PROCEEDINGS OF SCIENCE

Supersymmetric Cubic Interactions For Lower Spins From "Higher Spin" Approach

I.L. Buchbinder,[a] V. Krykhtin,[a] M. Tsulaia[b,*] and D. Weissman[b]

[a] *Center of Theoretical Physics, Tomsk State Pedagogical University, 634041, Tomsk, Russia*

[b] *Okinawa Institute of Science and Technology, 1919-1 Tancha, Onna-son, Okinawa 904-0495, Japan*

E-mail: joseph@tspu.edu.ru, krykhtin@tspu.edu.ru, mirian.tsulaia@oist.jp, dorin.weissman@oist.jp

We demonstrate how to reconstruct standard cubic vertices for $N = 1$ supersymmetric Yang-Mills and Supergravities using various techniques adopted for the description of cubic interactions between higher spin fields.

RDP online PhD school and workshop "Aspects of Symmetry"(Regio2021), 8-12 November 2021 Online

*Speaker

PoS(Regio2021)035

1. Introduction

The purpose of these notes is to demonstrate in detail various approaches of constructing supersymmetric cubic interaction vertices for higher spin fields. To this end we reconstruct well known cubic vertices for $N = 1$ Super Yang-Mills and for linearized $N = 1$ Supergravities. In particular, we consider the field theoretic limits of the pure $N = 1$, $\mathcal{D} = 10$ supergravity [1]–[2], of the $N = 1$, $\mathcal{D} = 4$ supergravity coupled with one chiral supermultiplet [3], and of the $N = 1$, $\mathcal{D} = 6$ supergravity coupled to one $(1, 0)$ tensor supermultiplet [4]– [5]. The reason for choosing these particular types of supergravity will become clear in the following. We shall also comment on a generalization of these vertices to higher spin "Yang-Mills-like" and "Supergravity- like" vertices [6], and on the higher spin generalizations of the corresponding free Lagrangians [7].

First, we shall describe the covariant BRST approach,[1] which is similar to the one of Open String Field Theory.[2] However, unlike String Field Theory, the BRST approach to higher spin fields is essentially a method for the construction of free and interacting Lagrangians using gauge invariance as the only guiding principle, without any recourse to a world-sheet description.

As the first step in this approach, one constructs free Lagrangians invariant under linear gauge transformations. Because of the presence of gauge symmetry, these Lagrangians contain both physical and non-physical degrees of freedom. Some of the non-physical degrees of freedom are removed using the equations of motion, the others are gauged away, and in the end one is left only with physical polarizations. In general, these systems can contain bosonic and fermionic fields, which are described by Young tableaux with mixed symmetries. However, at the free level one can consider Lagrangians for just one or several (a finite number of) representations of the Poincaré group [24]– [25]. A further requirement of supersymmetry singles out some particular representations of the Poincaré group, both in bosonic and fermionic sectors, so the corresponding bosonic and fermionic Lagrangians are related by supersymmetry transformations.

As the second step in the BRST approach, one promotes the original gauge symmetry to an interacting level by deforming the Lagrangian and gauge transformation rules with nonlinear terms, in such a way that the gauge invariance is kept order by order in the coupling constant. As in the case of free Lagrangians, supersymmetry singles out some particular subclass of the cubic vertices, which were found for non-supersymmetric systems [26]–[31].

To describe how this approach works on the examples of $N = 1$ Super Yang-Mills and linearized Supergravities, we start in Section 2 with a description of gauge invariant free Lagrangians. For the massless vector field the corresponding Lagrangian is the standard (Maxwell) one. A gauge invariant Lagrangian which contains the second rank symmetric massless tensor field as a physical component describes spins 2 and 0 simultaneously. Then we present a similar gauge invariant description for a massless spin-vector field which contains irreducible representations of the Poincaré group with spins $\frac{3}{2}$ and $\frac{1}{2}$.

In Section 3 we reformulate the results of Section 2 in the BRST approach. Then we impose an additional requirement of $N = 1$ supersymmetry on these systems. Using the technique developed in the Open Superstring Field Theory [32] we show that $N = 1$ supersymmetry requires some extra fields (both physical and auxiliary) in the bosonic sector [7]. The obtained Lagrangian provides a

[1] See [8] for a review of the BRST approach and [9] –[22] for reviews of different approaches to higher spin theories.
[2] See e.g. [23] for a recent review.

"unified" description of the above mentioned $N = 1$ linearized supergravities in $\mathcal{D} = 4, 6$ and 10 dimensions.

In Section 4 we turn to cubic interactions in the covariant formalism and present generic equations which determine the cubic vertices, as well as solutions to these equations for three massless bosonic fields with arbitrary spins [30], [33]. We also present the generic equations for determining cubic vertices for two massless fermions and one massless boson [6].

In Section 5 we describe $N = 1$ Super Yang-Mills in this formalism and in Section 6 we consider cubic vertices for linearized supergravities. We comment on the higher spin generalization of the vertices given in Sections 5 and 6. These cubic vertices [6] are covariant versions of the vertices for two fermionic and one bosonic higher spin fields in arbitrary dimensions, first derived in the light cone formalism [27].[3]

Finally, in Section 7 we describe the light cone approach to the construction of the cubic vertices. In this approach one splits the generators of the Poincaré (super)group into dynamical and kinematical operators. When using the field theoretic realization of the generators one takes the kinematical operators to be quadratic in terms of superfields, whereas the dynamical generators contain also cubic and higher order terms, i.e., cubic and higher order vertices. The requirement that the Poincaré superalgebra stays intact after the nonlinear deformation of the dynamical operators determines these vertices order by order in the coupling constant. We shall briefly review the construction of [47] (see [48],[49] for the case of an arbitrary N) for arbitrary spin supermultiplets in $N = 1$, $\mathcal{D} = 1$, and show how to obtain cubic vertices for four dimensional $N = 1$ super Yang-Mills and $N = 1$ supergravity in the light cone gauge[4] as a particular example.

2. Free Lagrangians

2.1 $s = 1$

Let us start with a massless vector field $\phi_\mu(x)$ with a standard gauge transformation rule

$$\delta\phi_\mu(x) = \partial_\mu\lambda(x) \tag{1}$$

The gauge invariant Klein -Gordon and transversality equations for the massless vector field can be written as

$$\Box\phi_\mu(x) = \partial_\mu C(x), \quad C(x) = \partial^\mu\phi_\mu(x) \tag{2}$$

where we introduced an auxiliary field $C(x)$, which transforms as

$$\delta C(x) = \Box\lambda(x) \tag{3}$$

The equations (2) can be obtained from the Lagrangian

$$\mathcal{L} = -\frac{1}{2}(\partial^\mu\phi^\nu)(\partial_\mu\phi_\nu) + C\partial^\mu\phi_\mu - \frac{1}{2}C^2 \tag{4}$$

After eliminating the field $C(x)$ via its own equation of motion one obtains the Maxwell Lagrangian for the vector field $\phi_\mu(x)$.

[3] Covariant cubic vertices with two fermions and electromagnetic and gravitational fields are given in [34]–[35]. Supersymmetric cubic interactions on flat and AdS backgrounds are given in [36]–[43].

[4] The cubic vertices for $\mathcal{D} = 4$, $N = 4$ super Yang-Mills in the light cone formulation are given in [50]–[52].

2.2 $s = 2$ and $s = 0$

One can repeat a similar consideration for a second rank symmetric tensor field $\phi_{\mu\nu}(x)$. Now the gauge invariant Klein-Gordon equation reads,

$$\Box\phi_{\mu\nu}(x) = \partial_\mu C_\nu(x) + \partial_\nu C_\mu(x) \tag{5}$$

where the physical field $\phi_{\mu\nu}(x)$ and the auxiliary field $C_\mu(x)$ transform as

$$\delta\phi_{\mu\nu}(x) = \partial_\mu\lambda_\nu(x) + \partial_\nu\lambda_\mu(x), \quad \delta C_\mu(x) = \Box\lambda_\mu(x) \tag{6}$$

In order to write gauge invariant transversality conditions one needs one more auxiliary field $D(x)$, which transforms as

$$\delta D(x) = \partial^\mu\lambda_\mu(x) \tag{7}$$

Then the gauge invariant transversality equation is

$$\partial^\nu\phi_{\mu\nu}(x) - \partial_\mu D(x) = C_\mu(x) \tag{8}$$

Finally, one can write a gauge invariant Klein-Gordon equation for the field $D(x)$

$$\Box D(x) = \partial^\mu C_\mu(x) \tag{9}$$

and "integrate" the equations (5), (8) and (9) back into the Lagrangian

$$\mathcal{L} = -\frac{1}{2}(\partial^\mu\phi^{\nu\rho})(\partial_\mu\phi_{\nu\rho}) + 2C^\mu\partial^\nu\phi_{\mu\nu} - C^\mu C_\mu + (\partial^\mu D)(\partial_\mu D) + 2D\partial^\mu C_\mu \tag{10}$$

Again, the only propagating degrees of freedom are the physical components of the field $\phi_{\mu\nu}(x)$. Its longitudinal components and the fields $C_\mu(x)$ and $D(x)$ are either pure gauge or zero on shell. Finally, since there is no zero trace condition involved, one obtains a gauge invariant description simultaneously for a spin 2 field $g_{\mu\nu}(x)$ and for a scalar $\phi(x)$, both packed in the field $\phi_{\mu\nu}(x)$.

2.3 $s = \frac{3}{2}$ and $s = \frac{1}{2}$

The Lagrangian describing only spin $\frac{1}{2}$ field is simply

$$\mathcal{L} = -i\,\bar\Psi\gamma^\mu\partial_\mu\Psi \tag{11}$$

The next simplest example is a spin-vector field $\Psi^a_\mu(x)$, where "a" is a spinorial index (see appendix A for the present conventions). Gauge invariant Dirac and transversality equations can be written by introducing one auxiliary field $\chi^a(x)$ as

$$\gamma^\nu\partial_\nu\Psi_\mu(x) + \partial_\mu\chi(x) = 0 \tag{12}$$

$$\partial^\mu\Psi_\mu(x) + \gamma^\nu\partial_\nu\chi(x) = 0 \tag{13}$$

The equations (12)– (12) are invariant under gauge transformations

$$\delta\Psi_\mu(x) = \partial_\mu\,\lambda'(x), \quad \delta\chi(x) = -\gamma^\nu\partial_\nu\lambda'(x) \tag{14}$$

and can be obtained from the Lagrangian

$$\mathcal{L} = -i\,\bar\Psi^\nu\gamma^\mu\partial_\mu\Psi_\nu - i\,\bar\Psi^\mu\partial_\mu\chi + i\,\bar\chi\partial^\mu\Psi_\mu + i\,\bar\chi\gamma^\mu\partial_\mu\chi \tag{15}$$

Similarly to the previous case, one can gauge away the auxiliary field $\chi^a(x)$ and the non-physical polarizations of $\Psi^a_\mu(x)$. Then, one has a gauge invariant description simultaneously of spins $\frac{3}{2}$ and $\frac{1}{2}$, the latter being the gamma-trace of the field $\Psi^a_\mu(x)$.

3. BRST invariant formulation

3.1 Set Up

A systematic way to obtain the systems described above is to use the BRST approach. Let us introduce an auxiliary Fock space spanned by one set of creation and annihilation operators. The commutation relations and the vacuum are defined in the usual way

$$[\alpha_\mu, \alpha_\nu^+] = \eta_{\mu\nu}, \quad \alpha_\mu|0\rangle_\alpha = 0 \tag{16}$$

A vector $|\Phi\rangle$ in this Fock space is a series expansion in terms of the creation operators α_μ^+. In the rest of this paper we shall take the maximal number of these oscillators to be equal to two. Using more than two oscillators will result in components of higher spin. The divergence, gradient and d'Alembertian operators are realized as

$$l = p\cdot\alpha, \quad l^+ = p\cdot\alpha^+, \quad l_0 = p\cdot p, \tag{17}$$

with $p_\mu = -i\partial_\mu$ and $A\cdot B \equiv \eta_{\mu\nu}A^\mu B^\nu$. The operators (17) form a simple algebra with only non-zero commutator, being

$$[l, l^+] = l_0 \tag{18}$$

Following a standard procedure (see [8] for a review) for each operator l^+, l and l_0 one introduces ghosts c, c^+, c_0 of ghost number +1 and the corresponding momenta b^+, b, b_0 which have ghost number -1 . These operators obey the anticommutation relations

$$\{b, c^+\} = \{b^+, c\} = \{b_0, c_0\} = 1 \tag{19}$$

We define the ghost vacuum as

$$c|0\rangle_{gh.} = b|0\rangle_{gh.} = b_0|0\rangle_{gh.} = 0, \tag{20}$$

the total vacuum being $|0\rangle = |0\rangle_\alpha \otimes |0\rangle_{gh.}$. Using the corresponding nilpotent BRST charge

$$Q = c_0 l_0 + c l^+ + c^+ l - c^+ c b_0, \quad Q^2 = 0 \tag{21}$$

this set up allows one to build gauge invariant free Lagrangians in a compact form,

$$\mathcal{L} = \int dc_0 \langle\Phi|Q|\Phi\rangle \tag{22}$$

where the gauge transformation is given by

$$\delta|\Phi\rangle = Q|\Lambda\rangle \tag{23}$$

The Grassmann integration is carried out using the standard rule of

$$\int dc_0\, c_0 = 1 \tag{24}$$

The requirement that the Lagrangian (22) has zero ghost number uniquely fixes the expansion of an arbitrary vector $|\Phi\rangle$ in terms of the ghost variables. Noticing that the number operator

$$N = \alpha^+\cdot\alpha + c^+ b + b^+ c \tag{25}$$

commutes with the BRST operator (21), one has the following expansion for the case of the vector field

$$|\Phi\rangle = (\phi^{\mu}(x)\alpha^{+}_{\mu} - iC(x)c_0 b^{+})|0\rangle \tag{26}$$

Since the BRST charge has the ghost number +1 and the vector $|\Phi\rangle$ has the ghost number zero, then due to (22) the parameter of gauge transformations has the ghost number -1. Therefore

$$|\Lambda\rangle = ib^{+}\lambda(x)|0\rangle \tag{27}$$

Using the equations (22), (21) and (26) one can recover the Lagrangian (4) after performing the normal ordering and integrating over c_0. Similarly, using (21), (26) and (27) one recovers the gauge transformation rules (1) and (3).

