$p_r = \frac{w + \bar{w}}{2}\sqrt{\frac{2}{A}}, \quad r = \sqrt{\frac{2}{A}}, \quad \pi_\alpha = \frac{z^\alpha \bar{z}^\alpha}{A}, \quad \varphi_\alpha = \arg(z^\alpha), \quad \chi^A = -\frac{\theta^A}{\sqrt{A}}, \quad c.c., \tag{34}$$

where

$$I = g + \sum_{\alpha=1}^{N-1}\pi_\alpha + \sum_{A=1}^{M}\imath\bar{\chi}^A\chi^A, \qquad A := \frac{\imath(w - \bar{w}) - z^\gamma\bar{z}^\gamma + \imath\theta^C\bar{\theta}^C}{g} = \frac{2}{r^2}. \tag{35}$$

In these canonical coordinates the isometry generators read

$$H = \frac{p_r^2}{2} + \frac{I^2}{2r^2}, \quad K = \frac{r^2}{2}, \quad D = p_r r, \tag{36}$$

$$H_\alpha = \sqrt{\frac{\pi_\alpha}{2}}\mathrm{e}^{-\imath\varphi_\alpha}\left(p_r - \imath\frac{I}{r}\right), \quad K_\alpha = r\sqrt{\frac{\pi_\alpha}{2}}\mathrm{e}^{-\imath\varphi_\alpha}, \quad h_{\alpha\bar{\beta}} = \sqrt{\pi_\alpha\pi_\beta}\mathrm{e}^{-\imath(\varphi_\alpha - \varphi_\beta)}, \tag{37}$$

$$Q_A = \frac{\bar{\chi}^A}{\sqrt{2}}\left(p_r - \imath\frac{\sqrt{2\mathcal{I}}}{r}\right), \quad S_A = \frac{\bar{\chi}^A}{\sqrt{2}}r, \quad \Theta_{A\bar{\alpha}} = \bar{\chi}^A\sqrt{\pi_\alpha}\mathrm{e}^{\imath\varphi_\alpha}, \quad R_{A\bar{B}} = \imath\bar{\chi}^A\chi^B. \tag{38}$$

5. Oscillator- and Coulomb-like Systems

In [6] there are an examples of superintegrable deformations of N-dimensional oscillator and Coulomb systems on noncompact projective space $\widetilde{\mathbb{CP}}^N$ playing the role of phase space were constructed. So, one can expect that on the phase superspace $\widetilde{\mathbb{CP}}^{N|M}$ one can construct the super-counterparts of that systems, which presumably, possess (deformed) $\mathcal{N} = 2M, d = 1$ Poincaré supersymmetry. Below we examine this question and show that our claim is corrects in some particular cases.

5.1 Oscillator-like systems

We define the supersymmetric oscillator-like system by the the phase space $\widetilde{\mathbb{CP}}^{N|M}$ (equipped with the Poisson brackets (1)) by the Hamiltonian

$$H_{osc} = H + \omega^2 K, \tag{39}$$

where the generators H, K are given by (5). In canonical coordinates (34)it reads

$$H_{osc} = \frac{p_r^2}{2} + \frac{(g + \sum_{\alpha=1}^{N-1} \pi_\alpha + \sum_{A=1}^{M} \imath \bar{\chi}^A \chi^A)^2}{r^2} + \frac{\omega^2 r^2}{2}. \tag{40}$$

This system possesses the $u(N)$ symmetry given by the generators $h_{\alpha\bar{\beta}}$ defined in (6)(among them $N-1$ constants of motion π_α are functionally independent), the $u(M)$ R-symmetry given by the generators $R_{A\bar{B}}$ (8) as well as $N-1$ hidden symmetries given by the generators

$$M_{\alpha\beta} = (H_\alpha + \imath\omega K_\alpha)(H_\beta - \imath\omega K_\beta) = \frac{\bar{z}^\alpha \bar{z}^\beta}{A^2}(w^2 + \omega^2) \ : \quad \{H_{osc}, M_{\alpha\beta}\} = 0, \tag{41}$$

