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Null Surface Thermodynamics

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Formulating D dimensional pure Einstein gravity in spacetimes with a null codimension one boundary requires addition of *boundary degrees of freedom (BDOF)* to the the usual bulk gravitons. We construct the solution space perturbatively around the null boundary in which *BDOF* are described by D functions on the codimension one null boundary. Employing covariant phase space formalism we compute charges associated with symmetries which rotate us within the solution phase space. We establish that the system admit a thermodynamical description and that the *null surface thermodynamics* is a consequence of the diffeomorphism invariance of the theory, not relying on other special features of the null surface or the gravity theory. In presence of flux of bulk gravitons through the null boundary we deal with an open thermodynamical system. Our analysis here extends the usual laws of black hole thermodynamics in two ways: it is true for any null surface (which is not necessarily horizon of a black hole) which may describe a system out of thermal equilibrium and that they are local equations over the null surface.

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1. Introduction

Einstein's formulation of Gravity, the General Relativity (GR), makes apparent its salient feature, the universality. That everything gravitates in the same way. This universality is encoded into the (strong) equivalence principle which manifests itself in the theory through diffeomorphism invariance of the theory. One may wonder how presence of a boundary in the spacetime, which may be a physical boundary or just a putative codimension one surface which cuts spacetime into two parts, affects the diffeomorphism invariance or the equivalence principle. It has been argued that in presence of boundaries equivalence principle should be amended [3] by adding in new degrees of freedom which only reside at the boundary, the *BDOF*.

In this talk we will explore this question further, by focusing on the cases where the boundary is a null surface. That is, we consider D dimensional pure Einstein theory described by Einstein-Hilbert action,

$$S_{\text{EH}} = \frac{1}{16\pi G}\int \mathrm{d}^D x\, \sqrt{\det -g_{\mu\nu}}\, L_{\text{EH}}, \qquad L_{\text{EH}} = R - 2\Lambda, \tag{1}$$

where $g_{\mu\nu}$ is the spacetime metric, R is the Ricci curvature and Λ is a possible cosmological constant and consider a spacetime bounded to one side of a $D-1$ dimensional null boundary. The null boundary can be in the asymptotic region, i.e. the asymptotic null infinity in the asymptotic flat spacetimes, or a generic null surface in the spacetime. In our analysis here, motivated by physics of black holes, where horizon is typically a null surface, we consider the latter case. We would like to formulate gravity on one side of this null surface and excise the spacetime at the null boundary, i.e. to formulate physics in the "outside region" of the horizon, see Fig. 1. A similar analysis has been tackled in [4–7], see also [1, 8–22]. We would like to study this problem with putting the focus on the boundary, rather than the bulk and explore how the boundary observer formulates physics.

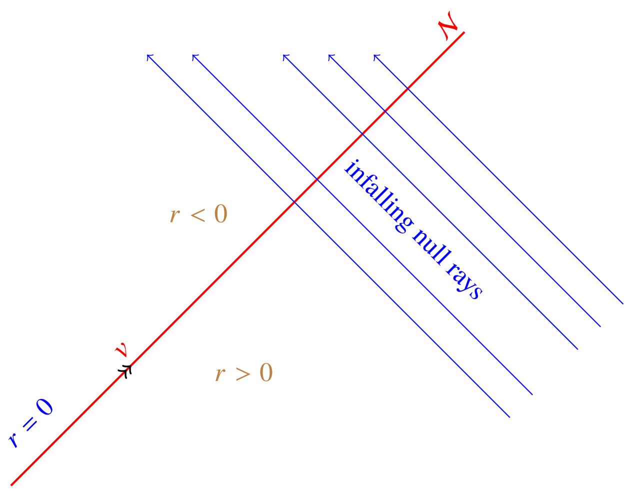

Figure 1: Depiction of null boundary. We would like to formulate physics in $r \geq 0$ region.

To this end, we set about constructing the solution space of the problem at hand. We will then observe that there is a set of diffeomorphisms which act non-trivially at the boundary. Such diffeomorphisms move us in the solution space, hence they are Noether symmetry generators and by virtue of Noether's theorem, there should be surface charges associated with them. We then systematically analyse the surface charges as functions over the solution space using the covariant phase space formalism, mainly developed by Wald and collaborators, e.g. [23, 24]. What we find through this analysis is that the solution space admits a natural thermodynamics description, especially once one tries to describe physics from the boundary viewpoint. This talk is based on [1, 2].