One can repeat the same procedure for the system described in the subsection 2.2. In particular, the expansion of the vector $|\Phi\rangle$ and of the parameter of gauge transformations $|\Lambda\rangle$ have the form

$$|\Phi\rangle = (\phi^{\mu\nu}(x)\alpha^{+}_{\mu}\alpha^{+}_{\nu} - ic_0 b^{+}C^{\mu}(x)\alpha^{+}_{\mu} + c^{+}b^{+}D(x))|0\rangle \tag{28}$$

$$|\Lambda\rangle = ib^{+}\lambda^{\mu}(x)\alpha^{+}_{\mu}|0\rangle \tag{29}$$

Using this expansion one obtains the Lagrangian (10) and the gauge transformation rules (6) – (7).

The BRST formulation for fermions is slightly more complicated, because of the anticommuting nature of the Dirac operator

$$g_0 = p\cdot\gamma \tag{30}$$

As a result, one has to introduce a commuting ghost variable, which in turn leads to an infinite expansion in its powers. One can, however, partially fix the BRST gauge to truncate the expansion of the vector in the Fock space to a finite form and write the Lagrangian

$$\begin{aligned}\mathcal{L}_F &= \frac{1}{\sqrt{2}}\,{}_a\langle\Psi_1|(g_0)^a{}_b|\Psi_1\rangle^b + {}_a\langle\Psi_2|\tilde{Q}_F|\Psi_1\rangle^a + \\ &+ {}_a\langle\Psi_1|\tilde{Q}_F|\Psi_2\rangle^a + \sqrt{2}\,{}_a\langle\Psi_2|c^{+}c(g_0)^a{}_b|\Psi_2\rangle^b\end{aligned} \tag{31}$$

where

$$\tilde{Q}_F = c^{+}_1 l_1 + c_1 l^{+}_1 \tag{32}$$

One can check, that the Lagrangian (31) is invariant under the gauge transformations

$$\begin{aligned}\delta|\Psi_1\rangle^a &= \tilde{Q}_F|\Lambda'\rangle^a \\ \delta|\Psi_2\rangle^a &= -\frac{1}{\sqrt{2}}(g_0)^a{}_b\,|\Lambda'\rangle^b\end{aligned} \tag{33}$$

and is equivalent to (15) with

$$|\Psi_1\rangle^a = \Psi^{\mu,a}(x)\alpha^{+}_{\mu}|0\rangle, \quad |\Psi_2\rangle^a = b^{+}\chi^a(x)|0\rangle \tag{34}$$

The gauge transformations are obtained by taking the gauge parameter $|\Lambda'\rangle$ to be of the form

$$|\Lambda'\rangle^a = ib^{+}\lambda'^a(x)|0\rangle \tag{35}$$

As in the case of the bosonic fields, the dependence on α^{+}_{μ} and on ghost variables is uniquely fixed by the choice that the field $|\Psi_1\rangle^a$ contains a maximal spin equal to $\frac{3}{2}$ and the requirement that the Lagrangian (31) has zero ghost number.

3.2 Supersymmetry. Linearized Supergravities

Let us notice that the systems considered in the subsections 2.2 and 2.3 can not be connected by supersymmetry transformations, because the fields $\phi_{\mu\nu}(x)$ and $\Psi_\mu(x)$ have different numbers of physical degrees of freedom on-shell.

In order to establish $N = 1$ supersymmetry, one can take a formulation of the Open Superstring Field Theory [32] as a hint and proceed as follows [7]. Consider two independent sets of α-oscillators

$$[\alpha_{\mu,m}, \alpha^+_{\nu,n}] = \eta_{\mu\nu}\delta_{mn}, \quad m,n = 1,2 \tag{36}$$

The corresponding divergence and gradient operators, as well as the ghost variables $c^\pm_m$ and $b^\mp_m$ will get the index "m" as well. Therefore, we have the algebra

$$[l_m, l^+_n] = \delta_{mn}\, l_0 \tag{37}$$

$$\{b_m, c^+_n\} = \{b^+_m, c_n\} = \delta_{mn}, \quad \{b_0, c_0\} = 1 \tag{38}$$

We take the fields in the fermionic sector to contain only the first set of oscillators. In other words, we consider the system described in the Subsection 2.3 without changes and in the corresponding BRST formulation in the Section 3 we assume that all oscillators belong to the first set ($m = 1$).

On the other hand, the vectors in the Fock space in the bosonic sector contain both types of oscillators. Taking physical component of the field $|\Phi\rangle$ to contain one oscillator of each type, we get the following expansions

$$\begin{aligned} |\Phi\rangle &= (\phi_{\mu,\nu}(x)\alpha^{\mu,+}_1\alpha^{\nu,+}_2 - A(x)c^+_1 b^+_2 - B(x)c^+_2 b^+_1 \\ &+ ic_0 b^+_1 C_\mu(x)\alpha^{\mu,+}_2 + ic_0 b^+_2 E_\mu(x)\alpha^{\mu,+}_1)|0\rangle. \end{aligned} \tag{39}$$

and

$$|\Lambda\rangle = (ib^+_2\lambda_\mu(x)\alpha^{\mu,+}_1 + ib^+_1\rho_\mu(x)\alpha^{\mu,+}_2 - c_0 b^+_1 b^+_2\tau(x))|0\rangle. \tag{40}$$

Using the corresponding nilpotent BRST charge

$$Q = c_0 l_0 + \sum_{m=1,2}(c_m l^+_m + c^+_m l_m - c^+_m c_m b_0), \quad Q^2 = 0 \tag{41}$$

it is straightforward to obtain the Lagrangian

$$\begin{aligned} L_B &= -\phi^{\mu,\nu}\Box\phi_{\mu,\nu} + B\Box A + A\Box B \\ &+E^\mu\partial_\mu B + C^\nu\partial^\mu\phi_{\nu,\mu} + C^\nu\partial_\nu A + E^\mu\partial^\nu\phi_{\nu,\mu} \\ &-B\partial_\mu E^\mu - \phi^{\nu,\mu}\partial_\mu C_\nu - A\partial_\mu C^\mu - \phi^{\mu,\nu}\partial_\mu E_\nu \\ &+C^\mu C_\mu + E^\mu E_\mu\,. \end{aligned} \tag{42}$$

by plugging the expressions (39) and (41) into (22), performing the normal ordering of oscillators and integrating over c_0 according to (24). Similarly, one can find, that Lagrangian (42) is invariant

under the gauge transformations

$$\begin{aligned}
&\delta\phi_{\nu,\mu}(x) = \partial_\mu\lambda_\nu(x) + \partial_\nu\rho_\mu(x),\\
&\delta A(x) = -\partial^\mu\rho_\mu(x) - \tau(x),\\
&\delta B(x) = -\partial^\mu\lambda_\mu(x) + \tau(x),\\
&\delta C_\mu(x) = -\Box\lambda_\mu(x) + \partial_\mu\tau(x),\\
&\delta E_\mu(x) = -\Box\rho_\mu(x) - \partial_\mu\tau(x).
\end{aligned} \tag{43}$$

The Lagrangian (42) is analogous to the one given in the equation (10). However, the present Lagrangian describes a physical field $\phi_{\mu,\nu}(x)$ with no symmetry between the indices μ and ν. As a result, the Lagrangian contains more auxiliary fields. In particular, the fields $C_\mu(x)$ and $E_\mu(x)$ in (42) are analogous to the field $C_\mu(x)$ in (10), and the fields $A(x)$ and $B(x)$ are analogous to the field $D(x)$. Again, after eliminating the auxiliary fields after gauge fixing and using the equations of motion one is left with only physical polarizations in the field $\phi_{\mu,\nu}(x)$. This means, that we have a description of a spin 2 field $g_{\mu\nu}(x)$, of an antisymmetric second rank tensor $B_{\mu\nu}(x)$ and of a scalar $\phi(x)$, all contained in the field $\phi_{\mu,\nu}(x)$.

Finally, one can check that the total Lagrangian

$$\begin{aligned}
L_{tot.} \quad &= -\phi^{\mu,\nu}\Box\phi_{\mu,\nu} + B\Box A + A\Box B\\
&+E^\mu\partial_\mu B + C^\nu\partial^\mu\phi_{\nu,\mu} + C^\nu\partial_\nu A + E^\mu\partial^\nu\phi_{\nu,\mu}\\
&-B\partial_\mu E^\mu - \phi^{\nu,\mu}\partial_\mu C_\nu - A\partial_\mu C^\mu - \phi^{\mu,\nu}\partial_\mu E_\nu\\
&+C^\mu C_\mu + E^\mu E_\mu\\
&-i\,\bar\Psi^\nu\gamma^\mu\partial_\mu\Psi_\nu - i\,\bar\Psi^\mu\partial_\mu\chi + i\,\bar\chi\partial^\mu\Psi_\mu + i\,\bar\chi\gamma^\mu\partial_\mu\chi
\end{aligned} \tag{44}$$

being a sum of the Lagrangians (15) and (42), is invariant under the supersymmetry transformations [7]

$$\delta\phi_{\nu,\mu}(x) = i\,\bar\Psi_\mu(x)\gamma_\nu\,\epsilon, \quad \delta C_\nu(x) = -i\,\partial_\mu\bar\chi(x)\gamma^\mu\gamma_\nu\,\epsilon, \quad \delta B(x) = -i\,\bar\chi(x)\,\epsilon, \tag{45}$$

$$\delta\Psi_\mu(x) = -\gamma^\nu\gamma^\rho\epsilon\,\partial_\nu\phi_{\rho,\mu}(x) - \epsilon E_\mu(x), \quad \delta\chi(x) = -\gamma^\nu\epsilon\,C_\nu(x)\,. \tag{46}$$

Let us note that we have not encountered any restriction on the number of space-time dimensions until now. The requirement that the algebra of supersymmetry transformations (45)–(46) closes on shell singles out the number of space-time dimensions to be $\mathcal{D} = 3$, 4, 6, or 10.

Decomposing the fields into irreducible representations of the Poincaré group as

$$\phi_{\mu,\nu} = \left(\phi_{(\mu,\nu)} - \eta_{\mu\nu}\frac{1}{\mathcal{D}}\phi^\rho{}_\rho\right) + \phi_{[\mu,\nu]} + \eta_{\mu\nu}\frac{1}{\mathcal{D}}\phi^\rho{}_\rho \equiv h_{\mu\nu} + B_{\mu\nu} + \frac{1}{\mathcal{D}}\eta_{\mu\nu}\varphi \tag{47}$$

and

$$\psi^a_\mu = \Psi^a_\mu + \frac{1}{\mathcal{D}}(\gamma_\mu)^{ab}(\gamma^\nu)_{bc}\psi^c_\nu \equiv \Psi^a_\mu + \frac{1}{\mathcal{D}}(\gamma_\mu)^{ab}\Xi_b \tag{48}$$

one obtains the following $N = 1$ supermultiplets:

- In $\mathcal{D} = 4$: a supergravity multiplet $\left(g_{\mu\nu}(x), \psi^a_\mu(x)\right)$ and a chiral multiplet $(\phi(x), a(x), \Xi(x))$ where $\partial_\mu a(x) = \frac{1}{3!}\epsilon_{\mu\nu\rho\sigma}\partial^\nu B^{\rho\sigma}(x)$.

- In $\mathcal{D} = 6$: a supergravity multiplet $\left(g_{\mu\nu}(x), B^{+}_{\mu\nu}(x), \psi^{a}_{\mu}(x)\right)$ and a $(1,0)$ tensor multiplet $\left(\phi(x), B^{-}_{\mu\nu}(x), \Xi(x)\right)$, where we decomposed $B_{\mu\nu}(x)$ into self-dual and anti self-dual parts.

- In $\mathcal{D} = 10$: one irreducible supergravity multiplet.

Therefore, one can say, that the Lagrangian (44) gives an "uniform" description of various linearized $N = 1$ supergravities. The case of $\mathcal{D} = 3$ contains no massless propagating degrees of freedom with spin 2, so we shall not consider it here.