The generators (41) and the $su(N)$ generators $h_{\alpha\bar{\beta}}$ form the following symmetry algebra

$$\{h_{\alpha\bar{\beta}}, M_{\gamma\delta}\} = \imath \left(M_{\alpha\delta}\delta_{\gamma\bar{\beta}} + M_{\gamma\alpha}\delta_{\delta\bar{\beta}} \right), \quad \{M_{\alpha\beta}, M_{\gamma\delta}\} = 0, \tag{42}$$

$$\{M_{\alpha\beta}, \overline{M}_{\gamma\delta}\} =$$
$$\imath \left(4\omega^2 I h_{\alpha\bar{\delta}} h_{\beta\bar{\gamma}} - \frac{M_{\alpha\beta}\bar{M}_{\gamma\delta}}{h_{\alpha\bar{\gamma}}}\delta_{\alpha\bar{\gamma}} - \frac{M_{\alpha\beta}\bar{M}_{\gamma\delta}}{h_{\alpha\bar{\delta}}}\delta_{\alpha\bar{\delta}} - \frac{M_{\alpha\beta}\bar{M}_{\gamma\delta}}{h_{\beta\bar{\gamma}}}\delta_{\beta\bar{\gamma}} - \frac{M_{\alpha\beta}\bar{M}_{\gamma\delta}}{h_{\beta\bar{\delta}}}\delta_{\beta\bar{\delta}} \right), \tag{43}$$

with I given by (16) and summation over repeated indices is not assumed.

Besides, this system has a fermionic constants of motion $\Theta_{A\bar{\alpha}}$ defined in (7).Hence, it is superintegrable system in the sense of super-Liouville theorem, i.e. it has $2N-1$ bosonic and $2M$ fermionic, functionally independent, constants of motion [12].

Let us show, that for the even $M = 2k$ this system possess the deformed $\mathcal{N} = 2k$ Poincaré supersymmetry, in the sense of papers [10]. For this purpose we choose the following Ansatz for supercharges

$$\mathcal{Q}_A = Q_A + \omega C_{AB}\bar{S}_B, \tag{44}$$

with the constant matrix C_{AB} obeying the conditions

$$C_{AB} + C_{BA} = 0, \qquad C_{AB}\overline{C}_{BD} = -\delta_{A\bar{D}} \tag{45}$$

For sure, the condition (45) assumes that M is an even number, $M = 2k$.

Calculating Poisson brackets of the functions (44) we get

$$\{\mathcal{Q}_A, \bar{\mathcal{Q}}_B\} = H_{osc}\delta_{AB}, \qquad \{\mathcal{Q}_A, \mathcal{Q}_B\} = -\imath\omega\mathcal{G}_{AB}, \qquad \{\bar{\mathcal{Q}}_A, \bar{\mathcal{Q}}_B\} = \imath\omega\bar{\mathcal{G}}_{AB}, \tag{46}$$

where

$$\mathcal{G}_{AB} := C_{AC}R_{B\bar{C}} + C_{BC}R_{A\bar{C}},$$
$$\mathcal{G}_{\bar{A}\bar{B}} := \bar{\mathcal{G}}_{AB} = \bar{C}_{AC}R_{C\bar{B}} + \bar{C}_{BC}R_{C\bar{A}}, \tag{47}$$

$$\bar{\mathcal{G}}_{AB} = \bar{C}_{AC}\bar{C}_{DB}\mathcal{G}_{DC}.$$

Then we get that the algebra of generators $Q_A, \mathcal{H}_{osc}, \mathcal{R}_A^B$ is closed indeed:

$$\{Q_A, H_{osc}\} = \omega C_{AB} Q_B, \qquad \{\mathcal{G}_{AB}, H_{osc}\} = 0, \tag{48}$$

$$\{Q_A, \mathcal{G}_{BC}\} = \imath(C_{AB}Q_C + C_{AC}Q_B), \qquad \{Q_A, \bar{\mathcal{G}}_{BC}\} = -\imath(\bar{C}_{BD}Q_D\delta_{A\bar{C}} + \bar{C}_{CD}Q_D\delta_{A\bar{B}}). \tag{49}$$

Hence, for the $M = 2k$ the above oscillator-like system (39) possesses deformed $\mathcal{N} = 4k$ supersymmetry. In the particular case $M = 2$ the choice of the matrix C_{AB} is unique(up to unessential phase factor): $C_{AB} := e^{\kappa}\varepsilon_{AB}$. In that case the above relations define the superalgebra $su(1|2)$-deformation of $\mathcal{N} = 4$ Poincaré supersymmetric mechanics studied in details in [10]. For the $k \geq 2$ the choice of matrices C_{AB} is not unique, and we get the family of deformed $\mathcal{N} = 4k$ Poincaré supersymmetric mechanics.

Let us present other deformed $\mathcal{N} = 2M$ Poincaré supersymmetric systems whose bosonic part is different from those of (39) but nevertheless, has the oscillator potential.