2. Null Surface Solution Phase Space

We start with a D dimensional ($D \geq 3$) generic metric adopting v, r, x^i, $i = 1, 2, \cdots, D-2$ coordinates,

$$\mathrm{d}s^2 = -V\mathrm{d}v^2 + 2\eta\mathrm{d}v\mathrm{d}r + g_{ij}\left(\mathrm{d}x^i + U^i\mathrm{d}v\right)\left(\mathrm{d}x^j + U^j\mathrm{d}v\right), \tag{2}$$

such that the null boundary $\mathcal{N}$ resides at $r = 0$. We assume metric coefficients admit the near $\mathcal{N}$ expansion, an expansion in powers of r around $r = 0$:

$$\begin{aligned}
V &= -\eta\left(\Gamma - \frac{2}{D-2}\frac{\mathcal{D}_v\Omega}{\Omega} + \frac{\mathcal{D}_v\eta}{\eta}\right)r + O(r^2) \\
U^i &= \mathcal{U}^i - r\frac{\eta}{\Omega}\mathcal{J}^i + O(r^2) \\
g_{ij} &= \Omega_{ij} + O(r)
\end{aligned} \tag{3}$$

where all the fields are functions of v, x^i,

$$\Omega_{ij} = \Omega^{2/(D-2)}\gamma_{ij}, \qquad \Omega := \sqrt{\det\Omega_{ij}}, \qquad \det\gamma_{ij} = 1. \tag{4}$$

and

$$\mathcal{D}_v := \partial_v - \mathcal{L}_{\mathcal{U}}, \tag{5}$$

where $\mathcal{L}_{\mathcal{U}}$ is the Lie derivative along $\mathcal{U}^i$. As we see, ∂_r is a null vector for generic r and $\underset{\sim}{r}$ is a null one-form at $r = 0$. One may view the coefficients of the metric as scalars, vector and a tensor in the $D-2$ dimensional sense. With this notation, the metric with coefficients as expanded above has three scalars Ω, Γ, η, two vectors $\mathcal{U}^i, \mathcal{J}_i$ and a determinant-free tensor, γ_{ij}.

These functions of v, x^i are subject to Einstein field equations, perturbatively imposed around $r = 0$. A careful analysis reveals that [1], there are $D-1$ first order differential equations governing them. These equations are Einstein equations projected at the boundary, at $r = 0$, namely the Raychaudhuri and Damour equations:

$$\mathcal{D}_v\Theta + \frac{1}{2}\left(\Gamma - \frac{\mathcal{D}_v\eta}{\eta}\right)\Theta + N_{ij}N^{ij} = 0, \tag{6a}$$

$$\mathcal{D}_v\mathcal{J}_i + \Omega\Theta\frac{\partial_i\eta}{\eta} + \Omega\partial_i\left(\Gamma - 2\Theta\right) + 2\Omega\bar{\nabla}^j N_{ij} = 0. \tag{6b}$$

In the above, Θ is the expansion of the null surface Θ and the *news tensor* N_{ij} which parameterizes gravitons passing through $\mathcal{N}$, and are defined as,

$$\Theta := \mathcal{D}_v \ln\Omega, \qquad N_{ij} := \frac{1}{2}\Omega^{2/(D-2)}\mathcal{D}_v\gamma_{ij} \tag{7}$$

We use Ω^{ij} and Ω_{ij} respectively for raising and lowering indices. Note that N_{ij} as defined above is a symmetric-traceless tensor. Moreover, $\bar{\nabla}_i$ denotes the covariant derivative with respect to the metric Ω_{ij}. One may replace η with the new variable $\mathcal{P}$,

$$\mathcal{P} := \ln \frac{\eta}{\Theta^2}, \tag{8}$$

in terms of which (6) simplify to

$$\mathcal{D}_v\Omega = \Omega\Theta, \tag{9a}$$

$$\mathcal{D}_v\mathcal{P} = \Gamma + \frac{2}{\Theta}N_{ij}N^{ij}, \tag{9b}$$

$$\mathcal{D}_v\mathcal{J}_i + \Omega\Theta\partial_i\mathcal{P} + \Omega\partial_i\Gamma + 2\Omega\bar{\nabla}^j N_{ij} = 0. \tag{9c}$$

Therefore, the solution space is completely specified by two scalars $\Omega, \mathcal{P}$, a vector $\mathcal{J}_i$ and the symmetric-traceless tensor N_{ij}, i.e. $2 + (D-2) + D(D-3)/2$ functions over $\mathcal{N}$. There are other graviton modes which vanish at $r = 0$ on $\mathcal{N}$ and do not enter our analysis here. One may also show that [1] higher order expansions of the metric do not introduce any new functions; Einstein's equations relate all those coefficients to two scalar, one vector and traceless tensor mode that we have here. So, these modes, plus the other graviton modes which vanish at $r = 0$ completely specify the solution space. Moreover, as discussed in [2], this solution space admits a well-defined symplectic form and is hence a phase space.

Null boundary symmetry generators. The vector field

$$\xi = T\,\partial_v + r\,(\mathcal{D}_vT - W)\,\partial_r + \left(Y^i - r\eta\partial^iT\right)\partial_i + \mathcal{O}(r^2)\,, \tag{10}$$

where T, W, Y^i are generic functions of v, x^i, preserves the form of metric (2), keeps $r = 0$ a null surface. T is generating local translations in v (v supertranslations), W is generating local scalings in r (r-superscalings), and Y^i generates superrotations. Vector field ξ therefore, consists of generic diffeomorphisms on $\mathcal{N}$ plus the superscalings generated by W. Under the above diffeomorphism fields on the solution phase space transform as

$$\delta_\xi\Omega = T\Omega\Theta + \mathcal{L}_{(Y+T\mathcal{U})}\Omega\,, \tag{11a}$$