Writing the supersymmetry transformations (45)–(46) in terms of auxiliary oscillators,

$$\begin{aligned}
&\langle 0|\delta\,\phi_{\mu,\nu}(x)\,\alpha_1^{\mu}\,\alpha_2^{\nu} = i\,\langle 0|\overline{\Psi}_{\mu}(x)\,\alpha_1^{\mu}\,(\gamma\cdot\alpha_2)\,\epsilon \\
&\langle 0|\delta\,C_{\mu}(x)\,\alpha_2^{\mu}\,b_1 = -\langle 0|\overline{\chi}(x)\,g_0\,(\gamma\cdot\alpha_2)\,\epsilon\,b_1 \\
&\langle 0|\delta\,B(x)\,b_1\,c_2 = -i\,\langle 0|\overline{\chi}(x)\,b_1\,c_2\,\epsilon
\end{aligned} \tag{49}$$

$$\begin{aligned}
&\delta\,\Psi_{\mu}(x)\,\alpha_1^{\mu,+}|0\rangle = (-i\,g_0\,(\gamma\cdot\alpha_2)\,\epsilon\,\phi_{\mu,\nu}(x)\,\alpha_1^{\mu,+}\,\alpha_2^{\nu,+} - \epsilon\,E_{\mu}(x)\,\alpha_1^{\mu,+})|0\rangle \\
&\delta\,\chi(x)\,b_1^{+}|0\rangle = -(\gamma\cdot\alpha_2)\,\epsilon\,b_1^{+}\,C^{\mu}(x)\,\alpha^{+}_{\mu,2}|0\rangle
\end{aligned} \tag{50}$$

one can see that supersymmetry is "generated" by the second set of oscillators ($m = 2$). In other words, to obtain the $N = 1$ supermultiplets one can start with the fermionic sector, which contains only the first set ($m = 1$), then apply the transformations (49)–(50) and require the closure of SUSY algebra.[5]

The description for the $N = 1$ supersymmetric vector multiplet is similar. Taking the fields in the bosonic and the fermionic sectors as

$$|\Phi\rangle = (\phi_{\mu}(x)\alpha_2^{\mu,+} - ic_0 b_2^{+}E(x))|0\rangle, \quad |\Psi\rangle = \Psi(x)|0\rangle, \tag{51}$$

one can check that the corresponding Lagrangian

$$\mathcal{L} = (\partial_{\mu}\phi^{\nu})(\partial_{\mu}\phi_{\nu}) - 2E\partial^{\mu}\phi_{\mu} + E^2 - i\,\bar{\Psi}\gamma^{\mu}\partial_{\mu}\Psi \tag{52}$$

is invariant under the supersymmetry transformations

$$\begin{aligned}
\langle 0|\delta\,\phi_{\mu}(x)\,\alpha_2^{\mu} &= i\,\langle 0|\overline{\Psi}(x)\,(\gamma\cdot\alpha_2)\,\epsilon \\
\delta\,\Psi(x)|0\rangle &= (-i\,g_0\,(\gamma\cdot\alpha_2)\,\epsilon\,\phi_{\mu}(x)\,\alpha_2^{\mu,+} - \epsilon\,E(x))|0\rangle
\end{aligned} \tag{53}$$

After eliminating the auxiliary field $E(x)$ via its own equations of motion one obtains the standard formulation of $N = 1$ vector supermultiplet in $\mathcal{D} = 4$, 6, or 10 with an on-shell supersymmetry.

[5]The same pattern persists for the higher spin supermultiplets: the fermionic sector contains only the first set of the oscillators, while the bosonic sector contains the oscillators from the first set and at most one oscillator from the second set, see [7] for the details.

4. Cubic Interactions

4.1 Three bosons

In order to construct cubic interactions for the fields considered in the previous sections, [6] we take three copies of the auxiliary Fock space and corresponding operators. The oscillators now obey the commutation relations

$$[\alpha^{(i)}_{\mu,m}, \alpha^{(j),+}_{\nu,n}] = \delta^{ij}\delta_{mn}\eta_{\mu\nu}, \tag{54}$$

$$\{c^{(i),+}_m, b^{(j)}_n\} = \{c^{(i)}_m, b^{(j),+}_n\} = \{c^{(i)}_{0,m}, b^{(j)}_{0,n}\} = \delta^{ij}\delta_{mn}\,, \tag{55}$$

$$i,j = 1,2,3, \quad m,n = 1,2, \quad \mu,\nu = 0,...,\mathcal{D}-1$$

Then, we can consider the cubic Lagrangian

$$\begin{aligned}\mathcal{L}_{3B,\text{int}} &= \sum_{i=1}^{3}\int dc^{(i)}_0\langle\Phi^{(i)}|Q^{(i)}|\Phi^{(i)}\rangle + \\ &+ g\left(\int dc^{(1)}_0 dc^{(2)}_0 dc^{(3)}_0\langle\Phi^{(1)}|\langle\Phi^{(2)}|\langle\Phi^{(3)}||V\rangle + h.c.\right)\end{aligned} \tag{56}$$

where g is a coupling constant and

$$|V\rangle = V(p^{(i)}_\mu, \alpha^{(i),+}_{\mu,m}, c^{(i),+}_m, b^{(i),+}_m, b^{(i)}_{0,m})\; c^{(1)}_0 c^{(2)}_0 c^{(3)}_0\; |0^{(1)}\rangle\otimes|0^{(2)}\rangle\otimes|0^{(3)}\rangle \tag{57}$$

where V is a function of the creation operators that is restricted as follows. An obvious requirement is that V must be Lorentz invariant. In order the Lagrangian to have the ghost number zero, the function V must have the ghost number equal to zero, and finally, the requirement of the invariance of (56) under the non-linear gauge transformations

$$\begin{aligned}\delta|\Phi^{(i)}\rangle &= Q^{(i)}|\Lambda^{(i)}\rangle - \\ &- g\int dc^{(i+1)}_0 dc^{(i+2)}_0\left((\langle\Phi^{(i+1)}|\langle\Lambda^{(i+2)}| + \langle\Phi^{(i+2)}|\langle\Lambda^{(i+1)}|)|V\rangle\right)\end{aligned} \tag{58}$$

up to the first power in g, implies that the vertex $|V\rangle$ is BRST invariant:

$$(Q^{(1)} + Q^{(2)} + Q^{(3)})|V\rangle = 0 \tag{59}$$

The same condition guarantees that the group structure of the gauge transformations is preserved up to the first order in g. Using momentum conservation

$$p^{(1)}_\mu + p^{(2)}_\mu + p^{(3)}_\mu = 0 \tag{60}$$

and the commutation relations (54), one can show that that the following expressions are BRST invariant for any values of the spins entering the cubic vertex [30], [33]

$$\mathcal{K}^{(i)}_m = (p^{(i+1)} - p^{(i+2)})\cdot\alpha^{(i),+}_m + (b^{(i+1)}_0 - b^{(i+2)}_0)\, c^{(i),+}_m, \tag{61}$$

[6] See [44], [45] for the details of construction for higher spin fields and [46] for the analogous construction in Open Bosonic String Field Theory.

$$O^{(i,i)}_{mn} = \alpha^{(i),+}_m \cdot \alpha^{(i),+}_n + c^{(i),+}_m b^{(i),+}_n + c^{(i),+}_n b^{(i),+}_m, \tag{62}$$

$$\mathcal{Z}_{mnp} = Q^{(1,2)}_{mn}\mathcal{K}^{(3)}_p + Q^{(2,3)}_{np}\mathcal{K}^{(1)}_m + Q^{(3,1)}_{pm}\mathcal{K}^{(2)}_n, \tag{63}$$

where

$$Q^{(i,i+1)}_{mn} = \alpha^{(i),+}_m \cdot \alpha^{(i+1),+}_n + \frac{1}{2} b^{(i),+}_m c^{(i+1),+}_n + \frac{1}{2} b^{(i+1),+}_n c^{(i),+}_m. \tag{64}$$

Before turning to a description of cubic vertices between bosonic and fermionic fields, let us note that one can consider the cubic vertices between three bosonic fields obeying some off-shell constraints. In particular, for the fields considered in Subsection 3.2 we shall impose off-shell transversality conditions

$$\partial^\mu \phi_{\mu,\nu}(x) = \partial^\nu \phi_{\mu,\nu}(x) = 0 \tag{65}$$

These conditions in turn restrict the parameters of gauge transformations

$$\partial^\mu \lambda_\mu(x) = \partial^\mu \rho_\mu(x) = 0, \quad \Box\lambda_\mu(x) = \Box\rho_\mu(x) = 0, \quad \tau(x) = 0. \tag{66}$$

The constraints can be rewritten as

$$l^{(i)}_1|\phi^{(i)}\rangle = l^{(i)}_2|\phi^{(i)}\rangle = 0\,, \quad l^{(i)}_1|\Lambda^{(i)}\rangle = l^{(i)}_2|\Lambda^{(i)}\rangle = l^{(i)}_0|\Lambda^{(i)}\rangle = 0 \tag{67}$$

As a result of these constraints, all auxiliary fields and the ghost dependence disappears in $|\Phi^{(i)}\rangle$ and the Lagrangian (56) reduces to

$$\mathcal{L}_{\rm 3B,int} = \sum_{i=1,2,3} \langle\phi^{(i)}|l^{(i)}_0|\phi^{(i)}\rangle + g\left(\langle\phi^{(1)}|\ \langle\phi^{(2)}|\ \langle\phi^{(3)}||V\rangle + h.c\right) \tag{68}$$

The formulation in terms of the constrained fields considerably simplifies the consideration of supersymmetry, as we shall see below.

4.2 Two fermions and one boson

For cubic interactions between two fermionic and one bosonic fields the procedure is similar. Again, in order to simplify the consideration one imposes an off-shell transversality constraint on the physical field,

$$\partial^\mu \Psi^a_\mu(x) = 0 \quad \Leftrightarrow \quad l_1|\Psi\rangle = 0 \tag{69}$$

thus putting to zero the auxiliary field $\chi(x)$ (see subsection 2.3). This constraint, in turn, restricts the parameter of gauge transformations to

$$\gamma^\mu \partial_\mu \lambda'(x) = 0 \quad \Leftrightarrow \quad g_0|\Lambda\rangle = 0 \tag{70}$$

The corresponding cubic Lagrangian which describes interactions between two fermionic and one bosonic fields has the form

$$\begin{aligned}\mathcal{L}_{\rm 2F1B,int} \quad = \quad & \sum_{i=1}^{2} {}_a\langle\Psi^{(i)}|(g^{(i)}_0)^a{}_b|\Psi^{(a)}\rangle^b + \langle\phi^{(3)}|l^{(3)}_0|\phi^{(3)}\rangle + \\ & + g\left(\ \langle\phi^{(3)}|\ {}_a\langle\Psi^{(1)}|\ {}_b\langle\Psi^{(2)}||\mathcal{V}\rangle^{ab} + h.c\right).\end{aligned} \tag{71}$$

The requirement that the Lagrangian (71) is invariant under the non-linear gauge transformations

$$\begin{aligned}\delta|\phi^{(3)}\rangle &= \tilde{Q}_B^{(3)}|\Lambda_B^{(3)}\rangle + \\ &+ g({}_a\langle\Psi^{(1)}|\ {}_b\langle\Lambda_F{}^{(2)}||\mathcal{W}_3^{1,2}\rangle^{ab} + {}_a\langle\Psi^{(2)}|\ {}_b\langle\Lambda_F^{(1)}|\ |\mathcal{W}_3^{2,1}\rangle^{ab}),\end{aligned} \tag{72}$$

$$\begin{aligned}\delta|\Psi^{(1)}\rangle^a &= \tilde{Q}_F^{(1)}|\Lambda_F{}^{(1)}\rangle^a + \\ &+ g({}_b\langle\Psi^{(2)}|\ \langle\Lambda_B^{(3)}||\mathcal{W}_1^{2,3}\rangle^{ab} + \langle\phi^{(3)}|\ {}_b\langle\Lambda_F^{(2)}||\mathcal{W}_1^{3,2}\rangle^{ab}),\end{aligned} \tag{73}$$

$$\begin{aligned}\delta|\Psi^{(2)}\rangle^a &= \tilde{Q}_F^{(2)}|\Lambda_F{}^{(2)}\rangle^a + \\ &+ g({}^A\langle\phi^{(3)}|\ {}_b^B\langle\Lambda_F^{(1)}||\mathcal{W}_2^{3,1}\rangle^{ab} + {}_b\langle\Psi^{(1)}|\ \langle\Lambda_B^{(3)}||\mathcal{W}_2^{1,3}\rangle^{ab}),\end{aligned} \tag{74}$$

as well as the requirement of preservation of the group structure for the gauge transformations up to the first power in the coupling constant g imposes conditions on the vertices $|\mathcal{V}\rangle^{ab}$ and $|\mathcal{W}\rangle^{ab}$ [6], which are similar to (59).

The vertex $|\mathcal{V}\rangle$ is again defined by a Lorentz invariant, ghost number zero function of the creation operators. Given $|\mathcal{V}\rangle$, gauge invariance of the Lagrangian (71) holds provided one can find transformation vertices $|\mathcal{W}\rangle$ such that

$$(g_0^{(1)})^a{}_b|\mathcal{W}_1^{2,3}\rangle^{bc} - (g_0^{(2)})^c{}_b|\mathcal{W}_2^{1,3}\rangle^{ba} + \tilde{Q}_B^{(3)}|\mathcal{V}\rangle^{ac} = 0 \tag{75}$$

$$(g_0^{(1)})^a{}_b|\mathcal{W}_1^{3,2}\rangle^{bc} + l_0^{(3)}|\mathcal{W}_3^{1,2}\rangle^{ac} + \tilde{Q}_F^{(2)}|\mathcal{V}\rangle^{ac} = 0 \tag{76}$$

$$(g_0^{(2)})^a{}_b|\mathcal{W}_2^{3,1}\rangle^{bc} + l_0^{(3)}|\mathcal{W}_3^{2,1}\rangle^{ac} - \tilde{Q}_F^{(1)}|\mathcal{V}\rangle^{ca} = 0 \tag{77}$$

The preservation of the group structure leads to another set of equations. For consistency, there must be some functions $|\mathcal{X}_i\rangle$ such that

$$\tilde{Q}_F^{(2)}|\mathcal{W}_1^{2,3}\rangle^{ab} + \tilde{Q}_B^{(3)}|\mathcal{W}_1^{3,2}\rangle^{ab} - \tilde{Q}_F^{(1)}|\mathcal{X}_1\rangle^{ab} = 0 \tag{78}$$

$$\tilde{Q}_F^{(1)}|\mathcal{W}_2^{1,3}\rangle^{ab} + \tilde{Q}_B^{(3)}|\mathcal{W}_2^{3,1}\rangle^{ab} - \tilde{Q}_F^{(2)}|\mathcal{X}_2\rangle^{ab} = 0 \tag{79}$$

$$\tilde{Q}_F^{(1)}|\mathcal{W}_3^{1,2}\rangle^{ab} - \tilde{Q}_F^{(2)}|\mathcal{W}_3^{2,1}\rangle^{ba} - \tilde{Q}_B^{(3)}|\mathcal{X}_3\rangle^{ab} = 0 \tag{80}$$

Note that the equations (75)-(80) should hold only when acting on fields and transformations satisfying the constraints of equations (67), (69), and (70). The generalization to the unconstrained case was written in [6].

