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RG flow between W_3 minimal models

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The RG flow between neighboring minimal CFT models $A_2^{(p)}$ and $A_2^{(p-1)}$ with W_3 symmetry is explored. Although in perturbed theory dilatation current is no longer conserved it is still possible to get an exact operator expression for its divergence. Exploring this anomalous conservation law one can express the leading order anomalous dimensions of local fields in terms of structure constants of OPE in the original CFT. We generalize these line of argument for the case when a higher spin W current is present. We introduce the notion of anomalous W zero mode matrix which again can be expressed in terms of OPE coefficients of the original CFT. Diagonalization of this matrix provides an additional independent confirmation that indeed $A_2^{(p)}$ flows to $A_2^{(p-1)}$.

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1. Introduction

In [1] A. Zamolodchikov investigated the RG flow from minimal model $\mathcal{M}_p$ to $\mathcal{M}_{p-1}$ initiated by the relevant field $\phi_{1,3}$. Using leading order perturbation theory valid for $p >> 1$ he calculated the mixing coefficients specifying the UV - IR map for several classes of local fields. The next to leading order perturbation was analyzed in [2]. A similar RG trajectory connecting $\mathcal{N} = 1$ super-minimal models $\mathcal{SM}_p$ to $\mathcal{SM}_{p-2}$ was found in [3] (see also [4–6]).

The mixing coefficients for several classes of fields were computed with the help of a RG domain wall [7] (see also [8, 9]) for both minimal CFT and $N = 1$ SCFT [10–13]. That the results agree with those of the perturbative analysis were shown [1, 2, 14, 15].

Our focus will be the RG flow for minimal A_2 CFTs. Namely here the minimal model $A_2^{(p)}$ flows to $A_2^{(p-1)}$ and the slightly relevant perturbation field is $\varphi\left[\begin{smallmatrix}1&2\\1&2\end{smallmatrix}\right]$ [16]. To find the matrices of anomalous dimensions we need several classes of OPE structure constants. Some of them were already derived for Toda CFTs in [17]. Still even in these cases the analytic continuation to the minimal models is quite subtle. This is why in [18] we preferred to derive these structure constants from very basics among many other previously unknown ones. Then we computed the matrices of anomalous dimensions for three RG invariant sets. The first set contains a single primary, the second one three primaries and the third includes six primaries and four level one secondary fields. We diagonalized the matrices of anomalous dimensions and found the explicit maps between UV and IR fields.

The perturbation under consideration preserves a subgroup of W transformations. The corresponding conserved current is derived explicitly. This was used to define the the W analog of anomalous dimensions.

In [18] we also constructed the RG domain wall using the coset construction [5, 19–21] of minimal models in terms of $\widehat{SU}(3)_k$ WZNW models [22, 23].

In this proceedings a systematic approach for calculation of the anomalous dimensions matrix Γ in perturbed CFT is demonstrated. This approach is also applied to define an operator $\hat{\mathfrak{W}}$ which extends the notion of W-current zero mode to the case with perturbation. We calculate this quantity in leading order for certain local fields.

2. Review on W_3 minimal CFTs and RG flow

In any CFT the energy-momentum tensor has two nonzero components: the holomorphic field $T(z)$ with conformal dimension $(2, 0)$ and its anti-holomorphic counterpart $\bar{T}(\bar{z})$ with dimensions $(0, 2)$. In W_3 CFTs one has in addition the currents $W(z)$ and $\bar{W}(\bar{z})$ with dimensions $(3, 0)$ and

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$(0,3)$ respectively. These fields satisfy the OPE rules[1]

$$T(z)T(0)=\frac{c/2}{z^4}+\frac{2T(0)}{z^2}+\frac{T'(0)}{z}+\cdots\,, \tag{1}$$

$$T(z)W(0)=\frac{3W(0)}{z^2}+\frac{W'(0)}{z}+\cdots\,, \tag{2}$$

$$W(z)W(0)=\frac{c/3}{z^6}+\frac{2T(0)}{z^4}+\frac{T'(0)}{z^3}+\frac{1}{z^2}\left(\frac{15c+66}{10(22+5c)}T''(0)+\frac{32}{22+5c}\Lambda(0)\right) \tag{3}$$
$$+\frac{1}{z}\left(\frac{1}{15}T'''(0)+\frac{16}{22+5c}\Lambda'(0)\right)+\cdots\,.$$