$$\delta_\xi\mathcal{P} = T\mathcal{D}_v\mathcal{P} + \mathcal{L}_{(Y+T\mathcal{U})}\mathcal{P} - W\,, \tag{11b}$$

$$\delta_\xi\mathcal{J}_i = T\mathcal{D}_v\mathcal{J}_i + \mathcal{L}_{(Y+T\mathcal{U})}\mathcal{J}_i + \Omega[\partial_iW - \Gamma\partial_iT - 2N_{ij}\partial^jT]\,, \tag{11c}$$

$$\delta_\xi N_{ij} = \mathcal{D}_v(TN_{ij}) + \mathcal{L}_{(Y+T\mathcal{U})}N_{ij}\,, \tag{11d}$$

where $\mathcal{L}_Y$ denote the Lie derivative along Y^i.

Surface charge variation. Charge variation associated with the boundary symmetry generator ξ, may be computed using covariant phase space formalism [23, 24]. Detailed analysis yields [1]

$$\not{\delta}Q_\xi = \frac{1}{16\pi G}\int_{\mathcal{N}_v} \mathrm{d}^{D-2}x\Big[(W - \Gamma T)\,\delta\Omega + (Y^i + \mathcal{U}^iT)\delta\mathcal{J}_i + T\Omega\Theta\delta\mathcal{P} - T\Omega\Omega^{ij}\delta N_{ij}\Big], \tag{12}$$

where $\mathcal{N}_v$ is a constant v section on $\mathcal{N}$. This charge variation is an integral over $\sum_{a=1}^4 \mathcal{G}_a\ \delta\boldsymbol{Q}_a$, where $\boldsymbol{Q}_a \in \{\Omega, \mathcal{P}, \mathcal{J}_i; N_{ij}\}$ parameterize the solution phase space and $\mathcal{G}_a$ are linear combinations

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of the symmetry generators with field dependent coefficients, i.e. $\delta_\xi \mathcal{G}_a \neq 0$. Among these the first three parameterize *BDOF* and N_{ij} the bulk degrees of freedom. $\Gamma, \mathcal{U}^i$ functions which appear in $\mathcal{G}_a$ are subject to field equations (9) and may hence be expressed in term of the charges $\boldsymbol{Q}_a$. The solution phase space is hence parametrized by the four tower of charges and their variations. We also note that the charge variation $\oint Q_\xi$, as stressed in the notation $\oint$, is hence not integrable.

Among the charges, we consider three 'zero mode' charges (or charge variations) associated with $\xi = -r\partial_r, \xi = \partial_i$ and $\xi = \partial_v$,

$$\begin{aligned} Q_{-r\partial_r} &:= \frac{\mathbf{S}}{4\pi} = \frac{1}{16\pi G}\int_{\mathcal{N}_v} \mathrm{d}^{D-2}x\, \Omega, \\ Q_{\partial_i} &:= \mathbf{J}_i = \frac{1}{16\pi G}\int_{\mathcal{N}_v} \mathrm{d}^{D-2}x\, \mathcal{J}_i, \\ \oint Q_{\partial_v} &:= \oint \mathbf{H} = \frac{1}{16\pi G}\int_{\mathcal{N}_v} \mathrm{d}^{D-2}x\, \left(-\Gamma\delta\Omega + \mathcal{U}^i\delta\mathcal{J}_i + \Omega\Theta\delta\mathcal{P} - \Omega\Omega^{ij}\delta N_{ij}\right). \end{aligned} \tag{13}$$

Surface charges and flux. The charge variation (12) is not integrable. It can be split into Noether (integrable) part Q^{N} and the 'flux' part F: $\oint Q_\xi = \delta Q^{\mathrm{N}}_\xi + F_\xi(\delta g; g)$. Q^{N} may be computed for the Einstein-Hilbert action using the standard Noether procedure, yielding

$$Q^{\mathrm{N}}_\xi = \frac{1}{16\pi G}\int_{\mathcal{N}_v} \mathrm{d}^{D-2}x \left[W\,\Omega + Y^i\,\mathcal{J}_i + T\,\left(-\Gamma\Omega + \mathcal{U}^i\,\mathcal{J}_i\right)\right], \tag{14}$$

and non-integrable flux part

$$F_\xi(\delta g; g) = \frac{1}{16\pi G}\int_{\mathcal{N}_v} \mathrm{d}^{D-2}x\, T\left(\Omega\delta\Gamma - \mathcal{J}_i\delta\mathcal{U}^i + \Omega\Theta\delta\mathcal{P} - \Omega\Omega^{ij}\delta N_{ij}\right). \tag{15}$$

Here we assumed symmetry generators T, W, Y^i to be field-independent, i.e. $\delta T = \delta W = 0 = \delta Y^i$. One may work out the expressions for the zero mode Noether charges, which are obtained as

$$Q^{\mathrm{N}}_{-r\partial_r} = \frac{\mathbf{S}}{4\pi}, \qquad Q^{\mathrm{N}}_{\partial_i} = \mathbf{J}_i, \qquad Q^{\mathrm{N}}_{\partial_v} := \mathbf{E} = \frac{1}{16\pi G}\int_{\mathcal{N}_v} \mathrm{d}^{D-2}x\, (-\Gamma\Omega + \mathcal{U}^i\mathcal{J}_i), \tag{16}$$