5. $N = 1$ Super Yang-Mills

Let us turn to particular examples.

For the case of Super Yang-Mills one introduces colour indices in the equations (56)–(58) and in (71)–(74) and takes the fields in the bosonic and the fermionic sectors as

$$|\Phi^{(i)}\rangle^A = (\phi_\mu^A(x)\alpha_2^{\mu(i),+} - iE^A(x)c_0^{(i),+}b_2^{(i),+})|0^{(i)}\rangle, \tag{81}$$

$$|\Psi^{(i)}\rangle^{a,A} = \Psi^{a,A}(x)|0^{(i)}\rangle \tag{82}$$

the only non-zero parameter of gauge transformations being

$$|\Lambda^{(i)}\rangle^A = ib_2^{(i),+}\lambda^A(x)|0^{(i)}\rangle \tag{83}$$

The full interacting cubic Lagrangian is a sum of (68) and of[7]

$$\mathcal{L}_{\rm int} = \sum_{i=1}^{3} {}^A\langle\Psi^{(i)}|g_0^{(i)}|\Psi^{(i)}\rangle_A + g\left({}^A\langle\Psi^{(1)}|{}^B\langle\Psi^{(2)}|{}^C\langle\Phi^{(3)}||\mathcal{V}\rangle_{ABC} + \text{ cyclic}\right) \tag{84}$$

The cubic interaction vertex between three bosons is given by the expression (63) with added colour indices

$$|V\rangle_{ABC} = -\frac{i}{12}f_{ABC}\mathcal{Z}_{222}\times c_0^{(1)}c_0^{(2)}c_0^{(3)}\ |0^{(1)}\rangle\otimes|0^{(2)}\rangle\otimes|0^{(3)}\rangle \tag{85}$$

The cubic interaction vertex between two fermions with spins one-half and one boson with spin one is

$$|\mathcal{V}\rangle^{ab}_{ABC} = \frac{i}{3}f_{ABC}(\alpha_2^{(3),+}\cdot\gamma)^{ab}\ |0^{(1)}\rangle\otimes|0^{(2)}\rangle\otimes|0^{(3)}\rangle + \text{ cyclic} \tag{86}$$

The only non-trivial $|\mathcal{W}\rangle$ vertices in the solution of equations (75)–(77) are given by

$$|\mathcal{W}_1^{2,3}\rangle^{ab}_{ABC} = |\mathcal{W}_2^{1,3}\rangle^{ab}_{ABC} = f_{ABC}\,C^{ab}c_2^+\,|0^{(1)}\rangle\otimes|0^{(2)}\rangle\otimes|0^{(3)}\rangle \tag{87}$$

The vertex (86) enters the Lagrangian and determines the interaction between the vector field and two fermions, whereas the vertices (87) express the nonlinear part of the gauge transformations.

After eliminating the auxiliary field $E^A(x)$ via its own equations of motion one obtains an action for $N = 1$ Super Yang-Mills up to the cubic order. Alternatively, one could have imposed an off-shell transversality constraint on the physical field $\phi^A_\mu(x)$ similarly to how it was done for the supergravity multiplets (see the discussion around the equations (65) – (68)). This would have put an auxiliary field $E^A(x)$ equal to zero and restricted the parameter of gauge transformations as $\Box\lambda(x) = 0$.

A higher spin generalization is given in [6], by multiplying the vertex (86) with an arbitrary function of the BRST invariant expressions (61)–(63) and then finding corresponding $|\mathcal{W}\rangle$ and $|\mathcal{X}\rangle$ vertices. These solutions are covariant versions of the vertices found in the light-cone formalism [27].

Because of the cubic interactions, the supersymmetry transformations for the fermion in (53) will be deformed with a nonlinear term

$$\begin{aligned}\delta'|\Psi^{(i)}\rangle^a_A &= g\,f_{ABC}\times \\ &\times\ {}_B\langle\phi^{(i+1)}|_C\langle\phi^{(i+2)}|(\gamma^{\mu\nu})^a{}_b\alpha^{(i+1),+}_{2,\mu}\alpha^{(i+2),+}_{2,\nu}\epsilon^b\ |0^{(i+1)}\rangle\otimes|0^{(i+2)}\rangle\otimes|0^{(i)}\rangle\end{aligned} \tag{88}$$

being the standard supersymmetry transformations for the $N = 1$ Yang-Mills supermultiplet. Let us notice, however, that if one imposes the off-shell transversality condition on the vector field, the supersymmetry transformations put the fields completely on shell. In this way one considers on-shell cubic vertices, which transform into each other under linear supersymmetry transformations (53).

[7]Note that in Section 4.2 we placed the boson in the third Fock space, whereas now we have three copies of the boson's Fock space in addition to three copies of the fermion's. To write the interaction in a symmetric way, we introduce a cyclic sum over the Fock space indices.

6. $N = 1$ Supergravities

In the following we show how we can write the cubic interaction vertices of $N = 1$ supergravity in $\mathcal{D} = 4, 6$ and 10 dimensions. These can be compared with the full Lagrangians of the theories which we include in Appendix B.

The cubic vertices for $N = 1$ supergravities, which describe a nonlinear deformations of the Lagrangian (44) can be divided into two types. The first type is universal in the same sense as is the Lagrangian (44), i.e., the vertices have the same form in $\mathcal{D} = 4, 6$ and 10. The second type of the vertices are specific to particular dimensions.

In this section we consider the cubic interaction vertices of supergravities whose free versions were written in Section 3.2. Namely, these are supergravity in $\mathcal{D} = 4$ coupled to a chiral multiplet, supergravity in $\mathcal{D} = 6$ coupled to a $(1, 0)$ tensor multiplet, and pure supergravity in $\mathcal{D} = 10$.

In Subsection 6.1 we write the "universal" vertices, which are present in $\mathcal{D} = 4$, 6, and 10. In Subsections 6.2 and 6.3 we write specific boson-fermion-fermion vertices for $\mathcal{D} = 10$ and 6, respectively.

6.1 Universal vertices

Let us start with the first type of cubic vertices. We impose off-shell transversality constraints on the physical fields $\phi_{\mu,\nu}(x)$ and $\Psi^a_\mu(x)$. Therefore, we have

$$|\phi^{(i)}\rangle = \phi_{\mu,\nu}(x)\,\alpha_1^{\mu(i),+}\alpha_2^{\nu(i),+}|0^{(i)}\rangle, \qquad |\Psi^{(i)}\rangle^a = \Psi^a_\mu(x)\,\alpha_1^{\mu(i),+}|0^{(i)}\rangle \tag{89}$$

We take the cubic vertices for three bosons

$$-2\,g\,\langle\phi^{(1)}|\,\langle\phi^{(2)}|\,\langle\phi^{(3)}|\mathcal{Z}_{111}\mathcal{Z}_{222}|0^{(1)}\rangle\otimes|0^{(2)}\rangle\otimes|0^{(3)}\rangle \tag{90}$$

where the expressions for $\mathcal{Z}_{mnp}$ are given in (63). For two fermions and one boson we take the vertex

$$g\,\langle\phi^{(3)}|\,{}_a\langle\Psi^{(1)}|\,{}_b\langle\Psi^{(2)}|\mathcal{Z}_{111}(\gamma\cdot\alpha_2^{(3),+})^{ab}|0^{(1)}\rangle\otimes|0^{(2)}\rangle\otimes|0^{(3)}\rangle + \text{ cyclic.} \tag{91}$$

which solves the equations (75)–(77) and (78)–(80) with

$$\begin{aligned}
(\mathcal{W}_3^{1,2})^{ab} &= c_1^{(2),+}(\gamma\cdot\alpha_2^+)^{ab}(\alpha_1^{(3),+}\cdot\alpha_1^{(1),+}) \\
(\mathcal{W}_3^{2,1})^{ab} &= c_1^{(1),+}(\gamma\cdot\alpha_2^+)^{ab}(\alpha_1^{(2),+}\cdot\alpha_1^{(3),+}) \\
(\mathcal{W}_1^{3,2})^{ab} &= -c_1^{(2),+}(p^{(1)}\cdot\gamma)^a{}_c(\gamma\cdot\alpha_2^+)^{cb}(\alpha_1^{(3),+}\cdot\alpha_1^{(1),+}) \\
(\mathcal{W}_2^{3,1})^{ab} &= -c_1^{(1),+}(p^{(2)}\cdot\gamma)^a{}_c(\gamma\cdot\alpha_2^+)^{cb}(\alpha_1^{(2),+}\cdot\alpha_1^{(3),+}) \\
(\mathcal{W}_1^{2,3})^{ab} &= -c_2^+C^{ab}\mathcal{Z}_{111} + c_1^{(3),+}(p^{(1)}\cdot\gamma)^a{}_c(\gamma\cdot\alpha_2^+)^{cb}(\alpha_1^{(1),+}\cdot\alpha_1^{(2),+}) \\
(\mathcal{W}_2^{1,3})^{ab} &= c_2^+C^{ab}\mathcal{Z}_{111} + c_1^{(3),+}(p^{(2)}\cdot\gamma)^a{}_c(\gamma\cdot\alpha_2^+)^{cb}(\alpha_1^{(1),+}\cdot\alpha_1^{(2),+})
\end{aligned} \tag{92}$$

and

$$\begin{aligned}
\mathcal{X}_1^{ab} &= c_1^{(2),+}c_2^+b_1^{(1),+}C^{ab}(p^{(1)}\cdot\alpha_1^{(3),+}) - c_1^{(2),+}c_1^{(3),+}b_1^{(1),+}(p^{(1)}\cdot\gamma)^a{}_c(\gamma\cdot\alpha_2^+)^{cb} \\
\mathcal{X}_2^{ab} &= -c_2^+c_1^{(1),+}b_1^{(2),+}C^{ab}(p^{(2)}\cdot\alpha_1^{(3),+}) + c_1^{(3),+}c_1^{(1),+}b_1^{(2),+}(p^{(2)}\cdot\gamma)^a{}_c(\gamma\cdot\alpha_2^+)^{cb} \\
\mathcal{X}_3^{ab} &= -c_1^{(1),+}c_1^{(2),+}b_1^{(3)+}(\gamma\cdot\alpha_2^+)^{ab}
\end{aligned} \tag{93}$$

The total Lagrangian, which is the sum of (68) and of

$$\mathcal{L}_{\rm int} = \sum_{i=1}^{3} \langle \Psi^{(i)} | g_0^{(i)} | \Psi^{(i)} \rangle + g \left(\langle \Psi^{(1)} | \langle \Psi^{(2)} | \langle \Phi^{(3)} || \mathcal{V} \rangle + \text{ cyclic} \right) \tag{94}$$

has the form

$$\begin{aligned} \mathcal{L} &= -\phi^{\mu,\nu} \Box \phi_{\mu,\nu} + 48 g (\partial_\rho \partial_\tau \phi_{\mu,\nu}) \phi^{\mu,\nu} \phi^{\rho,\tau} - 96 g (\partial_\rho \partial_\tau \phi_{\mu,\nu}) \phi^{\mu,\tau} \phi^{\rho,\nu} \\ &- \frac{1}{2} \bar{\Psi}^\mu \gamma^\nu \partial_\nu \Psi_\mu + 12 i g \phi^{\mu,\nu} \bar{\Psi}^\alpha \gamma_\nu \partial_\alpha \Psi_\mu - 6 i g \phi^{\mu,\nu} \bar{\Psi}^\alpha \gamma_\nu \partial_\mu \Psi_\alpha \end{aligned} \tag{95}$$

Let us note that the overall coefficients in the cubic vertices (90) and (91) are not fixed by the requirement of the gauge invariance and this particular choice is dictated by the supersymmetry transformations. However, due to the off-shell transversality conditions the supersymmetry transformations

$$\phi_{\nu,\mu}(x) = i \, \bar{\Psi}_\mu(x) \gamma_\nu \, \epsilon, \quad \delta \Psi_\mu(x) = -\gamma^\nu \gamma^\rho \epsilon \, \partial_\nu \phi_{\rho,\mu}(x) \tag{96}$$

put the fields completely on shell.

Similarly to the case of cubic vertices in Super Yang-Mills, the supergravity vertices of the first type (91) can be generalized to higher spins [6] by multiplying them by an arbitrary function of the BRST invariant expressions (61)–(63) and finding corresponding $|\mathcal{W}\rangle$ and $|\mathcal{X}\rangle$ vertices.