Here $\Lambda(z)$ is a quasi primary field defined as

$$\Lambda(z)=:TT:(z)-\frac{3}{10}T''(z) \tag{4}$$

where :: is regularization by means of subtraction of all OPE singular terms. We can expand these fields as Laurent series

$$T(z)=\sum_{n=-\infty}^{+\infty}\frac{L_n}{z^{n+2}}\,,\quad W(z)=\sum_{n=-\infty}^{+\infty}\frac{W_n}{z^{n+3}}\,,\quad \Lambda(z)=\sum_{n=-\infty}^{+\infty}\frac{\Lambda_n}{z^{n+4}}\,, \tag{5}$$

The OPE's (1), (2) and (3) are equivalent to the W_3 algebra relations

$$[L_n,L_m]=(n-m)L_{n+m}+\frac{c}{12}(n^3-n)\delta_{n+m,0}\,,\quad [L_n,W_m]=(2n-m)\,W_{n+m}\,, \tag{6}$$

$$[W_n,W_m]=\alpha(n,m)L_{n+m}+\frac{16(n-m)}{22+5c}\Lambda_{n+m}+\frac{c}{360}(n^2-4)(n^2-1)n\delta_{n+m,0}, \tag{7}$$

where

$$\alpha(n,m)=(n-m)\left(\tfrac{1}{15}(n+m+2)(n+m+3)-\tfrac{1}{6}(n+2)(m+2)\right)$$
$$\Lambda_n=d_nL_n+\textstyle\sum_{-\infty}^{+\infty}:L_mL_{n-m}: \tag{8}$$

here :: means normal ordering (i.e operators with smaller index come first) and

$$d_{2m}=\frac{1}{5}\left(1-m^2\right)\,,\quad d_{2m-1}=\frac{1}{5}\,(1+m)\,(2-m) \tag{9}$$

The central charge of Virasoro algebra in A_2-Toda CFT conventionally is parameterized as

$$c=2+12Q\cdot Q\,,\quad \text{where}\quad Q=\left(b+\tfrac{1}{b}\right)(\omega_1+\omega_2)\,, \tag{10}$$

with b being the (dimensionless) Toda coupling and in what follows it would be convenient to represent the roots, weights and Cartan elements of A_2 as 3-component vectors with the usual Kronecker scalar product, subject to the condition that sum of components is zero. Of course this is equivalent to more conventional representation of these quantities as diagonal traceless 3×3 matrices with the pairing given by trace. So

$$\omega_1=[2/3,-1/3,-1/3]^T\,,\qquad \omega_2=[1/3,1/3,-2/3]^T \tag{11}$$

[1]The corresponding expressions for the anti-chiral fields look exactly the same: one simply substitutes z by $\bar{z}$. This is why we'll mainly concentrate on the holomorphic part.

are the highest weights of two fundamental representations of $su(3)$. The weights of the fundamental representation are

$$h_1 = \begin{pmatrix} 2/3 \\ -1/3 \\ -1/3 \end{pmatrix}; \quad h_2 = \begin{pmatrix} -1/3 \\ 2/3 \\ -1/3 \end{pmatrix}; \quad h_3 = \begin{pmatrix} -1/3 \\ -1/3 \\ 2/3 \end{pmatrix} \tag{12}$$

Conformal (Virasoro) dimensions and w-weights of the exponential fields V_α with charge α are given by

$$\Delta(\alpha) = \tfrac{\alpha\cdot(2Q-\alpha)}{2}, \quad w(\alpha) = \tfrac{\sqrt{6}bi}{\sqrt{(3b^2+5)(5b^2+3)}} \prod_{i=1}^{3} ((\alpha - Q)\cdot h_i) \;. \tag{13}$$

The conjugate charge α^* is defined as ($i = 1, 2$): $\alpha \cdot \omega_i = \alpha^* \cdot \omega_{3-i}$. If represented as a three component vector, the conjugation amounts to reversing the direction and permuting the first and third components.