The expression of charges (14) involve an integration over codimension two surface $\mathcal{N}_v$ and hence they are functions of v as well as being a functional of functions parameterizing the solution space, while are linear functions of symmetry generators. Starting from the definition of Noether charge and flux as given above, one may readily check that

$$\frac{\mathrm{d}}{\mathrm{d}v} Q^{\mathrm{N}}_\xi \approx -F_{\partial_v}(\delta_\xi g; g), \tag{17}$$

where $\approx$ denotes on-shell equality. Eq.(17), the *balance equation*, may be viewed as (1) the boundary field equations (9) written in terms of charges; (2) 'generalized charge conservation equation' which relates time dependence, or non-conservation, of the charge (as viewed by the null boundary observer) to the flux passing through the boundary; (3) how the passage of flux through the null boundary is 'balanced' by the rearrangements in the charges. This viewpoint yields *null surface memory effects* [1].

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3. Null Surface Thermodynamics

We would now like to reinterpret the expressions for the charge variation and Noether charges as thermodynamical equations. To this end, let us briefly review some facts about usual thermodynamical systems. Consider a usual thermodynamical system with chemical potentials $\mu_A (A = 1, 2, \cdots, N)$ and temperature T. This system is specified with charges Q_A, the entropy S and the energy E; that is, there are $N+2$ charges and $N+1$ chemical potentials. The distinction between charges and associated chemical potentials is by convention and is specified with/specifies the ensemble. The first law takes the form

$$\mathrm{d}E = T\mathrm{d}S + \sum_{A=1}^{N} \mu_A \mathrm{d}Q_A, \tag{18}$$

implying that the LHS is an exact one-form over the thermodynamic space. Moreover, chemical potentials and the charges are related by the Gibbs-Duhem relation

$$S\mathrm{d}T + \sum_{A=1}^{N} Q_A \mathrm{d}\mu_A = 0. \tag{19}$$

Together with the first law (18) this yields $E = TS + \sum_A \mu_A Q_A$. This equation relates E to the other charges and chemical potentials, e.g. $E = E(S, Q_A)$ (in microcanonical description) or $E = E(T, \mu_A)$ (in grandcanonical description). Depending on the ensemble chosen, $N+1$ number of chemical potentials and/or charges may be taken to be 'independent' variables parameterizing the thermodynamical configuration space and the rest of $N+1$ of them as functions of the former $N+1$ variables. In other words, the thermodynamic configuration space is $(N+1)$ dimensional and the change of ensemble is basically a canonical transformation the generator of which is the difference between various 'energy' functions associated with each ensemble.

3.1 Null Boundary Thermodynamical Phase Space

From (12) one can recognize that functions parameterizing the solution space come in two categories: the bulk modes N_{ij} (and its conjugate 'chemical potential' determinant-free part of Ω^{ij}, γ^{ij}) and the boundary modes. The latter may also be separated into those whose variation appears $\Omega, \mathcal{P}$ and $\mathcal{J}_i$, and those which appear only in the coefficients, in chemical potentials $\Gamma, \mathcal{U}^i$. There are hence $D = 1 + 1 + (D-2)$ charges and $D - 1 = 1 + (D-2)$ chemical potentials. As already remarked, one may view (9) as equations for $\Gamma, \mathcal{U}^i$ giving them in terms charges.

A careful look into the charge expressions, can lead us to the following picture for the generic case.

I. Null boundary solution space relevant to the null boundary thermodynamics consists of three parts:

   I.1) $(D-1)$ dimensional 'thermodynamic sector' parametrized by $(\Gamma, \mathcal{U}^i)$ and conjugate charges $(\Omega, \mathcal{J}_i)$;

   I.2) $\mathcal{P}$, which only appears in the flux (15) and not in the Noether charge (14);

I.3) the bulk mode parameterized by determinant free part of Ω^{ij} and its 'conjugate charge' N_{ij} which appear in the flux (15).

II. N_{ij} parameterizes effects of the bulk and how they take the boundary system out-of-thermal-equilibrium (OTE) whereas $\mathcal{P}$ parameterizes OTE within the boundary dynamics. Put differently, OTE may come from inner boundary dynamics and/or from the gravity-waves passing through the null boundary, parameterized by N_{ij}.

III. Expansion parameter Θ is a measure of OTE, from both bulk and boundary viewpoints. When $\Theta = 0$ the system is completely specified by the $D-1$ dimensional thermodynamic phase space.

IV. The rest of the in-falling graviton modes parameterized through $O(r)$ terms in g_{ij}, do not enter in the boundary/thermo dynamics, as of course expected from usual causality and that the boundary is a null surface.

