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Klein-Gordon field dynamics on near horizon extremal Myers-Perry black hole

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A dynamical system with N degrees of freedom is called integrable if it possesses N independent integrals of motion. Integrable systems in the vicinity of near horizon geometries are interesting for both astrophysical and theoretical reasons. We study the integrability of probe particle and Klein-Gordon field on the background of Near Horizon Extreme Myers-Perry black hole.

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1. Introduction

A system with N degrees of freedom which possessing N independent symmetries is called integrable. If the number of independent symmetries is bigger, the system is called superintegrable and in the case the system possess the maximum number of $2N-1$ functionally independent constants of motion, the system is called maximally superintegrable.

Research of integrable systems in the vicinity of balck holes has both theoretical and astrophysical importance. Accretions around a black hole travel along the near horizon geodesic lines and might be the key to their fist direct observation. Such accretions might also be source of Very High Energy gamma-ray bursts. On the other hand, Extremal black holes have much simpler geometry than other black holes of the same type, which makes theoretical studies of such metrics less troublesome.

Dynamics of a probe particle in near horizon extremal black hole geometries have been studied extensively. Latest works in this direction include [1], where the Hamiltonian of the spherical mechanics associated with Near Horizon Extreme Myers-Perry (NHEMP) geometry has been constructed for the special case when all rotation parameters of the black hole are equal. In [2] the integrability of this system has been proven and the integrals of motion were presented. Extremal Myers-Perry black holes with nonequal nonvanishing rotation parameters in odd dimensions have been studied in [3] where the integrability of such systems was proven and separation of variables was carried out. In [4], the integrals of motion of a massive particle in this system for seven, nine and eleven dimensions have been derived. This work was further extended to arbitrary dimensions [5], where a unified description was introduces for even and odd dimensional geodesic motion and an intermediate case was studied, when only some of the rotational parameters of the black hole are equal. In addition, the integrability of the so called vanishing horizon extremal Myers-Perry black holes was studied, which corresponds to Myers-Perry black holes with one of the rotation parameters being zero. The separability of variables of Klein-Gordon field in the background of NHEMP and vanishing horizon MP balck holes have been studied in [6].

In this paper, we will study probe particle and KG field dynamics in the background of Near Horizon Extreme Myers-Perry black hole. In the following sections, we will study separation of variables of probe particle dynamics in arbitrary higher dimension and describe the unified approach for odd and even dimensions. The last section discusses the separation of variables of Klain-Gordon field in the same background.

2. Particle dynamics in NHEMP geometry

The NHEMP metric can be written in a unified form for odd and even dimensions in Boyer-Lindquist coordinates [5]

$$ds^2 = \frac{F_H}{b}\left(-r^2 d\tau^2 + \frac{dr^2}{r^2}\right) + \sum_{I=1}^{N_\sigma}(r_H^2 + a_I^2)d\mu_I^2 + \gamma_{ij}D\varphi^i D\varphi^j, \qquad D\varphi^i \equiv d\varphi^i + \frac{B^i}{b} r d\tau, \tag{1}$$

where $N_\sigma = N + \sigma$ and σ is a parameter equal to 0 for the odd and 1 for the even dimensional cases. b and B^i are functions of black hole radius r_H, σ and rotation parameters a_i, i.e. $b(\sigma, r_H, a_{1\ldots N})$,

$B^i(r_H, a_i)$. The black hole radius r_H satisfies the equation

$$\sum_{I=1}^{N_\sigma} \frac{r_H^2}{r_H^2 + a_I^2} = \frac{1+2\sigma}{1+\sigma}, \qquad \text{with} \quad a_{N+1} = 0 \tag{2}$$

and metric functions have the form

$$F_H = 1 - \sum_{i=1}^{N} \frac{a_i^2 \mu_i^2}{r_H^2 + a_i^2}, \qquad \gamma_{ij} = (r_H^2 + a_i^2)\mu_i^2 \delta_{ij} + \frac{1}{F_H} a_i \mu_i^2 a_j \mu_j^2, \quad \text{with} \qquad \sum_{I=1}^{N_\sigma} \mu_I^2 = 1. \tag{3}$$

Lowercase Latin indices i, j run from 1 to N and uppercase Latin indices I, J take values from 1 to N_σ. We can see that the metric has one additional latitudinal coordinates x_I in even dimensions but has N number of rotation parameters a_i in both odd and even dimensions. For the further discussion, it is also important to note that the metric has $SL(2,\mathbb{R}) \times U(1)^N$ isometry. When non of the rotation parameters is 0, it is convenient to introduce new parameters m_i instead of them and re-scale the x_I coordinates.