For this purpose we choose another ansatz for supercharges (in contrast with previous case M is not restricted to be even number)

$$\widetilde{Q}_A = Q_A + \imath\omega S_A. \tag{50}$$

These supercharges generates the $su(1|M)$ superalgebra, and thus generalizes the systems considered in [10] to arbitrary M,

$$\{\widetilde{Q}_A, \bar{\widetilde{Q}}_B\} = \mathcal{H}_{osc}\delta_{AB} - \omega\mathcal{R}_B^A, \quad \{\widetilde{Q}_A, \widetilde{Q}_B\} = 0, \quad \{\mathcal{R}_A^{\ B}, \mathcal{R}_C^{\ D}\} = \imath(\mathcal{R}_A^{\ D}\delta_C^B - \mathcal{R}_C^{\ B}\delta_A^D) \tag{51}$$

$$\{\widetilde{Q}_A, \mathcal{R}_B^C\} = \imath\left(\frac{1}{M}\widetilde{Q}_A\delta_{B\bar{C}} + \widetilde{Q}_B\delta_{A\bar{C}}\right), \quad \{\widetilde{Q}_A, \mathcal{H}_{osc}\} = \imath\omega\frac{2M-1}{M}\widetilde{Q}_A, \tag{52}$$

where

$$\mathcal{H}_{osc} := H_{osc} - \omega(I + \frac{1}{M}\sum_C R_{C\bar{C}}), \qquad \mathcal{R}_A^{\ B} := R_{A\bar{B}} - \frac{1}{M}\delta_A^B\sum_C R_{C\bar{C}} \tag{53}$$

with I defined by (16). Hence, the Hamiltonian get the additional bosonic term proportional to the Casimir of conformal group. In canonical coordinates (34) it reads

$$\mathcal{H}_{osc} = \frac{p_r^2}{2} + \frac{\mathcal{I}}{r^2} + \frac{\omega^2 r^2}{2} - \omega\left(\sqrt{2\mathcal{I}} + \frac{1}{M}(\bar{\chi}\chi)\right). \tag{54}$$

This Hamiltonian, seemingly, describes the oscillator-like systems specified by the presence of external magnetic field.

So, choosing $\widetilde{\mathbb{CP}}^{N|M}$ as a phase superspace, we can easily construct superintegrable oscillator-like systems which possess deformed $\mathcal{N} = 2M, d = 1$ Poincaré supersymmetry.

5.2 Coulomb-like systems

Now, let us construct on the phase space $\widetilde{\mathbb{CP}}^{N|M}$ with the Poisson bracket relations (1), the Coulomb-like system given by the Hamiltonian

$$H_{Coul} = H + \frac{\gamma}{\sqrt{2K}}, \tag{55}$$

where the generators H, K are defined by (5).

The bosonic constants of motion of this system are given by the $u(N-1)$ symmetry generators $h_{\alpha\beta}$, and by the $N-1$ additional constants of motion

$$R_\alpha = H_\alpha + \imath\gamma \frac{K_\alpha}{I\sqrt{2K}} \quad : \quad \{H_{Coul}, R_\alpha\} = \{H_{Coul}, h_{\alpha\bar{\beta}}\} = 0, \tag{56}$$

where $H_\alpha, K_\alpha, \eta_{\alpha\bar{\beta}}$ are defined by (6). These generators form the algebra

$$\{R_\alpha, \bar{R}_{\bar{\beta}}\} = -\imath\delta_{\alpha\bar{\beta}}\left(H_{Coul} - \frac{\imath\gamma^2}{2I^2}\right) + \frac{\imath\gamma^2 h_{\alpha\bar{\beta}}}{2I^3}, \quad \{h_{\alpha\bar{\beta}}, R_\gamma\} = \imath\delta_{\gamma\bar{\beta}}R_\alpha, \quad \{R_\alpha, R_\beta\} = 0. \tag{57}$$

Besides, proposed system has $2M$ fermionic constants of motion given by $\Theta_{A\bar{\alpha}}$, and $u(M)$ R-symmetry given by $R_{A\bar{B}}$. Hence, it is superintegrable in the sense of super-Liouville theorem [12]. So, we constructed the maximally superintegrable Coulomb problem with dynamical $SU(1,N|M)$ superconformal symmetry which inherits all symmetries of initial bosonic system.

One can expect, that in analogy with oscillator-like system, our Coulomb-like system would possess (deformed) $\mathcal{N} = 2M$-super-Poincaré symmetry for $M = 2k$ and $\gamma > 1$. However, it is not a case.