6.2 Vertices of $\mathcal{D} = 10$ supergravity

In order to consider the cubic vertices of the second type it is easier to decompose the fermionic fields into irreducible representations of Poincaré group according to (48). Then in ten dimensions we have the vertex

$$\langle \phi^{(3)} | \, {}_a\langle \psi^{(1)} | \, {}_b\langle \psi^{(2)} || \mathcal{V}_{\mathcal{J}} \rangle^{ab}, \tag{97}$$

where

$$\begin{aligned} \mathcal{V}^{ab}_{\mathcal{J}} &= (\gamma_{\mu\tau\sigma\lambda\nu})^{ab} \alpha_1^{\mu(1),+} \alpha_1^{\nu(2),+} p^{\tau,(3)} \alpha_1^{\sigma(3),+} \alpha_2^{\lambda(3),+} + \\ &+ (\alpha_1^{(3),+} \cdot \gamma)^{ab} [(p^{(3)} \cdot \alpha_1^{(1),+})(\alpha_1^{(2),+} \cdot \alpha_2^{(3),+}) - (p^{(3)} \cdot \alpha_1^{(2),+})(\alpha_1^{(1),+} \cdot \alpha_2^{(3),+})] - \\ &- (\alpha_2^{(3),+} \cdot \gamma)^{ab} [(p^{(3)} \cdot \alpha_1^{(1),+})(\alpha_1^{(2),+} \cdot \alpha_1^{(3),+}) - (p^{(3)} \cdot \alpha_1^{(2),+})(\alpha_1^{(1),+} \cdot \alpha_1^{(3),+})] + \\ &+ (p^{(3)} \cdot \gamma)^{ab} [(\alpha_1^{(1),+} \cdot \alpha_2^{(3),+})(\alpha_1^{(2),+} \cdot \alpha_1^{(3),+}) - (\alpha_1^{(2),+} \cdot \alpha_2^{(3),+})(\alpha_1^{(1),+} \cdot \alpha_1^{(3),+})] \end{aligned} \tag{98}$$

This vertex corresponds to the coupling of the field $B_{\mu\nu}(x)$ with two gravitini. The corresponding $\mathcal{W}$ vertices have the form

$$\begin{aligned} (\mathcal{W}_3^{2,1})^{ab} &= \frac{1}{2} c_1^{(1),+} \alpha_1^{\nu(2),+} \alpha_1^{\sigma(3),+} \alpha_2^{\lambda(3),+} (\gamma_{\sigma\lambda\gamma\nu})^{ab} \\ (\mathcal{W}_3^{1,2})^{ab} &= \frac{1}{2} c_1^{(2),+} \alpha_1^{\nu(1),+} \alpha_1^{\sigma(3),+} \alpha_2^{\lambda(3),+} (\gamma_{\sigma\lambda\gamma\nu})^{ab} \\ (\mathcal{W}_2^{3,1})^{ab} &= -c_1^{(1),+} [\frac{1}{2} (p^{(2)} \cdot \gamma)^a{}_c \alpha_1^{\nu(2),+} \alpha_1^{\sigma(3),+} \alpha_2^{\lambda(3),+} (\gamma_{\sigma\lambda\gamma\nu})^{cb} + \\ &\quad + p^{(1),\mu} \alpha_1^{\nu(2),+} \alpha_1^{\sigma(3),+} \alpha_2^{\lambda(3),+} (\eta_{\mu\nu}\gamma_\lambda\gamma_\nu - \eta_{\mu\lambda}\gamma_\sigma\gamma_\nu + \eta_{\mu\nu}\gamma_{\sigma\lambda})^{ab}] \\ (\mathcal{W}_1^{3,2})^{ab} &= c_1^{(2),+} [\frac{1}{2} (p^{(2)} \cdot \gamma)^b{}_c \alpha_1^{\nu(1),+} \alpha_1^{\sigma(3),+} \alpha_2^{\lambda(3),+} (\gamma_{\sigma\lambda\gamma\nu})^{ca} + \\ &\quad + p^{(2),\mu} \alpha_1^{\nu(1),+} \alpha_1^{\sigma(3),+} \alpha_2^{\lambda(3),+} (\eta_{\mu\nu}\gamma_\lambda\gamma_\nu - \eta_{\mu\lambda}\gamma_\sigma\gamma_\nu + \eta_{\mu\nu}\gamma_{\sigma\lambda})^{ba}] \end{aligned} \tag{99}$$

The solution for the group structure equations includes

$$\begin{aligned}
\mathcal{X}_1^{ab} &= -b_1^{(1),+}[c_1^{(3),+}c_1^{(2),+}(\gamma\cdot\alpha_2^{(3),+})^a{}_c(\gamma\cdot p^{(1)})^{cb} - c_2^{(3),+}c_1^{(2),+}(\gamma\cdot\alpha_1^{(3),+})^a{}_c(\gamma\cdot p^{(1)})^{cb}]\\
\mathcal{X}_2^{ab} &= b_1^{(2),+}[c_1^{(3),+}c_1^{(1),+}(\gamma\cdot\alpha_2^{(3),+})^a{}_c(\gamma\cdot p^{(2)})^{cb} - c_2^{(3),+}c_1^{(1),+}(\gamma\cdot\alpha_1^{(3),+})^a{}_c(\gamma\cdot p^{(2)})^{cb}]\\
\mathcal{X}_3^{ab} &= b_1^{(3),+}c_1^{(3),+}c_1^{(1),+}(\gamma\cdot\alpha_2^{(3),+})^{ab} - b_2^{(3),+}c_2^{(3),+}c_1^{(1),+}(\gamma\cdot\alpha_1^{(3),+})^{ab}
\end{aligned} \tag{100}$$

The remaining cubic vertices in $\mathcal{D} = 10$ are those which correspond to the coupling of the $B_{\mu\nu}(x)$ to one gravitino and one dilatino,

$$\langle\phi^{(3)}|\ {}_a\langle\psi^{(1)}|\ {}^b\langle\Xi^{(2)}||\mathcal{V}_{\mathcal{L}}\rangle_b^a, \tag{101}$$

There are two such couplings. The first is

$$(\mathcal{V}_{\mathcal{L}_1})^a{}_b = (\gamma_{\mu\nu\tau})^{ac}p^{\mu,(3)}\alpha_1^{\nu(3),+}\alpha_2^{\tau(3),+}(\alpha_1^{(1),+}\cdot\gamma)_{cb} \tag{102}$$

The non-trivial $\mathcal{W}$ vertices are

$$\begin{aligned}
(\mathcal{W}_3^{2,1})^{ab} &= -c_1^{(1),+}(\gamma_{\nu\tau})^{ab}\alpha_1^{\nu(3),+}\alpha_2^{\tau(3),+}\\
(\mathcal{W}_2^{3,1})^{ab} &= c_1^{(1),+}\left[(\gamma_{\nu\tau})^a{}_c\alpha_1^{\nu(3),+}\alpha_2^{\tau(3),+}(p_2\cdot\gamma)^{cb}\right]
\end{aligned} \tag{103}$$

The second vertex of this type is

$$(\mathcal{V}_{\mathcal{L}_2})^a{}_b = \delta^a{}_b(\alpha_1^{(3),+}\cdot\alpha_2^{(3),+})(\alpha_1^{(1),+}\cdot p^{(3)}) \tag{104}$$

for which

$$\begin{aligned}
(\mathcal{W}_3^{2,1})^{ab} &= -\frac{1}{2}c_1^{(1),+}C^{ab}\alpha_1^{(3),+}\cdot\alpha_2^{(3),+}\\
(\mathcal{W}_2^{3,1})^{ab} &= \frac{1}{2}c_1^{(1),+}(p_2\cdot\gamma)^{ab}\alpha_1^{(3),+}\cdot\alpha_2^{(3),+}
\end{aligned} \tag{105}$$

For the last two vertices the solutions for the group structure equations are with $|\mathcal{X}_i\rangle = 0$.

6.3 Vertices of $\mathcal{D} = 6$ supergravity

Most of the vertices of $\mathcal{D} = 6$ supergravity have already been described above. They include the universal vertices of Subsection 6.1, as well as the vertices (102) and (104), which are both present in $\mathcal{D} = 6$.

The vertex that has a different form is the coupling of the B-field to two gravitini, which is now given by

$$\mathcal{V}^{ab} = (\gamma_\tau\gamma_{\mu\nu\rho}\gamma_\lambda)^{ab}\alpha_1^{\lambda(1),+}\alpha_1^{\tau(2),+}p^{\mu(3)}\alpha_1^{\mu(3),+}\alpha_2^{\nu(3),+} \tag{106}$$

The corresponding $\mathcal{W}$ vertices have the form

$$\begin{aligned}
(\mathcal{W}_3^{2,1})^{ab} &= c_1^{(1),+}\alpha_2^{\mu(2),+}\alpha_1^{\nu(3),+}\alpha_1^{\rho(3),+}(\gamma_{\nu\rho}\gamma_\mu)^{ab}\\
(\mathcal{W}_3^{1,2})^{ab} &= c_1^{(2),+}\alpha_1^{\mu(2),+}\alpha_1^{\nu(3),+}\alpha_1^{\rho(3),+}(\gamma_{\nu\rho}\gamma_\mu)^{ab}\\
(\mathcal{W}_2^{3,1})^{ab} &= -c_1^{(1),+}\big[\alpha_1^{\nu(3),+}\alpha_2^{\rho(3),+}(\gamma_{\nu\rho})^a{}_c p^{\mu(2)}\alpha_1^{\sigma(2),+}(\gamma_\mu\gamma_\sigma)^{cb} +\\
&\qquad +2(p^{(1)}\cdot\alpha_1^{(2),+})\alpha_1^{\nu(3),+}\alpha_2^{\rho(3),+}(\gamma_{\nu\rho})^{ab}\big]\\
(\mathcal{W}_1^{3,2})^{ab} &= c_1^{(2),+}\big[\alpha_1^{\nu(3),+}\alpha_2^{\rho(3),+}(\gamma_{\nu\rho})^a{}_c p^{\mu(1)}\alpha_1^{\sigma(1),+}(\gamma_\mu\gamma_\sigma)^{cb} +\\
&\qquad +2(p^{(2)}\cdot\alpha_1^{(1),+})\alpha_1^{\nu(3),+}\alpha_2^{\rho(3),+}(\gamma_{\nu\rho})^{ab}\big]
\end{aligned} \tag{107}$$

with

$$\begin{aligned}
\mathcal{X}_1^{ab} &= -b_1^{(1),+}[c_1^{(3),+}c_1^{(2),+}(\gamma\cdot\alpha_2^{(3),+})^a{}_c(\gamma\cdot p^{(1)})^{cb} - c_2^{(3),+}c_1^{(2),+}(\gamma\cdot\alpha_1^{(3),+})^a{}_c(\gamma\cdot p^{(1)})^{cb}]\\
\mathcal{X}_2^{ab} &= b_1^{(2),+}[c_1^{(3),+}c_1^{(1),+}(\gamma\cdot\alpha_2^{(3),+})^a{}_c(\gamma\cdot p^{(2)})^{cb} - c_2^{(3),+}c_1^{(1),+}(\gamma\cdot\alpha_1^{(3),+})^a{}_c(\gamma\cdot p^{(2)})^{cb}]\\
\mathcal{X}_3^{ab} &= 2b_1^{(3),+}c_1^{(1),+}c_1^{(2),+}(\gamma\cdot\alpha_2^{(3),+})^{ab} - 2b_2^{(3),+}c_1^{(1),+}c_1^{(2),+}(\gamma\cdot\alpha_1^{(3),+})^{ab}
\end{aligned} \tag{108}$$

solving the group structure equations.

There is also the coupling of $B_{\mu\nu}$ to two dilatini,

$$\langle\phi^{(3)}|\ {}^a\langle\Xi^{(1)}|\ {}^b\langle\Xi^{(2)}||\mathcal{V}_{\mathcal{X}}\rangle_{ab}, \tag{109}$$

which is of the form

$$(\mathcal{V}_{\mathcal{X}})_{ab} = (\gamma_{\mu\nu\rho})_{ab}p^{\mu(3)}\alpha_1^{\nu(3),+}\alpha_2^{\rho(3),+} \tag{110}$$

In this case all the $\mathcal{W}$ vertices are trivial.

7. Light Cone Formalism

In this Section we describe how to construct cubic vertices of $\mathcal{D} = 4$ $N = 1$ super Yang-Mills and Supergravity in the light cone approach following [47] (see also [53]–[54] for a brief review of the light cone approach).

7.1 Set Up

To construct cubic interaction vertices in the light cone approach let us consider a field theoretic realization of the $\mathcal{D} = 4$ $N = 1$ super Poincaré algebra

$$[Q^a, Q^b] = \frac{1}{2}(\gamma^\mu)^{ab}P_\mu\,, \tag{111}$$

$$[Q^a, J^{\mu\nu}] = \frac{1}{2}(\gamma^{\mu\nu})^a{}_bQ^b\,, \tag{112}$$

$$[J^{\mu\nu}, P^\rho] = P^\mu\eta^{\nu\rho} - P^\nu\eta^{\mu\rho}\,, \tag{113}$$

$$[J^{\mu\nu}, J^{\rho\sigma}] = J^{\mu\sigma}\eta^{\nu\rho} - J^{\nu\sigma}\eta^{\mu\rho} - J^{\mu\rho}\eta^{\nu\sigma} + J^{\nu\rho}\eta^{\mu\sigma}\,. \tag{114}$$

Here $J^{\mu\nu}$ are generators of Lorentz transformations, P^μ are generators of translations, and Q^a are generators of Supersymmetry transformations. These generators are split into kinematical and dynamical generators. Kinematical generators preserve the Cauchy surface (the light cone) and are quadratic in fields both on free and interacting levels. The other generators are dynamical and they receive higher order corrections in fields. These corrections are determined from the requirement that the Poincaré algebra is preserved at the interacting level.