$$m_i = \frac{r_H^2 + a_i^2}{r_H^2} > 1, \qquad m_{N+1} = 1 \qquad \Longrightarrow \qquad \sum_{I=1}^{N_\sigma} \frac{1}{m_I} = \frac{1+2\sigma}{1+\sigma}, \tag{4}$$

$$x_I = \sqrt{m_I}\mu_I \quad \Longrightarrow \qquad \sum_{I=1}^{N_\sigma} \frac{x_I^2}{m_I} = 1. \tag{5}$$

With m_i parameters and rescaled coordinates, the metric becomes

$$\frac{ds^2}{r_H^2} = A(x)\left(-r^2 d\tau^2 + \frac{dr^2}{r^2}\right) + \sum_{I=1}^{N_\sigma} dx_I dx_I + \sum_{i,j=1}^{N} \tilde{\gamma}_{ij} x_i x_j D\varphi^i D\varphi^j, \qquad D\varphi^i \equiv d\varphi^i + k^i r d\tau, \tag{6}$$

where $k^i = k^i(\sigma, m_{1\ldots N})$ and metric functions read

$$A(x) = \frac{\sum_{I=1}^{N_\sigma} x_I^2 / m_I^2}{\frac{\sigma}{1+\sigma} + 4\sum_{i<j}^{N} \frac{1}{m_i}\frac{1}{m_j}}, \qquad \tilde{\gamma}_{ij} = \delta_{ij} + \frac{1}{\sum_I^{N_\sigma} x_I^2 / m_I^2} \frac{\sqrt{m_i - 1}x_i}{m_i} \frac{\sqrt{m_j - 1}x_j}{m_j} \tag{7}$$

2.1 Angular mechanics

We can see that part of the (6) geometry has the form of AdS_2 metric which gives rise to $SL(2,\mathbb{R})$ isometry. $SL(2,\mathbb{R})$ isometry is also reffered to as conformal symmetry [1, 2, 7–13] whose algebra has three generators denoted by H, D, K:

$$\{H, D\} = H, \quad \{H, K\} = 2D, \quad \{D, K\} = K, \qquad \mathcal{I} = HK - D^2. \tag{8}$$

and $\mathcal{I}$ is the Casimir of the generators. To study the dynamics of a massive particle in the background of NHEMP, we should construct its mass-shell equation

$$m_0^2 = -\sum_{A,B=1}^{2N+1+\sigma} g^{AB} p_A p_B, \tag{9}$$

where p_A and p_B represent the momenta $p_0, p_a, p_{\varphi_i}, p_r$ conjugate to t, x_a, φ_i and r with canonical Poisson brackets. After calculating the inverse components of the metric, the mass-shell equation becomes

$$m_0^2 r_H^2 = \frac{1}{A}\left[\left(\frac{p_0}{r} - \sum_{i=1}^{N} k_i p_{\varphi_i}\right)^2 - (rp_r)^2\right] - \sum_{a,b=1}^{N_\sigma - 1} h^{ab} p_a p_b - \sum_{i,j=1}^{N} \tilde{\gamma}^{ij} \frac{p_{\varphi_i}}{x_i}\frac{p_{\varphi_j}}{x_j}, \tag{10}$$

where

$$h^{ab} = \delta^{ab} - \frac{1}{\sum_{I=1}^{N_\sigma} x_I^2/m_I^2}\frac{x_a}{m_a}\frac{x_b}{m_b}, \qquad \tilde{\gamma}^{ij} = \delta^{ij} - x_i\frac{\sqrt{m_i - 1}}{m_i} x_j \frac{\sqrt{m_j - 1}}{m_j}, \tag{11}$$

with $a, b = 1, \cdots N_\sigma - 1,\ i, j = 1, \cdots, N$. Using (10), we can construct the Hamiltonian $H = p_0$, the other generators of the conformal algebra and their Casimir element as in [3]. In an appropriately chosen frame, H can be written in nonrelativistic form [1, 2, 8–13]

$$H = \frac{p_R^2}{2} + \frac{2\mathcal{I}}{R^2}, \tag{12}$$

where $R = \sqrt{2K}$, $p_R = \frac{2D}{\sqrt{2K}}$ are the effective “radius" and its canonical conjugate “radial momentum” and

$$\mathcal{I} = A\left[\sum_{a,b=1}^{N_\sigma - 1} h^{ab} p_a p_b + \sum_{i=1}^{N} \frac{p_{\varphi_i}^2}{x_i^2} + g_0\right] - \mathcal{I}_0 \tag{13}$$

is the Casimir element of the conformal algebra with

$$g_0 = -\left(\sum_{i=1}^{N} \frac{\sqrt{m_i - 1} p_{\varphi_i}}{m_i}\right)^2 + m_0^2 r_H^2, \qquad \mathcal{I}_0 = \left(\sum_{i}^{N} k_i p_{\varphi_i}\right)^2. \tag{14}$$

Because of the nonrelativistic form of (12), the radial part of the Hamiltonian is already separated and the information about the particle dynamics is embedded in the Casimir element itslef. So, one can look at Casimir operator of the conformal mechanics as an independent dynamical system, whose solution is the key to the description of particle motion in the background of NHEMP. The reduced mechanics of Casimir $\mathcal{I}$ is called angular or sperical mechanics [14–18]. We can see from (12), that the azimuthal angular variables φ^i are cyclic in the reduced spherical mechanics. As a result, the corresponding conjugate momenta p_{φ_i} are constants of motion, N in total for both odd and even dimensions.