Indeed, let us choose the following Ansatz for supercharges

$$\mathcal{Q}_A = Q_A + \sqrt{2\gamma}C_{AB}\frac{\bar{S}_B}{(2K)^{3/4}}, \tag{58}$$

with the constant matrix C_{AB} obeying the conditions (45), $M = 2k$ and $\gamma > 0$.

Calculating their Poisson brackets we find

$$\{\mathcal{Q}_A, \bar{\mathcal{Q}}_B\} = H_{Coul}\delta_{A\bar{B}} + \frac{3}{2}\frac{\sqrt{2\gamma}}{(2K)^{7/4}}\left(S_A\bar{C}_{BD}S_D + \bar{S}_B C_{AD}\bar{S}_D\right), \tag{59}$$

$$\{\mathcal{Q}_A, \mathcal{Q}_B\} = -\frac{\imath\sqrt{2\gamma}}{2(2K)^{3/4}}(C_{BD}\mathcal{R}^D_A + C_{AC}\mathcal{R}^D_B), \quad \{\mathcal{Q}_A, \mathcal{R}^C_B\} = -\imath\mathcal{Q}_B\delta_{A\bar{C}}, \tag{60}$$

where $\mathcal{R}^A_B$ is defined in (53).

Further calculating the Poisson brackets of $\mathcal{Q}_A$ with the generators appearing in the r.h.s. of the above expressions we get that the superalgebra is not closed. For example,

$$\{\mathcal{Q}_A, H_{Coul}\} = \frac{3\gamma}{(2K)^{3/2}}S_A + \frac{\sqrt{2\gamma}}{(2K)^{3/4}}C_{AB}\left(\bar{Q}_B - \frac{3}{4K}\bar{S}_B D\right). \tag{61}$$

Hence, proposed supercharges do not yield closed deformation of $\mathcal{N} = 2M$-super-Poincaré algebra.

Let us choose another ansatz for supercharges (as above we assume that $\gamma > 0$)

$$\widetilde{\mathcal{Q}}_A = Q_A + \imath\sqrt{2\gamma}e^{\imath\frac{\pi}{2}}\frac{S_A}{(2K)^{3/4}}, \tag{62}$$

which yields

$$\{\widetilde{\mathcal{Q}}_A, \bar{\widetilde{\mathcal{Q}}}_B\} = \mathcal{H}_{Coul}\delta_{A\bar{B}} + \frac{\sqrt{2\gamma}}{2(2K)^{3/4}}\mathcal{R}^B_A, \qquad \{\widetilde{\mathcal{Q}}_A, \widetilde{\mathcal{Q}}_B\} = 0,$$

$$\{\widetilde{Q}_A, \mathcal{R}^C_B\} = \imath\left(\frac{1}{M}\widetilde{Q}_A\delta_{B\bar{C}} - \widetilde{Q}_B\delta_{A\bar{C}}\right), \tag{63}$$

where

$$\mathcal{H}_{Coul} = H_{Coul} - \frac{\sqrt{2\gamma}}{(2K)^{3/4}}\left(I - \frac{1}{2M}\sum_C R_{C\bar{C}}\right), \tag{64}$$

with I and $\mathcal{R}^A_B$ are defined, respectively, in (16) and (53). In canonical coordinates (34) this Hamiltonian reads

$$\mathcal{H}_{Coul} = \frac{p_r}{2} + \frac{\mathcal{I}}{r^2} + \frac{\gamma}{r} - \frac{\sqrt{2\gamma}}{r^{3/2}}\left(g + \sum_\alpha \pi_\alpha + \frac{2M-1}{2M}(\bar{\chi}\chi)\right). \tag{65}$$

However, one can easily check that proposed supercharges do not yield closed deformation of Poincaré superalgebra as well, e.g.

$$\{\widetilde{Q}_A, \frac{\mathcal{R}^C_B}{(2K)^{3/4}}\} = \frac{\imath}{(2K)^{3/4}}\left(\frac{1}{M}\widetilde{Q}_A\delta_{B\bar{C}} - \widetilde{Q}_B\delta_{A\bar{C}}\right) + \frac{3}{2}\frac{S_A}{(2K)^{7/4}}\mathcal{R}^C_B \tag{66}$$

So, proposed superextensions of Coulomb-like systems, being well-defined from the viewpoint of superintegrability, do not possess neither $\mathcal{N} = 2M$ supersymmetry, no its deformation. The $su(1, N|M)$ superalgebra plays the role of dynamical algebra of that systems.

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