We now rewrite the above equations as a local first law and a local Gibbs-Duhem equation and then discuss a notion of local zeroth law. We use the notation $\boldsymbol{\mathcal{X}}$ to notify the density of the quantity $\mathbf{X}$,

$$\mathbf{X} := \int_{\mathcal{N}_v} \mathrm{d}^{D-2}x\, \boldsymbol{\mathcal{X}}. \tag{20}$$

3.2 Local First Law at Null Boundary

Defining $\boldsymbol{\mathcal{P}} := \mathcal{P}/(16\pi G)$ and $\boldsymbol{\mathcal{N}}_{ij} := (16\pi G)^{-1} N_{ij}$, (13) implies,

$$\boxed{\not{\delta}\boldsymbol{\mathcal{H}} = T_{\mathcal{N}}\delta\boldsymbol{\mathcal{S}} + \boldsymbol{\mathcal{U}}^i\delta\boldsymbol{\mathcal{J}}_i + \Omega\Theta\delta\boldsymbol{\mathcal{P}} - \Omega\Omega^{ij}\delta\boldsymbol{\mathcal{N}}_{ij}, \qquad T_{\mathcal{N}} := -\frac{\Gamma}{4\pi}} \tag{21}$$

The above is true at each v, x^i over the null surface and represents the local null boundary first law. The LHS, unlike the usual first law (18), is not a complete variation, a manifestation of the fact that we are dealing with an open thermodynamic system due to the existence of the expansion and the flux. The above reduces to a usual first law for closed systems when $N_{ij} = 0$ or in the non-expanding $\Theta = 0$ case.

$\Gamma = -2\kappa + \mathcal{D}_v \ln(\eta\Omega^{\frac{2}{D-2}})$ where κ is the non-affinity parameter (surface gravity) associated the vector field generating the null surface $\mathcal{N}$ [1]. $-\Gamma/2$ is the local acceleration of an observer for whom $r = 0$ is locally the Rindler horizon. So, $T_{\mathcal{N}} = \frac{\kappa}{2\pi} - \frac{1}{4\pi}\mathcal{D}_v \ln(\eta\Omega^{\frac{2}{D-2}})$. For non-expanding $\Theta = 0$ case where one may put $\eta = 1$ or when we have a Killing horizon, $T_{\mathcal{N}}$ equals the usual Unruh/Hawking temperature, *cf.* section 3.4 for more discussions.

3.3 Local Extended Gibbs-Duhem Equation at Null Boundary

Given the expressions for the zero mode charges (16) and for the densities in the same notation as in (20) we have

$$\boxed{\boldsymbol{\mathcal{E}} = T_{\mathcal{N}}\boldsymbol{\mathcal{S}} + \boldsymbol{\mathcal{U}}^i\boldsymbol{\mathcal{J}}_i} \tag{22}$$

The above is an analogue of the Gibbs-Duhem equation if $\boldsymbol{\mathcal{E}}$ is viewed as energy, $\boldsymbol{\mathcal{S}}$ as entropy and $\boldsymbol{\mathcal{J}}_i$ as other conserved charges and $\Gamma, \boldsymbol{\mathcal{U}}^i$ as the respective chemical potentials. This manifests the picture we outlined in section 3.1. One should, however, note that (22) is a local equation at the null boundary, unlike its usual thermodynamic counterpart. This equation also holds for non-stationary/non-adiabatic cases when the system is out-of-thermal-equilibrium (OTE). So, we call (22) 'local extended Gibbs-Duhem' (LEGD) equation at the null boundary.

Since the integrable parts of the charge are (by definition) independent of the bulk flux N_{ij} and also of $\mathcal{P}$, the LEGD equation (22), also do not involve $\mathcal{P}$ and N_{ij}. Nonetheless, the chemical potentials in (22), Γ and $\mathcal{U}^i$, implicitly depend on N_{ij} and $\mathcal{P}$ through Raychaudhuri and Damour equations.

We remark that the local first law (21) and LEGD equation are a manifestation of diffeomorphism invariance of the theory. While the explicit expressions for the charges do depend on the theory, we expect (22) to be universally true for any diff-invariant theory of gravity in any dimension. This equation is on par with the first law of thermodynamics but extends it in two important ways: it is a local equation in v, x^i and holds also for OTE.

3.4 Local Zeroth Law

Zeroth law is a statement of thermal equilibrium; zeroth law stipulates that two (sub)systems with the same temperature and chemical potentials are in thermal equilibrium. As flow of charges is proportional to the gradient of associated chemical potentials, the absence of such fluxes can be taken as a statement of the zeroth law. In our case, we are dealing with a system parameterized by chemical potentials $\Gamma, \mathcal{U}^i$ and γ^{ij} which are functions of charges $\Omega, \mathcal{P}, \mathcal{J}_i, N_{ij}$. This system is not in general in equilibrium but there could be special subsectors which are. The zeroth law is to specify such subsectors.