3. Separation of variables in angular mechniacs

It is convenient to study the separation of variables in separate special cases when non of the rotation parameters is equal to the rest (fully non-isotropic case) and all rotation parameters of the metric are equal (fully isotropic case). The fully non-isotropic metric is separable in elipsoidial coordinates:

$$x_I^2 = (m_I - \lambda_I) \prod_{J=1, J\neq I}^{N_\sigma} \frac{m_I - \lambda_J}{m_I - m_J}, \qquad \lambda_{N_\sigma} < m_{N_\sigma} < \ldots < \lambda_2 < m_2 < \lambda_1 < m_1. \tag{15}$$

where the Casimir of $SL(2,\mathbb{R})$ takes the following form after rescaling and shifting by a constant:

$$\mathcal{I} = \lambda_1 \ldots \lambda_{N_\sigma-1} \left[-\sum_{a}^{N_\sigma-1} \frac{4\prod_{I=1}^{N_\sigma}(m_I-\lambda_a)\pi_a^2}{\lambda_a \prod_{b=1,a\neq b}^{N_\sigma-1}(\lambda_b-\lambda_a)} + \sum_{i=1}^{N_\sigma} \frac{g_I^2}{\prod_{a=1}^{N_\sigma-1}(m_I-\lambda_a)} + g_0 \right]. \tag{16}$$

Here, π_a are the canonically conjugate momenta to λ_b, g_I are constants constructed from p_{φ_I} and rotation parameters m_J and $g_{N+1} \equiv 0$. The angular mechanics results into $N_\sigma - 1$ second rank killing tensors. Together with the inverse metric, which is associated with the trivial constant of motions of the particle mass we have N_σ second rank killing tensors. Also, taking into account the $N+1$ mutually commuting Killing vectors $\partial/\partial\varphi_i, \partial/\partial_\tau$, we can see that the fully non-isotropic case of NHEMP is integrable in arbitrary higher dimensions. In fully isotropic case, the form of the casimir operator is different in odd and even dimensions:

$$\mathcal{I} = \sum_{i,j=1}^{N-1+\sigma} (\eta^2(x)\delta_{ij} - x_i x_j) p_i p_j + \sum_{i=1}^{N} \frac{\eta^2(x) p_{\varphi_i}^2}{x_i^2} + \omega(p_{\varphi_i}) \sum_{i=1}^{N} x_i^2,$$

In odd dimensions the system is a generalization of Higgs oscillator, known as Rossochatius system.

$$\eta^2 = N, \qquad \omega = 0$$

The system does not correspond to Higgs oscillator in even dimensions, though.

$$\eta^2 = 4N^2 - (2N-1)\sum_{i=1}^{N} x_i^2, \qquad \omega = \left(1 - \frac{1}{2N}\right)^2 \sum_{i,j=1}^{N} p_{\varphi_i} p_{\varphi_j} - m_0^2\,(2N-1). \tag{17}$$

The fully isotropic system admits separation of variables in spherical coordinates in both odd and even dimensions:

$$x_{N_\sigma} = \sqrt{N_\sigma}\cos\theta_{N_\sigma-1}, \quad x_a = \sqrt{N_\sigma}\tilde{x}_a \sin\theta_{N_\sigma-1}, \quad \sum_{a=1}^{N_\sigma-1} \tilde{x}_a^2 = 1,$$

The systems also contain hidden symmetries, which make the odd dimensional Rossochatius system maximally superintegrable and make the even dimensional system superintegrable (lacking one constant of motion to be maximally superintegrable).