We choose the four dimensional coordinates as

$$x^\pm = \frac{1}{\sqrt{2}}(x^3 \pm x^0), \quad z = \frac{1}{\sqrt{2}}(x^1 + ix^2), \quad \bar{z} = \frac{1}{\sqrt{2}}(x^1 - ix^2) \tag{115}$$

The coordinate x^+ is treated as the time direction and $H = P^-$ is the Hamiltonian[8]. The generators of the super Poincaré algebra are split according to

$$\text{kinematical}: \quad P^+, P^z, P^{\bar{z}}, J^{z+}, J^{\bar{z}+}, J^{+-}, J^{z\bar{z}}, Q^+, \bar{Q}^+, \quad : 9 \tag{116}$$

$$\text{dynamical}: \quad P^-, J^{z-}, J^{\bar{z}-}, Q^-, \bar{Q}^- \quad : 5 \tag{117}$$

It is sufficient to construct the Poincare algebra at $x^+ = 0$ and then evolve all the generators according to $\dot{G} = i[H, G]$. The equations to be solved are

$$[Q^-, P^-] = [\bar{Q}^-, P^-] = 0\,, \quad [J^{z,-}, P^-] = [J^{\bar{z},-}, P^-] = 0\,. \tag{118}$$

The spectrum consists of bosonic $\phi_\lambda(x)$ and fermionic $\psi_\lambda(x)$ fields[9] with the helicities $\lambda = \pm 1, \pm 2$ for bosons, $\lambda = \pm\frac{1}{2}, \pm\frac{3}{2}$ for fermions. It is convenient to work with partial Fourier transforms

$$\phi_\lambda(x) = (2\pi)^{-\frac{3}{2}} \int e^{+i(x^-\beta + z\bar{p} + \bar{z}p)} \phi_\lambda(\vec{p})\, d^3p\,, \tag{119}$$

$$\psi_\lambda(x) = (2\pi)^{-\frac{3}{2}} \int e^{+i(x^-\beta + z\bar{p} + \bar{z}p)} \psi_\lambda(\vec{p})\, d^3p \tag{120}$$

with $d^3p = d\beta\, dp\, d\bar{p}$. The fields obey the following conjugation rules

$$\phi^\dagger_\lambda(\vec{p}) = \phi_{-\lambda}(-\vec{p}), \quad \psi^\dagger_\lambda(\vec{p}) = \psi_{-\lambda}(-\vec{p}) \tag{121}$$

Introducing a Grassman momentum p_θ, one can combine the bosonic and fermionic fields into superfields

$$\Phi_\lambda = \phi_\lambda + \frac{p_\theta}{\beta}\psi_{\lambda-\frac{1}{2}}, \quad \Phi_{-\lambda+\frac{1}{2}} = \psi_{-\lambda+\frac{1}{2}} + p_\theta \phi_{-\lambda}, \tag{122}$$

with conjugation properties

$$\overline{\Phi}_{-\lambda} = \phi^\dagger_{-\lambda} + \frac{p_\theta}{\beta}\psi^\dagger_{-\lambda+\frac{1}{2}}, \quad \overline{\Phi}_{\lambda-\frac{1}{2}} = -\psi^\dagger_{\lambda-\frac{1}{2}} + p_\theta \phi^\dagger_\lambda, \tag{123}$$

The equal time Poisson brackets between the fields

$$[\phi_\lambda(\vec{p}), \phi^\dagger_{\lambda'}(\vec{p}')] = \delta_{\lambda,\lambda'} \frac{\delta^3(\vec{p} - \vec{p}')}{2\beta}, \quad [\psi_\lambda(\vec{p}), \psi^\dagger_{\lambda'}(\vec{p}')] = \delta_{\lambda,\lambda'} \frac{\delta^3(\vec{p} - \vec{p}')}{2} \tag{124}$$

read in terms of the superfields as

$$[\Phi_\lambda(\vec{p}, p_\theta), \overline{\Phi}_{\lambda'}(\vec{p}', p'_\theta)] = (-)^{\epsilon_{\lambda+\frac{1}{2}}} \delta_{\lambda,\lambda'+\frac{1}{2}} \frac{\delta^3(\vec{p} - \vec{p}')\, \delta(p_\theta - p'_\theta)}{2\beta} \tag{125}$$

where ϵ_λ is 0 for integer λ and is 1 for half-integer λ.

The kinematical generators, which are the same both on free and interacting levels have the form

$$P^+ = \beta\,, \quad P^z = p, \quad P^{\bar{z}} = \bar{p}, \quad J^{z+} = -\beta\frac{\partial}{\partial \bar{p}}, \quad J^{\bar{z}+} = -\beta\frac{\partial}{\partial p}, \tag{126}$$

[8]Note that β is used instead of p^+ in order to simplify the form of the equations.We shall also put $x^+ = 0$ for now on.

[9]In this Section, unlike the previous ones, the index λ denotes a helicity of a field, rather than its Lorentz index.

$$J^{-+} = -\frac{\partial}{\partial\beta}\beta - \frac{1}{2}p_\theta\frac{\partial}{\partial p_\theta} + \frac{1}{2}\epsilon_\lambda, \quad J^{z\bar{z}} = p\partial_p - \bar{p}\frac{\partial}{\partial\bar{p}} + \lambda - p_\theta\frac{\partial}{\partial p_\theta}$$

$$Q^+ = (-)^{\epsilon_\lambda}\beta\frac{\partial}{\partial p_\theta}, \quad \bar{Q}^+ = (-)^{\epsilon_\lambda}p_\theta$$

The dynamical generators at the free level are

$$\begin{aligned}
H_2 &= -\frac{p\bar{p}}{\beta}\,, \\
J_2^{z-} &= -\frac{\partial}{\partial\bar{p}}\frac{p\bar{p}}{\beta} + p\frac{\partial}{\partial\beta} - \left(\lambda - \frac{1}{2}p_\theta\frac{\partial}{\partial p_\theta}\right)\frac{p}{\beta} + \left(\frac{1}{2}p_\theta\frac{\partial}{\partial p_\theta} - \frac{1}{2}\epsilon_\lambda\right)\frac{p}{\beta} \\
J_2^{\bar{z}-} &= -\frac{\partial}{\partial p}\frac{p\bar{p}}{\beta} + \bar{p}\frac{\partial}{\partial\beta} + \left(\lambda - \frac{1}{2}p_\theta\frac{\partial}{\partial p_\theta}\right)\frac{\bar{p}}{\beta} + \left(\frac{1}{2}p_\theta\frac{\partial}{\partial p_\theta} - \frac{1}{2}\epsilon_\lambda\right)\frac{p}{\beta} \\
Q_2^- &= (-)^{\epsilon_\lambda}\frac{p}{\beta}p_\theta \\
\bar{Q}_2^- &= (-)^{\epsilon_\lambda}\bar{p}\frac{\partial}{\partial p_\theta}
\end{aligned} \tag{127}$$

At the level of cubic interactions one assumes the following expansion for the dynamical generators

$$\begin{aligned}
H_3 &= H_2 + \int d\Gamma_{[3]}\,\overline{\Phi}^{\lambda_1\lambda_2\lambda_3}_{q_1q_2q_3}\,h^{q_1q_2q_3}_{\lambda_1\lambda_2\lambda_3} \\
Q_3^- &= Q_2^- + \int d\Gamma_{[3]}\,\overline{\Phi}^{\lambda_1\lambda_2\lambda_3}_{q_1q_2q_3}\,q^{q_1q_2q_3}_{\lambda_1\lambda_2\lambda_3} \\
\bar{Q}_3^- &= \bar{Q}_2^- + \int d\Gamma_{[3]}\,\overline{\Phi}^{\lambda_1\lambda_2\lambda_3}_{q_1q_2q_3}\,\bar{q}^{q_1q_2q_3}_{\lambda_1\lambda_2\lambda_3} \\
J_3^{z-} &= J_2^{z-} + \int d\Gamma_{[3]}\times \\
&\times \left[\overline{\Phi}^{\lambda_1\lambda_2\lambda_3}_{q_1q_2q_3}\,j^{q_1q_2q_3}_{\lambda_1\lambda_2\lambda_3} - \frac{1}{3}\left(\sum_{k=1}^{3}\frac{\partial\overline{\Phi}^{\lambda_1\lambda_2\lambda_3}_{q_1q_2q_3}}{\partial\bar{q}_k}\right)h^{q_1q_2q_3}_{\lambda_1\lambda_2\lambda_3} - \frac{1}{3}\left(\sum_{k=1}^{3}\frac{\partial\overline{\Phi}^{\lambda_1\lambda_2\lambda_3}_{q_1q_2q_3}}{\partial q_{\theta,k}}\right)q^{q_1q_2q_3}_{\lambda_1\lambda_2\lambda_3}\right] \\
J_3^{\bar{z}-} &= J_2^{\bar{z}-} + \int d\Gamma_{[3]}\times \\
&\times \left[\overline{\Phi}^{\lambda_1\lambda_2\lambda_3}_{q_1q_2q_3}\,\bar{j}^{q_1q_2q_3}_{\lambda_1\lambda_2\lambda_3} - \frac{1}{3}\left(\sum_{k=1}^{3}\frac{\partial\overline{\Phi}^{\lambda_1\lambda_2\lambda_3}_{q_1q_2q_3}}{\partial q_k}\right)h^{q_1q_2q_3}_{\lambda_1\lambda_2\lambda_3} + \frac{1}{3}\left(\sum_{k=1}^{3}\frac{q_{\theta,k}}{\beta_k}\right)\overline{\Phi}^{\lambda_1\lambda_2\lambda_3}_{q_1q_2q_3}\bar{q}^{q_1q_2q_3}_{\lambda_1\lambda_2\lambda_3}\right]
\end{aligned} \tag{128}$$

where $\overline{\Phi}^{\lambda_1\lambda_2\lambda_3}_{q_1q_2q_3} \equiv \overline{\Phi}^{\lambda_1}_{q_1}\overline{\Phi}^{\lambda_2}_{q_2}\overline{\Phi}^{\lambda_3}_{q_3}$ and

$$d\Gamma_{[3]} = (2\pi)^3\prod_{k=1}^{3}\frac{d^3q_k}{(2\pi)^{\frac{3}{2}}}\delta^3\left(\sum_{i=1}^{3}q_i\right)\prod_{l=1}^{3}dq_{\theta,l}\,\delta\left(\sum_{j=1}^{3}q_{\theta,j}\right) \tag{129}$$

is an integration measure.

7.2 $\mathcal{D} = 4$, $N = 1$ Super Yang Mills and Pure Supergravity

The cubic vertices which are present in the interaction part of dynamical generators (128) are determined from the requirement of preservation of the algebra (111). A solution which contains

the $N = 1$ Super Yang Mills and Supergravity vertices has the form [47]

$$h^{q_1q_2q_3}_{\lambda_1\lambda_2\lambda_3} = C^{\lambda_1\lambda_2\lambda_3}(\overline{\mathbb{P}})^{M_\lambda+1}\prod_{i=1}^{3}\beta_i^{-\lambda_i-\frac{1}{2}\epsilon_{\lambda_i}} + \overline{C}^{\lambda_1\lambda_2\lambda_3}(\mathbb{P})^{-M_\lambda-\frac{1}{2}}\mathbb{P}_\theta\prod_{i=1}^{3}\beta_i^{\lambda_i-\frac{1}{2}\epsilon_{\lambda_i}} \tag{130}$$

$$q^{q_1q_2q_3}_{\lambda_1\lambda_2\lambda_3} = -C^{\lambda_1\lambda_2\lambda_3}(\overline{\mathbb{P}})^{M_\lambda}\mathbb{P}_\theta\prod_{i=1}^{3}\beta_i^{-\lambda_i-\frac{1}{2}\epsilon_{\lambda_i}} \tag{131}$$

$$j^{q_1q_2q_3}_{\lambda_1\lambda_2\lambda_3} = 2C^{\lambda_1\lambda_2\lambda_3}(\overline{\mathbb{P}})^{M_\lambda}\chi\prod_{i=1}^{3}\beta_i^{-\lambda_i-\frac{1}{2}\epsilon_{\lambda_i}} \tag{132}$$

and

$$M_\lambda = \lambda_1+\lambda_2+\lambda_3, \quad \lambda_1 = s_1-\frac{1}{2}, \quad \lambda_2 = s_2-\frac{1}{2}, \quad \lambda_3 = -s_3 \tag{133}$$

In these equations $C^{\lambda_1\lambda_2\lambda_3}$, $\overline{C}^{\lambda_1\lambda_2\lambda_3}$ are coupling constants and

$$\mathbb{P} = \frac{1}{3}\left[(\beta_1-\beta_2)p_3+(\beta_2-\beta_3)p_1+(\beta_3-\beta_1)p_2\right], \tag{134}$$

$$\mathbb{P}_\theta = \frac{1}{3}\left[(\beta_1-\beta_2)p_{\theta,3}+(\beta_2-\beta_3)p_{\theta,1}+(\beta_3-\beta_1)p_{\theta,2}\right], \tag{135}$$

$$\chi = \beta_1(\lambda_2-\lambda_3)+\beta_2(\lambda_3-\lambda_1)+\beta_3(\lambda_1-\lambda_2). \tag{136}$$

The momenta $\beta_i, p_i, \bar{p}_i$ and $p_{\theta,i}$ obey the conservation properties as in (60). The vertices $\bar{q}^{q_1q_2q_3}_{\lambda_1\lambda_2\lambda_3}$ and $\bar{j}^{q_1q_2q_3}_{\lambda_1\lambda_2\lambda_3}$ can be obtained from (131)–(132) by relevant hermitean conjugation.

The cubic vertices for $N = 1$ Super Yang Mills can be recovered by choosing $s_1 = s_2 = s_3 = 1$ in the equations above. Similarly, pure $N = 1$ Supergravity vertices can be recovered by putting $s_1 = s_2 = s_3 = 2$.

Acknowledgments. M.T. would like to thank the organizers of the online conference "Quarks 2020" and of the online workshop "Aspects of Symmetry" for the invitation to give talks. M.T. would like to thank the Department of Mathematics, the University of Auckland for the hospitality during the final stage of the project. D.W. would like to thank the Yukawa Institute of Theoretical Physics (YITP) for its support in the period when this work was completed. The work of I.L.B. and V.A.K. was partially supported by the Ministry of Education of Russian Federation, project FEWF-2020-0003. The work of M.T. and D.W. was supported by the Quantum Gravity Unit of the Okinawa Institute of Science and Technology Graduate University (OIST).

A. Conventions

We mainly follow the notations of [55].

The Latin letters $a, b \ldots$ label spinorial indices. The Greek letters $\mu, \nu, \ldots$ label flat space-time vector indices and Greek letters with "hat" $\hat{\mu}, \hat{\nu}, \ldots$ label vector indices in curved space-time.