Eq.(17) implies that flow of charges vanish on subsystems over which $F_{\partial_v}(\delta_\xi g, g)$ vanishes. On the other hand, one can show that [25] this flux has the same expression as the on-shell variation of the action and while the charge variation (12) is invariant under the addition of a total derivative term to the Lagrangian, the Noether charge and hence the flux are not. In particular, upon addition of a boundary Lagrangian $L_{\mathcal{B}} = \partial_\mu \mathcal{B}^\mu$, the on-shell variation of the action and hence the flux F are shifted by $\delta\mathcal{B}^r$. Let us denote $\mathcal{B}^r = \mathcal{G}$. This opens up the possibility of (partially) removing the flux by an appropriate boundary term. The question is hence what are the subsectors in the solution phase space for which flux can be removed by an appropriate boundary term. To formulate this idea, we start with the variation of on-shell action. A direct computation leads to

$$\delta S_{\text{EH}}|_{\text{on-shell}} = \frac{1}{16\pi G}\int_{\mathcal{N}} \text{v}\,\text{d}^{D-2}x \left(\Omega\Theta\delta\mathcal{P} + \Omega\delta\Gamma - \mathcal{J}_i\delta\mathcal{U}^i - \Omega N^{ij}\delta\Omega_{ij}\right) = \int \text{v} F_{\partial_v}(\delta g; g), \tag{23}$$

where F_{∂_v} may be readily read from (15). Next, let us add a boundary term to the Lagrangian upon which $\delta S_{\text{EH}}|_{\text{on-shell}} \to \delta S_{\text{EH}}|_{\text{on-shell}} + \int_{\mathcal{N}} \delta\mathcal{G}$. As the statement of the zeroth law we require there exists a $\mathcal{G} = \mathcal{G}(\Omega, \mathcal{P}, \mathcal{J}_A, N_{AB})$ such that,

$$\delta\mathcal{G} = -\boldsymbol{S}(\delta T_{_\mathcal{N}} - 4G\Theta\delta\boldsymbol{\mathcal{P}}) - \boldsymbol{\mathcal{J}}_i\delta\boldsymbol{\mathcal{U}}^i + \Omega\boldsymbol{N}_{ij}\delta\Omega^{ij} \tag{24}$$

admits non-zero solutions. Integrability condition for the zeroth law (24) is $\delta(\delta\mathcal{G}) = 0$, which yields an equation like $\sum_{a,b} C_{ab}\delta Q_a \wedge \delta Q_b = 0$, where Q_a are generic charges and C_{ab} is skew-symmetric. This equation is satisfied only for $C_{ab} = 0$. One can immediately see $N_{ij} = 0 = \delta N_{ij}$ is a necessary (but not sufficient) condition for (24) to have non-trivial solutions.

Note that when (24) is fulfilled the charge $\mathcal{H}$, which appears in the LHS of the local first law (21), becomes integrable and we obtain

$$\boxed{\mathcal{H} = \mathcal{G} + T_{\mathcal{N}}\mathcal{S} + \mathcal{U}^i\mathcal{J}_i} \tag{25}$$

Let us now turn again to the implications of the zeroth law (24). Besides $N_{ij} = 0$, in terms of $\mathcal{H} = \mathcal{H}(\mathcal{S}, \mathcal{J}_i, \mathcal{P})$ local zeroth law implies,

$$\boxed{T_{\mathcal{N}} = \frac{\delta\mathcal{H}}{\delta\mathcal{S}}, \qquad \mathcal{U}^i = \frac{\delta\mathcal{H}}{\delta\mathcal{J}_i}, \qquad \mathcal{D}_v\mathcal{S} = \mathcal{S}\Theta = \frac{1}{4G}\frac{\delta\mathcal{H}}{\delta\mathcal{P}}} \tag{26}$$

We next want to solve the above equations.

Generic $\Theta \neq 0$ case. For $N_{ij} = 0$ (9) reduce to

$$T_{\mathcal{N}} = -4G\mathcal{D}_v\mathcal{P}, \qquad \mathcal{D}_v\left[\mathcal{J}_i + 4G\bar{\nabla}_i(\mathcal{S}\mathcal{P})\right] = 0. \tag{27}$$

The above imply that zeroth law (26) is satisfied for any $\mathcal{H} = \mathcal{H}(\mathcal{S}, \mathcal{P}, \mathcal{J}_i)$, when $\mathcal{S}, \mathcal{P}$ and $\mathcal{J}_i$ have the following basic Poisson brackets [1, 2]

$$\begin{aligned}
&\{\mathcal{S}(x,v), \mathcal{P}(y,v)\} = \frac{1}{4G}\delta^{D-2}(x-y), \quad \{\mathcal{S}(x,v), \mathcal{S}(y,v)\} = \{\mathcal{P}(x,v), \mathcal{P}(y,v)\} = 0,\\
&\{\mathcal{S}(x,v), \mathcal{J}_i(y,v)\} = \mathcal{S}(y,v)\frac{\partial}{\partial x^i}\delta^{D-2}(x-y),\\
&\{\mathcal{P}(x,v), \mathcal{J}_i(y,v)\} = \left(\mathcal{P}(y,v)\frac{\partial}{\partial x^i} + \mathcal{P}(x,v)\frac{\partial}{\partial y^i}\right)\delta^{D-2}(x-y),\\
&\{\mathcal{J}_i(x,v), \mathcal{J}_j(y,v)\} = \frac{1}{16\pi G}\left(\mathcal{J}_i(y,v)\frac{\partial}{\partial x^j} - \mathcal{J}_j(x,v)\frac{\partial}{\partial y^i}\right)\delta^{D-2}(x-y)
\end{aligned} \tag{28}$$

and

$$\partial_v\mathcal{X} = \{\mathcal{H}, \mathcal{X}\}. \tag{29}$$

That is, $\mathcal{H}$ is the Hamiltonian over this phase space and (26) do not impose any restrictions on $\mathcal{H}$ which is a scalar over $\mathcal{N}$.