4. Klein-Gordon equation

Seperation of variables of Klelin-Gordon equation [6] will be studied in odd dimensions

$$\Box\Phi = \frac{1}{\sqrt{-g}}\partial_\alpha(\sqrt{-g}g^{\alpha\beta}\partial_\beta\Phi) = M^2\Phi\,, \tag{18}$$

where M is the mass of the scalar field and g is the determinant of the metric. Separation of variables of Klein-Gordon equation in the background of odd dimensional NHEMP can be carried out in elliptic coordinates λ_a which is related to x_i with the following equation

$$x_i^2 = (m_i - \lambda_i) \prod_{j=1,j\neq i}^{N} \frac{m_i - \lambda_j}{m_i - m_j}, \tag{19}$$

where $\lambda_N < m_N < \ldots < \lambda_2 < m_2 < \lambda_1 < m_1$. In this coordinates the NHEMP metric becomes

$$\frac{ds^2}{r_H^2} = A(\lambda)\left(-r^2 d\tau^2 + \frac{dr^2}{r^2}\right) + \sum_{a=1}^{N-1} h_a(\lambda) d\lambda_a^2 + \sum_{i,j=1}^{N} \tilde{\gamma}_{ij} x_i(\lambda) x_j(\lambda) D\varphi^i D\varphi^j, \tag{20}$$

where

$$A(\lambda) = \frac{1}{b} \frac{\prod_{a=1}^{N-1} \lambda_a}{\prod_{i=1}^{N} m_i}. \tag{21}$$

After calculating the inverse and the determinant of the metric, the Klein-Gordon equation becomes

$$\begin{aligned} \sum_{a=1}^{N-1} h^a \partial^2_{\lambda_a} \Phi - \sum_{a=1}^{N-1} \sum_{i=1}^{N} \frac{h^a}{m_i - \lambda_a} \partial_{\lambda_a} \Phi + \frac{1}{A(\lambda)} \left(-\frac{1}{r^2} \left[\frac{\partial}{\partial \tau} - \sum_{i=1}^{N} r k^i \frac{\partial}{\partial \varphi^i} \right]^2 \Phi + r^2 \partial_r^2 \Phi + 2r \partial_r \Phi \right) \\ + \sum_{i=1}^{N} \frac{1}{x_i^2} \partial^2_{\varphi_i} \Phi - \sum_{i,j=1}^{N} \frac{\sqrt{m_i - 1}}{m_i} \frac{\sqrt{m_j - 1}}{m_j} \partial_{\varphi_i} \partial_{\varphi_j} \Phi = M^2 \Phi. \end{aligned} \tag{22}$$

To separate the variables in this equation, we apply the following ansatz

$$\Phi = R_r(r) \cdot \prod_{a=1}^{N-1} R_{\lambda_a}(\lambda_a) \cdot e^{i\omega\tau} \cdot \prod_{i=1}^{N} e^{iL_i \varphi_i}, \tag{23}$$

where ω and L_i are arbitrary constants. The application of this anstaz results into the following separated equations:

$$\begin{gathered} r^2 \frac{R_r''}{R_r} + 2r \frac{R_r'}{R_r} + \frac{1}{r^2} (\omega - r \sum_{i=1}^{N} k^i L_i)^2 = C, \\ P_{\lambda_a}(\lambda_a) = \sum_{\alpha=1}^{N-1} k_\alpha \lambda_a^{\alpha-1}, \end{gathered} \tag{24}$$

where C and k_α $(\alpha = 1, ..., N-2)$ are arbitrary constants, $k_{N-1} = (-1)^{N-2} M^2$ and

$$\begin{aligned} P_{\lambda_a}(\lambda_a) \equiv -\frac{4}{\lambda_a} \left(\frac{R''_{\lambda_a}}{R_{\lambda_a}} - \frac{R'_{\lambda_a}}{R_{\lambda_a}} \sum_{i}^{N} \frac{1}{m_i - \lambda_a} \right) \prod_{j=1}^{N} (m_j - \lambda_a) + \frac{b}{\lambda_a} C \prod_{i=1}^{N} m_i \\ + (-1)^{N-1} \sum_{i=1}^{N} \frac{g_{\varphi_i}}{m_i - \lambda_a} + g_0 (-\lambda_a)^{N-2}. \end{aligned} \tag{25}$$

where g_0 and g_{φ_i} are constants depending on rotation parameters and constants of motion. After a proper transformation, the radial equation in (24) takes the familiar form of Whittaker's equation

$$\frac{d^2 \mathcal{W}}{dz^2} + \left(-\frac{1}{4} + \frac{K}{z} + \frac{(1/4 - \mu^2)}{z^2} \right) \mathcal{W} = 0, \tag{26}$$

with

$$\mathcal{W}(z) = R_r(z), \quad K = i \sum_{j=1}^{N} k^j L_j, \quad \text{and} \quad \mu^2 = \frac{1}{4} - \left(\sum_{i=1}^{N} k^i L_i \right)^2 + C_2. \tag{27}$$

The general solutions to the Whittaker's equation are Whittaker's functions : $\mathcal{M}_{K,\mu}(z)$ and $\mathcal{W}_{K,\mu}(z)$. Their behavior of Whittaker's functions at $r \to 0$ and $r \to \infty$ strongly depend on the values of k^i, L_i and C_2 and can be used to put physical restrictions on these constants.

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