To pass from the Toda theory to the minimal models, one specifies the parameter b as: $b = i\sqrt{\frac{p}{p+1}}$, where integers $p = 4, 5, 6, \cdots$ enumerate the infinite series of unitary models denoted as $A_2^{(p)}$. From (10) for the central charge we get

$$c_p = 2 - \tfrac{24}{p(p+1)} \;. \tag{14}$$

Furthermore, all primary fields of the minimal models are doubly-degenerated, a condition that is satisfied only for the following finite set of allowed charges

$$\alpha \left[\begin{smallmatrix} n & m \\ n' & m' \end{smallmatrix}\right] = \frac{i\,(((n-1)(p+1) + (1-m)p)\omega_1 + ((n'-1)(p+1) + (1-m')p)\omega_2)}{\sqrt{p(p+1)}} \tag{15}$$

where, n, n', m, m' are positive integers subject to the additional constraints $n + n' \leq p - 1$, $m + m' \leq p$. We can see from the definition of conjugate that the charge of a conjugate field is given by

$$\alpha^* \left[\begin{smallmatrix} n & m \\ n' & m' \end{smallmatrix}\right] = \alpha \left[\begin{smallmatrix} n' & m' \\ n & m \end{smallmatrix}\right] \tag{16}$$

In view of (13) the conformal and w dimensions are given explicitly by

$$\Delta \left[\begin{smallmatrix} n & m \\ n' & m' \end{smallmatrix}\right] = \frac{((p+1)(n-n') - p(m-m'))^2 + 3((p+1)(n+n') - p(m+m'))^2 - 12}{12p(p+1)} \tag{17}$$

$$\begin{aligned} w \left[\begin{smallmatrix} n & m \\ n' & m' \end{smallmatrix}\right] &= \sqrt{\frac{2}{3}}((p+1)(n'-n) - p(m'-m)) \times \\ &\times \frac{((p+1)(n+2n') - p(m+2m'))((p+1)(2n+n') - p(2m+m'))}{9p(p+1)\sqrt{(2p+5)(2p-3)}} \end{aligned} \tag{18}$$

One can check that both dimensions and W_3 weights of fields $\phi \left[\begin{smallmatrix} n & m \\ n' & m' \end{smallmatrix}\right]$ and $\phi \left[\begin{smallmatrix} n' & m' \\ p-n-n' & p+1-m-m' \end{smallmatrix}\right]$ are the same so one identifies them. In what follows a special role is played by the field $\varphi(x)$ with the charge parameter $\alpha \left[\begin{smallmatrix} 1 & 2 \\ 1 & 2 \end{smallmatrix}\right] = -b(\omega_1 + \omega_2)$ and conformal dimension

$$\Delta(-b(\omega_1 + \omega_2)) = \frac{p-2}{p+1} \equiv 1 - \epsilon\,, \qquad \epsilon = 3/(p+1) \tag{19}$$

Notice also that the field φ is w-neutral: $w(-b(\omega_1 + \omega_2)) = 0$. Consider the family generated from this field by multiple application of OPE. It is important that φ is the only member of this family (besides the identity operator) which is relevant. This fact allows one to construct a consistent perturbed CFT with a single coupling:

$$A = A_{CFT} + \lambda \int \varphi(x) d^2x \tag{20}$$

At large values of p, $\epsilon \ll 1$ and the perturbing field is only slightly relevant and the conformal perturbation theory becomes applicable along a large portion of RG flow. The case of positive values of the coupling $\lambda > 0$ has been investigated in [16] and it was shown that in infrared our initial theory $A_2^{(p)}$ flows to $A_2^{(p-1)}$. Our aim here is to investigate this RG trajectory in more details. In particular we investigate the mixing matrices of some families of fields. The technique is well established by now, and the most non-trivial part of our work was computation of some structure constants of OPE, which where not available in literature until now. First let us identify the IR fixed point. The three point function of the field φ according to [16] is

$$C_{\varphi,\varphi}^{\varphi} = \frac{2(4-5\rho)^2}{(3\rho-2)(4\rho-3)} \frac{\gamma^2\left(2-\frac{3\rho}{2}\right)\sqrt{\gamma(4-4\rho)\gamma(2-2\rho)}}{\gamma\left(1-\frac{\rho}{2}\right)\gamma\left(3-\frac{5\rho}{2}\right)\gamma(3-3\rho)} \, . \tag{21}$$