$\Theta = 0$ case. In this case trace of the extrinsic curvature of the null surface $\mathcal{N}$ vanishes. In this case η drops out from the charge variation (12) and we lose the charge $\mathcal{P}$, and the associated symmetry generator becomes a pure gauge. We hence remain with T, Y^i generators which form Diff($\mathcal{N}$) symmetry algebra. EoM (6) reduces to $\mathcal{D}_v\mathcal{J}_i = \mathcal{S}\partial_i T_{\mathcal{N}}$ and $\mathcal{D}_v\mathcal{S} = 0$, which may be viewed as equations for spatial derivatives of the chemical potentials.

Local zeroth law (26) is satisfied by any scalar Hamiltonian $\mathcal{H} = \mathcal{H}(\mathcal{S}, \mathcal{J}_i)$, together with basic Poisson brackets (28) but with $\mathcal{P}$ dropped [1] and again with $\partial_v\mathcal{X} = \{\mathcal{H}, \mathcal{X}\}$.

To summarize this part, Local zeroth law (26) is just defining the Poisson bracket structure over our charge space and existence of Hamiltonian dynamics, but does not specify a Hamiltonian. Choice of Hamiltonian fixes a boundary Lagrangian and the boundary dynamical equations which in turn specifies local dynamics of charges on the null boundary $\mathcal{N}$. Our zeroth law is a weaker condition than stationarity as ∂_v of the chemical potentials need not vanish. The usual zeroth law of

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black hole mechanics (for Killing horizons) that $\mathcal{U}^i$ and Γ are constants over the horizon (our null boundary $\mathcal{N}$) is a very special case which obeys our local zeroth law. For the stationary asymptotic flat black hole solutions to the vacuum Einstein gravity, i.e. the Myers-Perry solutions, we get $\mathcal{E} = \left(\frac{D-3}{D-2}\right)\mathcal{H}$, and we have the usual Smarr formula.

4. Outlook

We discussed that boundary degrees of freedom residing on a null surface in D dimensional Einstein gravity are governed by local laws of thermodynamics which are local equations over the $D-1$ dimensional null surface $\mathcal{N}$. We are in general dealing with an open thermodynamical system and from the boundary viewpoint a graviton which goes through the boundary gets *dissolved* into the boundary and the BDOF readjust themselves according to (9) or equivalently (17); these readjustments are nothing but our first law of thermodynamics and the Gibbs-Duhem equations. We should emphasize that our analysis does not fix the boundary dynamics, boundary Hamiltonian may still be chosen.

The zeroth law necessitates vanishing of the flux of bulk gravitons through $\mathcal{N}$ and establishes basic Poisson brackets on the thermodynamic phase space. That is, the zeroth law establishes we have a well define phase space.

We did not discussed the second law of thermodynamics, but establish a full thermodynamic description it is crucial to have a notion of the second law. It is hence of great interest to explore a notion of local second law.

As we discussed our thermodynamic description is a manifestation of diffeomorphism invariance of the theory, the equivalence principle, and not the details of the gravity theory. It is a straightforward exercise to verify that all our discussions and results are valid for any generally invariant theory of gravity. Moreover, being a result of diffeomorphism invariance, one may check that our analysis and results also extends when we include generic matter field. However, it is expected that an extension of the second law would require the matter to respect null energy condition.

Finally, our main motivation to consider null boundaries came from black hole physics. Our analysis here is a modern and more rigorous reincarnation of the membrane paradigm [26–28]. It would be desirable to connect the two more systematically, first steps towards this were taken in [29]. Moreover, our analysis provides a theoretical setup to formulate more precisely the soft hair proposal [30] and explore identification of black hole microstates. For the latter, one would presumably need to quantize boundary phase space and the boundary theory.

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References

[1] H. Adami, D. Grumiller, M. M. Sheikh-Jabbari, V. Taghiloo, H. Yavartanoo, and C. Zwikel, "Null boundary phase space: slicings, news & memory," *JHEP* **11** (2021) 155, 2110.04218.

[2] H. Adami, M. M. Sheikh-Jabbari, V. Taghiloo and H. Yavartanoo, "Null surface thermodynamics," Phys. Rev. D **105**, no.6, 066004 (2022) 2110.04224.

[3] M. M. Sheikh-Jabbari, "Residual diffeomorphisms and symplectic soft hairs: The need to refine strict statement of equivalence principle," *Int. J. Mod. Phys.* **D25** (2016), no. 12, 1644019, 1603.07862.

[4] H. Adami, D. Grumiller, S. Sadeghian, M. Sheikh-Jabbari, and C. Zwikel, "T-Witts from the horizon," *JHEP* **04** (2020) 128, 2002.08346.

[5] D. Grumiller, M. M. Sheikh-Jabbari, and C. Zwikel, "Horizons 2020," *Int. J. Mod. Phys. D* **29** (2020), no. 14, 2043006, 2005.06936.