We choose a real representation for Majorana spinors

$$(\lambda^a)^\star = \lambda^a, \quad \bar{\lambda}_a = \lambda^b C_{ba} \tag{A.1}$$

The spinor indices can be raised and lowered by anti-symmetric charge conjugation matrices C_{ab} and C^{ab} as

$$\lambda^a = C^{ab}\lambda_b, \quad \lambda_a = \lambda^b C_{ba}, \quad C^{ab}C_{bc} = -\delta^a_c. \tag{A.2}$$

The γ–matrices satisfy the following anti-commutation relations

$$(\gamma^\mu)^a{}_c(\gamma^\nu)^c{}_b + (\gamma^\nu)^a{}_c(\gamma^\mu)^c{}_b = 2\eta^{\mu\nu}\delta^a_b. \tag{A.3}$$

In $\mathcal{D} = 4$ the matrices γ_μ and $\gamma_{\mu\nu}$ with both spinorial indices up (down) are symmetric and the matrices C, γ_5 and $\gamma_5\gamma_\mu$ are anisymmetric. In $\mathcal{D} = 10$ the matrices γ_μ and $\gamma_{\mu_1,\ldots,\mu_5}$ with both spinorial indices up (down) are symmetric, and the matrices $\gamma_{\mu_1\mu_2\mu_3}$ are antisymmetric.

For checking the on-shell closure of the supersymmetry algebra and of the supersymmetry of the vertices we have used the following gamma-matrix identities

$$(\gamma^\nu)_{ab}(\gamma_\nu)_{cd} + (\gamma^\nu)_{ac}(\gamma_\nu)_{db} + (\gamma^\nu)_{ad}(\gamma_\nu)_{bc} = 0, \tag{A.4}$$

$$\gamma^\mu\gamma^{\nu_1,\nu_2,\ldots\nu_r}\gamma_\mu = (-1)^r(D-2r)\gamma^{\nu_1,\nu_2,\ldots\nu_r}. \tag{A.5}$$

For a product of gamma matrices we have

$$\gamma^{\nu_1,\ldots,\nu_i}\gamma_{\mu_1,\ldots,\mu_j} = \sum_{k=0}^{k=min(i,j)} \frac{i!j!}{(i-k)!(j-k)!k!}\gamma^{[\nu_1,\ldots,\nu_{i-k}}{}_{[\mu_{k+1},\ldots,\mu_j}\delta^{\nu_i}_{\mu_1}\delta^{\nu_{i-1}}_{\mu_2}\ldots\delta^{\nu_{n-k+1}]}_{\mu_k]} \tag{A.6}$$

and in particular

$$\gamma_\mu\gamma_{\nu_1,\ldots\nu_k} = \gamma_{\mu,\nu_1,\ldots,\nu_k} + k\eta_{\mu[\nu_1}\gamma_{\nu_2,\ldots,\nu_k]} \tag{A.7}$$

B. $N = 1$ Supergravities in $\mathcal{D} = 4, 6, 10$

The Lagrangian for $\mathcal{D} = 10$ $N = 1$ Supergravity is [1] – [2]

$$\begin{aligned} L &= -\frac{1}{2}R - \frac{1}{2}\bar\psi_{\hat\mu}\gamma^{\hat\mu\hat\nu\hat\rho}\mathcal{D}_{\hat\nu}\psi_{\hat\rho} - \frac{3}{4}\phi^{-\frac{3}{2}}H^{\hat\mu\hat\nu\hat\rho}H_{\hat\mu\hat\nu\hat\rho} - \\ &- \frac{1}{2}\bar\Xi\gamma^{\hat\mu}\mathcal{D}_{\hat\mu}\Xi - \frac{9}{16}\frac{\partial^{\hat\mu}\phi\,\partial_{\hat\mu}\phi}{\phi^2} - \frac{3\sqrt{2}}{8}\bar\psi_{\hat\mu}\gamma^{\hat\nu}\gamma^{\hat\mu}\Xi\frac{\partial_{\hat\nu}\phi}{\phi} + \\ &+ \frac{\sqrt{2}}{16}\phi^{-\frac{3}{4}}H_{\hat\nu\hat\rho\hat\tau}(\bar\psi_{\hat\mu}\gamma^{\hat\mu\hat\nu\hat\rho\hat\tau\hat\lambda}\psi_{\hat\lambda} + 6\bar\psi^{\hat\nu}\gamma^{\hat\rho}\psi^{\hat\tau} - \sqrt{2}\bar\psi_{\hat\mu}\gamma^{\hat\nu\hat\rho\hat\tau}\gamma^{\hat\mu}\Xi) + \\ &+ (\text{fermion})^4 \end{aligned}$$

where $H_{\hat\mu\hat\nu\hat\rho} = \partial_{[\hat\mu}B_{\hat\nu\hat\rho]}$ and $\mathcal{D}_{\hat\mu}$ is a covariant derivative

$$\mathcal{D}_{\hat\mu}\Psi^a_{\hat\nu} = \partial_{\hat\mu}\Psi^a_{\hat\nu} + \frac{1}{4}\omega_{\hat\mu}{}^{\rho\sigma}(\gamma_{\rho\sigma}\Psi_{\hat\nu})^a \tag{B.1}$$

The supersymmetry transformations with a local parameter $\epsilon^a(x)$ are

$$\begin{aligned}
\delta e^{\mu}_{\hat{\mu}} &= \frac{1}{2}\bar{\epsilon}\,\gamma^{\mu}\,\psi_{\hat{\mu}} \qquad (B.2)\\
\delta\phi &= -\frac{\sqrt{2}}{3}\phi\,\bar{\epsilon}\,\Xi\\
\delta B_{\hat{\mu}\hat{\nu}} &= \frac{\sqrt{2}}{4}\phi^{\frac{3}{4}}\,(\bar{\epsilon}\,\gamma_{\hat{\mu}}\,\psi_{\hat{\nu}} - \bar{\epsilon}\,\gamma_{\hat{\nu}}\,\psi_{\hat{\mu}} - \frac{\sqrt{2}}{2}\,\bar{\epsilon}\,\gamma_{\hat{\mu}\hat{\nu}}\Xi)\\
\delta\psi_{\hat{\mu}} &= \left(\mathcal{D}_{\hat{\mu}} + \frac{\sqrt{2}}{32}\phi^{-\frac{3}{4}}(\gamma_{\hat{\mu}}{}^{\hat{\nu}\hat{\rho}\hat{\sigma}} - 9\,\delta^{\hat{\nu}}_{\hat{\mu}}\,\gamma^{\hat{\rho}\hat{\sigma}})\,H_{\hat{\nu}\hat{\rho}\hat{\sigma}}\right)\epsilon + (\text{fermion})^2\\
\delta\Xi &= -\frac{3\sqrt{2}}{8}\phi^{-1}\,\gamma^{\hat{\mu}}\,\partial_{\hat{\mu}}\phi\,\epsilon + \frac{1}{8}\phi^{-\frac{3}{4}}\,\gamma^{\hat{\mu}\hat{\nu}\hat{\rho}}\,\epsilon\,H_{\hat{\mu}\hat{\nu}\hat{\rho}} + (\text{fermion})^2 \qquad (B.3)
\end{aligned}$$

The Lagrangian for $\mathcal{D} = 4$ $N = 1$ Supergravity coupled with one chiral supermultiplet, with no superpotential and the cannonical kinetic term for the scalars is [3]

$$\begin{aligned}
L &= -\frac{1}{2}R - \frac{1}{2}\bar{\psi}_{\hat{\mu}}\gamma^{\hat{\mu}\hat{\nu}\hat{\rho}}\left(\mathcal{D}_{\hat{\nu}} + \frac{1}{8}((\partial_{\hat{\nu}}z)z^{\star} - (\partial_{\hat{\nu}}z^{\star})z)\gamma_5\right)\psi_{\hat{\rho}} -\\
&- \frac{1}{2}(\partial_{\mu}z)(\partial_{\mu}z^{\star}) - \frac{1}{2}\bar{\Xi}\left(\gamma^{\hat{\mu}}\mathcal{D}_{\hat{\mu}} - \frac{1}{8}((\partial_{\hat{\nu}}z)z^{\star} - (\partial_{\hat{\nu}}z^{\star})z)\right)\Xi\\
&+ (\text{fermion})^4
\end{aligned}$$

which is invariant under supersymmetry transformations

$$\begin{aligned}
\delta e^{\mu}_{\hat{\mu}} &= \frac{1}{2}\bar{\epsilon}\,\gamma^{\mu}\,\psi_{\hat{\mu}} \qquad (B.4)\\
\delta z &= \frac{1}{2}\,\bar{\epsilon}\,\Xi\\
\delta\psi_{\hat{\mu}} &= \left(\mathcal{D}_{\hat{\mu}} + \frac{1}{8}((\partial_{\hat{\nu}}z)z^{\star} - (\partial_{\hat{\nu}}z^{\star})z)\right)\epsilon + (\text{fermion})^2\\
\delta\Xi &= \frac{1}{2}(1+\gamma_5)\,\gamma^{\hat{\mu}}(\partial_{\hat{\mu}}z)\epsilon + (\text{fermion})^2
\end{aligned}$$

with $\gamma_5 = -\gamma_0\gamma_1\gamma_2\gamma_3$.

The Lagrangian for $\mathcal{D} = 6$ $N = 1$ Supergravity coupled to one $(1,0)$ tensor multiplet is

$$\begin{aligned}
L &= -\frac{1}{2}R - \frac{i}{2}\bar{\psi}_{\hat{\mu}}\gamma^{\hat{\mu}\hat{\nu}\hat{\rho}}\mathcal{D}_{\hat{\nu}}\psi_{\hat{\rho}} + \frac{1}{12}e^{2\sqrt{2}\phi}\,H^{\hat{\mu}\hat{\nu}\hat{\rho}}H_{\hat{\mu}\hat{\nu}\hat{\rho}} +\\
&+ \frac{i}{2}\bar{\Xi}\gamma^{\hat{\mu}}\mathcal{D}_{\hat{\mu}}\Xi + \frac{1}{2}\partial^{\hat{\mu}}\phi\,\partial_{\hat{\mu}}\phi - \frac{1}{\sqrt{2}}\bar{\Xi}\gamma^{\hat{\nu}}\gamma^{\hat{\mu}}\psi_{\hat{\nu}}\,\partial_{\hat{\nu}}\phi +\\
&- \frac{i}{24}e^{\sqrt{2}\phi}H_{\hat{\mu}\hat{\nu}\hat{\rho}}(-\bar{\psi}^{\hat{\lambda}}\gamma_{[\hat{\lambda}}\gamma^{\hat{\mu}\hat{\nu}\hat{\rho}}\gamma_{\hat{\tau}]}\psi^{\hat{\tau}} + 2i\bar{\psi}_{\hat{\lambda}}\gamma^{\hat{\mu}\hat{\nu}\hat{\rho}}\gamma^{\hat{\lambda}}\Xi - \bar{\Xi}\gamma^{\hat{\mu}\hat{\nu}\hat{\rho}}\Xi) +\\
&+ (\text{fermion})^4
\end{aligned}$$

where $H_{\hat{\mu}\hat{\nu}\hat{\rho}} = \partial_{[\hat{\mu}}B_{\hat{\nu}\hat{\rho}]}$ and $\mathcal{D}_{\hat{\mu}}$ is a covariant derivative

$$\mathcal{D}_{\hat{\mu}}\Psi^{a}_{\hat{\nu}} = \partial_{\hat{\mu}}\Psi^{a}_{\hat{\nu}} + \frac{1}{4}\omega_{\hat{\mu}}{}^{\rho\sigma}(\gamma_{\rho\sigma}\Psi_{\hat{\nu}})^{a} \qquad (B.5)$$

The supersymmetry transformations with a local parameter $\epsilon^a(x)$ are

$$\delta e^{\mu}_{\hat{\mu}} = -i\bar{\epsilon}\,\gamma^{\mu}\,\psi_{\hat{\mu}} \tag{B.6}$$

$$\delta\phi = \frac{1}{\sqrt{2}}\,\bar{\epsilon}\,\Xi$$

$$\delta B_{\hat{\mu}\hat{\nu}} = -\frac{i}{2}e^{-\sqrt{2}\phi}\,(\bar{\epsilon}\,\gamma_{\hat{\mu}}\,\psi_{\hat{\nu}} - \bar{\epsilon}\,\gamma_{\hat{\nu}}\,\psi_{\hat{\mu}} - i\,\bar{\epsilon}\,\gamma_{\hat{\mu}\hat{\nu}}\Xi)$$

$$\delta\psi_{\hat{\mu}} = \left(\mathcal{D}_{\hat{\mu}} - \frac{1}{24}e^{\sqrt{2}\phi}\gamma^{\hat{\rho}\hat{\sigma}\hat{\tau}}\gamma_{\hat{\mu}}\,H_{\hat{\rho}\hat{\sigma}\hat{\tau}}\right)\epsilon + (\text{fermion})^2$$

$$\delta\Xi = -\frac{i}{\sqrt{2}}\gamma^{\hat{\mu}}\epsilon\,\partial_{\hat{\mu}}\,\phi - \frac{i}{12}e^{\sqrt{2}\phi}\gamma^{\hat{\mu}\hat{\nu}\hat{\rho}}\epsilon H_{\hat{\mu}\hat{\nu}\hat{\rho}} + (\text{fermion})^2 \tag{B.7}$$

We linearize around a flat background

$$e^{\mu}_{\hat{\mu}}(x) = \delta^{\mu}_{\hat{\mu}} + \frac{1}{2}h^{\mu}_{\hat{\mu}}(x), \quad e^{\hat{\mu}}_{\mu}(x) = \delta^{\hat{\mu}}_{\mu} - \frac{1}{2}h^{\hat{\mu}}_{\mu}(x), \tag{B.8}$$

and consider a cubic Lagrangian with global supersymmetry.