In the limit when $p \gg 1$ we get (remind that $\rho = p/(p+1)$ and $\epsilon = 3/(p+1)$)

$$C_{\varphi,\varphi}^{\varphi} = \frac{3\sqrt{2}}{2} - \frac{3\sqrt{2}}{2}\,\epsilon - \frac{4\sqrt{2}}{3}\,\epsilon^2 + O(\epsilon^3) \tag{22}$$

$$\beta(g) = \epsilon\, g - \frac{\pi}{2}\, C_{\varphi,\varphi}^{\varphi}\, g^2 + O(g^3) \tag{23}$$

Thus at $g = g_* = \frac{2\sqrt{2}\,\epsilon}{3\pi} + O\left(\epsilon^2\right)$ the beta-function vanishes and we get an infrared fixed point. The shift of the central charge is given by

$$c_p - c_* = 12\pi^2 \int_0^{g_*} \beta(g) dg = \frac{16}{9}\,\epsilon^3 + O(\epsilon^4)\, . \tag{24}$$

On the other hand

$$c_p - c_{p-1} = \frac{48}{p(p^2-1)} = \frac{16}{9}\,\epsilon^3 + O(\epsilon^4)\, , \tag{25}$$

which strongly supports the identification of the IR fixed point with $A_2^{(p-1)}$ as proposed in [16]. Furthermore it is well known that the slope of the beta function at a fixed point is directly related to the dimension of the perturbing field. In our case

$$\Delta_* = 1 - \left.\frac{d\beta}{dg}\right|_{g=g_*} = 1 + \epsilon + O(\epsilon^2) \tag{26}$$

As expected the perturbing slightly relevant field φ at UV becomes slightly irrelevant at IR. Remind that the W weight of φ is zero. It is possible to show that this weight should not get perturbative corrections at the IR point so that $w^{IR} = 0$. Examining (17) and (18) we see that the only primary field of $A_2^{(p-1)}$ with required properties is the field with charge $\alpha \left[\begin{smallmatrix} 2 & 1 \\ 2 & 1 \end{smallmatrix}\right]$ indeed

$$\Delta \left[\begin{smallmatrix} 2 & 1 \\ 2 & 1 \end{smallmatrix}\right]\Big|_{p\to p-1} = \frac{p+2}{p-1} = 1 + \epsilon + O(\epsilon^2)\, , \quad w \left[\begin{smallmatrix} 2 & 1 \\ 2 & 1 \end{smallmatrix}\right]\Big|_{p\to p-1} = 0 \tag{27}$$

so that it can be identified with the perturbing field at the IR fixed point.

3. Matrix of anomalous dimensions

In perturbed theory $T = T_{zz}$ is no longer holomorphic, indeed

$$\begin{aligned}\bar{\partial}\langle T_{z,z}(z,\bar{z})\rangle_\lambda = \bar{\partial}\langle T(z)e^{-\int\lambda\varphi d^2x}\rangle_{CFT} = \\ \sum_{n=0}^{\infty}\frac{(-\lambda)^n}{n!}\int\bar{\partial}\langle T(z)\varphi(x_1)\varphi(x_2)\ldots\varphi(x_n)\rangle d^2x_1\ldots d^2x_n = \\ \sum_{n=0}^{\infty}\sum_{i=1}^{n}\frac{(-\lambda)^n}{n!}\int\bar{\partial}\langle\varphi(x_1)\ldots\left(\frac{\Delta}{(z-x_i)^2}+\frac{1}{(z-x_i)}\frac{\partial}{\partial x_i}\right)\varphi(x_i)\ldots\varphi(x_n)\rangle d^2x_1\ldots d^2x_2\end{aligned} \tag{28}$$

Where in the last step we have used the standard Ward identity. Now we can use

$$\bar{\partial}(z-x_i)^{-1} = \pi\delta^{(2)}(z-x_i), \quad \text{hence} \quad \bar{\partial}(z-x_i)^{-2} = -\pi\delta'(z-x_i) \tag{29}$$

and evaluate the integral. The result can be represented as

$$\bar{\partial}\langle T(z)e^{-\int\lambda\varphi d^2x}\rangle = -\pi\lambda(1-\Delta)\langle\varphi'(z)e^{-\int\lambda\varphi d^2x}\rangle \tag{30}$$