[6] H. Adami, M. M. Sheikh-Jabbari, V. Taghiloo, H. Yavartanoo, and C. Zwikel, "Symmetries at null boundaries: two and three dimensional gravity cases," *JHEP* **10** (2020) 107, 2007.12759.

[7] H. Adami, M. M. Sheikh-Jabbari, V. Taghiloo, H. Yavartanoo, and C. Zwikel, "Chiral Massive News: Null Boundary Symmetries in Topologically Massive Gravity," *JHEP* **05** (2021) 261, 2104.03992.

[8] D. Grumiller, A. Pérez, M. Sheikh-Jabbari, R. Troncoso, and C. Zwikel, "Spacetime structure near generic horizons and soft hair," *Phys. Rev. Lett.* **124** (2020), no. 4, 041601, 1908.09833.

[9] L. Donnay, G. Giribet, H. A. González, and M. Pino, "Supertranslations and Superrotations at the Black Hole Horizon," *Phys. Rev. Lett.* **116** (2016), no. 9, 091101, 1511.08687.

[10] L. Donnay, G. Giribet, H. A. González, and M. Pino, "Extended Symmetries at the Black Hole Horizon," *JHEP* **09** (2016) 100, 1607.05703.

[11] H. Afshar, S. Detournay, D. Grumiller, W. Merbis, A. Perez, D. Tempo, and R. Troncoso, "Soft Heisenberg hair on black holes in three dimensions," *Phys. Rev.* **D93** (2016), no. 10, 101503, 1603.04824.

[12] H. Afshar, D. Grumiller, and M. M. Sheikh-Jabbari, "Near horizon soft hair as microstates of three dimensional black holes," *Phys. Rev.* **D96** (2017), no. 8, 084032, 1607.00009.

[13] H. Afshar, D. Grumiller, W. Merbis, A. Perez, D. Tempo, and R. Troncoso, "Soft hairy horizons in three spacetime dimensions," *Phys. Rev.* **D95** (2017), no. 10, 106005, 1611.09783.

[14] F. Hopfmüller and L. Freidel, "Gravity Degrees of Freedom on a Null Surface," *Phys. Rev. D* **95** (2017), no. 10, 104006, 1611.03096.

[15] F. Hopfmüller and L. Freidel, "Null Conservation Laws for Gravity," *Phys. Rev. D* **97** (2018), no. 12, 124029, 1802.06135.

[16] L. Donnay, G. Giribet, H. A. González, and A. Puhm, "Black hole memory effect," *Phys. Rev. D* **98** (2018), no. 12, 124016, 1809.07266.

[17] V. Chandrasekaran, É. É. Flanagan, and K. Prabhu, "Symmetries and charges of general relativity at null boundaries," *JHEP* **11** (2018) 125, 1807.11499.

[18] V. Chandrasekaran and K. Prabhu, "Symmetries, charges and conservation laws at causal diamonds in general relativity," *JHEP* **10** (2019) 229, 1908.00017.

[19] V. Chandrasekaran and A. J. Speranza, "Anomalies in gravitational charge algebras of null boundaries and black hole entropy," *JHEP* **01** (2021) 137, 2009.10739.

[20] L. Ciambelli and R. G. Leigh, "Isolated surfaces and symmetries of gravity," *Phys. Rev. D* **104** (2021), no. 4, 046005, 2104.07643.

[21] L. Freidel, R. Oliveri, D. Pranzetti, and S. Speziale, "The Weyl BMS group and Einstein's equations," *JHEP* **07** (2021) 170, 2104.05793.

[22] L. Freidel, R. Oliveri, D. Pranzetti, and S. Speziale, "Extended corner symmetry, charge bracket and Einstein's equations," *JHEP* **09** (2021) 083, 2104.12881.

[23] J. Lee and R. M. Wald, "Local symmetries and constraints," *J. Math. Phys.* **31** (1990) 725–743.

[24] V. Iyer and R. M. Wald, "Some properties of Nöther charge and a proposal for dynamical black hole entropy," *Phys. Rev.* **D50** (1994) 846–864, gr-qc/9403028.

[25] H. Adami, M. Sheikh-Jabbari, V. Taghiloo, and H. Yavartanoo, "Noether charge vs modified bracket, ambiguities and all that," *Work in preparation* (2022).

[26] R. H. Price and K. S. Thorne, "Membrane Viewpoint on Black Holes: Properties and Evolution of the Stretched Horizon," *Phys. Rev.* **D33** (1986) 915–941.

[27] K. S. Thorne, R. Price, and D. Macdonald, *Black Holes: The Membrane Paradigm*. Yale University Press, 1986.

[28] M. K. Parikh, "Membrane horizons: The Black hole's new clothes," thesis, Princeton University, 10, 1998.

[29] D. Grumiller and M. M. Sheikh-Jabbari, "Membrane Paradigm from Near Horizon Soft Hair," *Int. J. Mod. Phys.* **D27** (2018) 1847006, 1805.11099.

[30] S. W. Hawking, "The Information Paradox for Black Holes," 9, 2015. 1509.01147.