References

[1] A. H. Chamseddine, "N=4 Supergravity Coupled to N=4 Matter," Nucl. Phys. B **185**, 403 (1981) doi:10.1016/0550-3213(81)90326-6

[2] E. Bergshoeff, M. de Roo, B. de Wit and P. van Nieuwenhuizen, "Ten-Dimensional Maxwell-Einstein Supergravity, Its Currents, and the Issue of Its Auxiliary Fields," Nucl. Phys. B **195**, 97-136 (1982) doi:10.1016/0550-3213(82)90050-5

[3] E. Cremmer, B. Julia, J. Scherk, S. Ferrara, L. Girardello and P. van Nieuwenhuizen, "Spontaneous Symmetry Breaking and Higgs Effect in Supergravity Without Cosmological Constant," Nucl. Phys. B **147**, 105 (1979) doi:10.1016/0550-3213(79)90417-6

[4] L. J. Romans, "Selfduality for Interacting Fields: Covariant Field Equations for Six-dimensional Chiral Supergravities," Nucl. Phys. B **276**, 71 (1986) doi:10.1016/0550-3213(86)90016-7

[5] H. Nishino and E. Sezgin, "The Complete $N = 2$, $d = 6$ Supergravity With Matter and Yang-Mills Couplings," Nucl. Phys. B **278**, 353-379 (1986) doi:10.1016/0550-3213(86)90218-X

[6] I. L. Buchbinder, V. A. Krykhtin, M. Tsulaia and D. Weissman, "Cubic Vertices for N=1 Supersymmetric Massless Higher Spin Fields in Various Dimensions," Nucl. Phys. B **967**, 115427 (2021) doi:10.1016/j.nuclphysb.2021.115427 [arXiv:2103.08231 [hep-th]].

[7] D. Sorokin and M. Tsulaia, "Supersymmetric Reducible Higher-Spin Multiplets in Various Dimensions," Nucl. Phys. B **929**, 216-242 (2018) [arXiv:1801.04615 [hep-th]].

[8] A. Fotopoulos and M. Tsulaia, "Gauge Invariant Lagrangians for Free and Interacting Higher Spin Fields. A Review of the BRST formulation," Int. J. Mod. Phys. A **24**, 1-60 (2009) doi:10.1142/S0217751X09043134 [arXiv:0805.1346 [hep-th]].

[9] M. A. Vasiliev, "Higher spin gauge theories: Star product and AdS space," doi:10.1142/9789812793850_0030 [arXiv:hep-th/9910096 [hep-th]].

[10] X. Bekaert, S. Cnockaert, C. Iazeolla and M. A. Vasiliev, "Nonlinear higher spin theories in various dimensions," [arXiv:hep-th/0503128 [hep-th]].

[11] D. Sorokin, "Introduction to the classical theory of higher spins," AIP Conf. Proc. **767**, no.1, 172-202 (2005) doi:10.1063/1.1923335 [arXiv:hep-th/0405069 [hep-th]].

[12] N. Bouatta, G. Compere and A. Sagnotti, "An Introduction to free higher-spin fields," [arXiv:hep-th/0409068 [hep-th]].

[13] A. Campoleoni, "Metric-like Lagrangian Formulations for Higher-Spin Fields of Mixed Symmetry," Riv. Nuovo Cim. **33**, no.3-4, 123-253 (2010) doi:10.1393/ncr/i2010-10053-2 [arXiv:0910.3155 [hep-th]].

[14] D. Francia, "String theory triplets and higher-spin curvatures," Phys. Lett. B **690**, 90-95 (2010) doi:10.1016/j.physletb.2010.05.006 [arXiv:1001.5003 [hep-th]].

[15] X. Bekaert, N. Boulanger and P. Sundell, "How higher-spin gravity surpasses the spin two barrier: no-go theorems versus yes-go examples," Rev. Mod. Phys. **84**, 987-1009 (2012) doi:10.1103/RevModPhys.84.987 [arXiv:1007.0435 [hep-th]].

[16] A. Sagnotti, "Notes on Strings and Higher Spins," J. Phys. A **46**, 214006 (2013) doi:10.1088/1751-8113/46/21/214006 [arXiv:1112.4285 [hep-th]].

[17] E. Joung, L. Lopez and M. Taronna, "Solving the Noether procedure for cubic interactions of higher spins in (A)dS," J. Phys. A **46**, 214020 (2013) doi:10.1088/1751-8113/46/21/214020 [arXiv:1207.5520 [hep-th]].

[18] V. E. Didenko and E. D. Skvortsov, "Elements of Vasiliev theory," [arXiv:1401.2975 [hep-th]].

[19] R. Rahman and M. Taronna, "From Higher Spins to Strings: A Primer," [arXiv:1512.07932 [hep-th]].

[20] C. Sleight, "Interactions in Higher-Spin Gravity: a Holographic Perspective," J. Phys. A **50**, no.38, 383001 (2017) doi:10.1088/1751-8121/aa820c [arXiv:1610.01318 [hep-th]].

[21] D. Sorokin and M. Tsulaia, "Higher Spin Fields in Hyperspace. A Review," Universe **4**, no.1, 7 (2018) doi:10.3390/universe4010007 [arXiv:1710.08244 [hep-th]].

[22] T. V. Tran, "Higher Spin Gravity: Quantization and Algebraic Structures," doi:10.5282/edoc.26543 [arXiv:2008.12582 [hep-th]].

[23] T. Erler, "Four Lectures on Analytic Solutions in Open String Field Theory," [arXiv:1912.00521 [hep-th]].

[24] A. Sagnotti and M. Tsulaia, "On higher spins and the tensionless limit of string theory," Nucl. Phys. B **682**, 83-116 (2004) doi:10.1016/j.nuclphysb.2004.01.024 [arXiv:hep-th/0311257 [hep-th]].

[25] D. Francia and A. Sagnotti, "On the geometry of higher spin gauge fields," Comment. Phys. Math. Soc. Sci. Fenn. **166**, 165-189 (2004) doi:10.1088/0264-9381/20/12/313 [arXiv:hep-th/0212185 [hep-th]].

[26] R. R. Metsaev, "Cubic interaction vertices of massive and massless higher spin fields," Nucl. Phys. B **759**, 147-201 (2006) doi:10.1016/j.nuclphysb.2006.10.002 [arXiv:hep-th/0512342 [hep-th]].

[27] R. R. Metsaev, "Cubic interaction vertices for fermionic and bosonic arbitrary spin fields," Nucl. Phys. B **859**, 13-69 (2012) doi:10.1016/j.nuclphysb.2012.01.022 [arXiv:0712.3526 [hep-th]].

[28] R. Manvelyan, K. Mkrtchyan and W. Ruhl, "General trilinear interaction for arbitrary even higher spin gauge fields," Nucl. Phys. B **836**, 204-221 (2010) doi:10.1016/j.nuclphysb.2010.04.019 [arXiv:1003.2877 [hep-th]].

[29] A. Sagnotti and M. Taronna, "String Lessons for Higher-Spin Interactions," Nucl. Phys. B **842**, 299-361 (2011) doi:10.1016/j.nuclphysb.2010.08.019 [arXiv:1006.5242 [hep-th]].

[30] A. Fotopoulos and M. Tsulaia, "On the Tensionless Limit of String theory, Off - Shell Higher Spin Interaction Vertices and BCFW Recursion Relations," JHEP **11**, 086 (2010) doi:10.1007/JHEP11(2010)086 [arXiv:1009.0727 [hep-th]].

[31] R. Manvelyan, K. Mkrtchyan and W. Ruehl, "A Generating function for the cubic interactions of higher spin fields," Phys. Lett. B **696**, 410-415 (2011) doi:10.1016/j.physletb.2010.12.049 [arXiv:1009.1054 [hep-th]].

[32] Y. Kazama, A. Neveu, H. Nicolai and P. C. West, "Space-time Supersymmetry of the Covariant Superstring," Nucl. Phys. B **278**, 833-850 (1986) doi:10.1016/0550-3213(86)90421-9

[33] R. R. Metsaev, "BRST-BV approach to cubic interaction vertices for massive and massless higher-spin fields," Phys. Lett. B **720**, 237-243 (2013) doi:10.1016/j.physletb.2013.02.009 [arXiv:1205.3131 [hep-th]].

[34] M. Henneaux, G. Lucena Gómez and R. Rahman, "Higher-Spin Fermionic Gauge Fields and Their Electromagnetic Coupling," JHEP **08**, 093 (2012) doi:10.1007/JHEP08(2012)093 [arXiv:1206.1048 [hep-th]].

[35] M. Henneaux, G. Lucena Gómez and R. Rahman, "Gravitational Interactions of Higher-Spin Fermions," JHEP **01**, 087 (2014) doi:10.1007/JHEP01(2014)087 [arXiv:1310.5152 [hep-th]].

[36] K. B. Alkalaev and M. A. Vasiliev, "N=1 supersymmetric theory of higher spin gauge fields in AdS(5) at the cubic level," Nucl. Phys. B **655**, 57-92 (2003) doi:10.1016/S0550-3213(03)00061-0 [arXiv:hep-th/0206068 [hep-th]].

[37] K. Alkalaev, "FV-type action for AdS_5 mixed-symmetry fields," JHEP **03**, 031 (2011) doi:10.1007/JHEP03(2011)031 [arXiv:1011.6109 [hep-th]].

[38] I. L. Buchbinder, S. J. Gates and K. Koutrolikos, "Higher Spin Superfield interactions with the Chiral Supermultiplet: Conserved Supercurrents and Cubic Vertices," Universe **4**, no.1, 6 (2018) doi:10.3390/universe4010006 [arXiv:1708.06262 [hep-th]].

[39] I. L. Buchbinder, S. J. Gates and K. Koutrolikos, "Conserved higher spin supercurrents for arbitrary spin massless supermultiplets and higher spin superfield cubic interactions," JHEP **08**, 055 (2018) doi:10.1007/JHEP08(2018)055 [arXiv:1805.04413 [hep-th]].

[40] J. Hutomo and S. M. Kuzenko, "Non-conformal higher spin supercurrents," Phys. Lett. B **778**, 242-246 (2018) doi:10.1016/j.physletb.2018.01.045 [arXiv:1710.10837 [hep-th]].

[41] E. I. Buchbinder, J. Hutomo and S. M. Kuzenko, "Higher spin supercurrents in anti-de Sitter space," JHEP **09**, 027 (2018) doi:10.1007/JHEP09(2018)027 [arXiv:1805.08055 [hep-th]].

[42] M. V. Khabarov and Y. M. Zinoviev, "Cubic interaction vertices for massless higher spin supermultiplets in $d = 4$," JHEP **02**, 167 (2021) doi:10.1007/JHEP02(2021)167 [arXiv:2012.00482 [hep-th]].

[43] R. Fujii, H. Kanehisa, M. Sakaguchi and H. Suzuki, "Interacting Higher-Spin Gauge Models in BRST-antifield Formalism," [arXiv:2110.04990 [hep-th]].

[44] A. K. H. Bengtsson, "BRST approach to interacting higher spin gauge fields," Class. Quant. Grav. **5**, 437 (1988) doi:10.1088/0264-9381/5/3/005

[45] I. L. Buchbinder, A. Fotopoulos, A. C. Petkou and M. Tsulaia, "Constructing the cubic interaction vertex of higher spin gauge fields," Phys. Rev. D **74**, 105018 (2006) doi:10.1103/PhysRevD.74.105018 [arXiv:hep-th/0609082 [hep-th]].

[46] A. Neveu and P. C. West, "Symmetries of the Interacting Gauge Covariant Bosonic String," Nucl. Phys. B **278**, 601-631 (1986) doi:10.1016/0550-3213(86)90054-4

[47] R. R. Metsaev, "Cubic interaction vertices for N=1 arbitrary spin massless supermultiplets in flat space," JHEP **08**, 130 (2019) doi:10.1007/JHEP08(2019)130 [arXiv:1905.11357 [hep-th]].

[48] R. R. Metsaev, "Cubic interactions for arbitrary spin $\mathcal{N}$ -extended massless supermultiplets in 4d flat space," JHEP **11**, 084 (2019) doi:10.1007/JHEP11(2019)084 [arXiv:1909.05241 [hep-th]].

[49] A. K. H. Bengtsson, I. Bengtsson and L. Brink, "Cubic Interaction Terms for Arbitrarily Extended Supermultiplets," Nucl. Phys. B **227**, 41-49 (1983) doi:10.1016/0550-3213(83)90141-4

[50] L. Brink, O. Lindgren and B. E. W. Nilsson, "N=4 Yang-Mills Theory on the Light Cone," Nucl. Phys. B **212**, 401-412 (1983) doi:10.1016/0550-3213(83)90678-8

[51] L. Brink, O. Lindgren and B. E. W. Nilsson, "The Ultraviolet Finiteness of the N=4 Yang-Mills Theory," Phys. Lett. B **123**, 323-328 (1983) doi:10.1016/0370-2693(83)91210-8

[52] S. Mandelstam, "Light Cone Superspace and the Ultraviolet Finiteness of the N=4 Model," Nucl. Phys. B **213**, 149-168 (1983) doi:10.1016/0550-3213(83)90179-7

[53] L. Brink, "Particle physics as representations of the Poincare algebra," [arXiv:hep-th/0503035 [hep-th]].

[54] D. Ponomarev and E. D. Skvortsov, "Light-Front Higher-Spin Theories in Flat Space," J. Phys. A **50**, no.9, 095401 (2017) doi:10.1088/1751-8121/aa56e7 [arXiv:1609.04655 [hep-th]].

[55] A. Van Proeyen, "Tools for supersymmetry," Ann. U. Craiova Phys. **9**, no.I, 1-48 (1999) [arXiv:hep-th/9910030 [hep-th]].