Thus energy momentum conservation in the perturbed theory takes the form

$$\bar{\partial}T_{z,z}(z,\bar{z}) + \pi\lambda\epsilon\partial\varphi(z,\bar{z}) = 0 \tag{31}$$

Using this we immediately get

$$\bar{\partial}(zT) + \pi\epsilon\lambda\partial(z\varphi) = \pi\lambda\epsilon\varphi \tag{32}$$

The left hand side is the divergence of the unconserved current corresponding to dilatation. By definition the charge

$$\hat{Q}_T = \int_{\partial\Lambda}\left(zT\tfrac{dz}{2\pi i} - \pi\epsilon\lambda z\varphi\tfrac{d\bar{z}}{2\pi i}\right) \tag{33}$$

where Λ is a region of $\mathbb{R}^2$. Consider

$$\begin{aligned}\pi\epsilon\lambda\int_{\mathbb{R}^2}\varphi d\bar{z}dz = \pi\epsilon\lambda\int_\Lambda\varphi d\bar{z}dz + \pi\epsilon\lambda\int_{\mathbb{R}^2\setminus\Lambda}\varphi d\bar{z}dz = \pi\epsilon\lambda\int_\Lambda\varphi d\bar{z}dz + \\ +\int_{\mathbb{R}^2\setminus\Lambda}\left(\bar{\partial}(zT)+\pi\epsilon\lambda\partial(z\varphi)\right)d\bar{z}dz = \pi\epsilon\lambda\int_\Lambda\varphi d\bar{z}dz - \int_{\partial\Lambda}(zTdz - \pi\epsilon\lambda z\varphi dz)\end{aligned} \tag{34}$$

the initial integral was independent of Λ thus the final expression is independent too. So we can consider the ultraviolet limit taking Λ very small, notice also that due to the irrelevance of perturbation the effective coupling nearly vanishes, leading to the equality

$$\begin{aligned}-\int_{\partial\Lambda}\left(zT\phi_\beta(0)\tfrac{dz}{2\pi i} - \pi\epsilon\lambda z\varphi\phi_\beta(0)\tfrac{dz}{2\pi i}\right) + \pi\epsilon\lambda\int_\Lambda\varphi\phi_\beta(0)\tfrac{d\bar{z}dz}{2\pi i} = \\ = -\int_{\partial\Lambda}zT\phi_\beta(0)\tfrac{dz}{2\pi i} = -\Delta_\beta\phi_\beta(0)\end{aligned} \tag{35}$$

From (33)

$$\hat{Q}_T(\phi_\beta(0)) = \Delta_\beta\phi_\beta(0) + \pi\epsilon\lambda\int_\Lambda\varphi\phi_\beta(0)\tfrac{d\bar{z}dz}{2\pi i} \tag{36}$$

which simply implies that

$$\hat{Q}_T(\phi_\beta) = \Delta_\beta\phi_\beta - \epsilon\lambda\partial_\lambda\phi_\beta \tag{37}$$

Let us change the basis of fields to such where the new fields satisfy $\langle\phi^\lambda_\alpha(1)\phi^\lambda_\beta(0)\rangle_\lambda = \delta_{\alpha\beta}$. It is not difficult to see that

$$\phi^\lambda_\beta = B_{\beta\gamma}\phi_\gamma, \quad \text{where} \quad B_{\beta\gamma} = \delta_{\beta\gamma} + \frac{\pi\lambda C^\beta_{\varphi,\gamma}}{\epsilon+\Delta_{\gamma\beta}} + O(\lambda^2) \tag{38}$$

more details can be found in appendix A. From here by straightforward computation we get

$$\hat{Q}_T(\phi^\lambda_\beta) = \left(B_{\beta n}\Delta_n B^{-1}_{nm} - \epsilon\lambda B_{\beta n}\partial_\lambda B^{-1}_{nm}\right)\phi^\lambda_m - \epsilon\lambda\partial_\lambda\phi^\lambda_\beta \tag{39}$$

The matrix of anomalous dimensions is defined as

$$\hat{\Gamma} = B\Delta B^{-1} - \epsilon\lambda B(\partial_\lambda B^{-1}) \tag{40}$$

inserting here B from (38) we get $\Gamma_{\alpha\beta} = \Delta_\alpha\delta_{\alpha\beta} + \pi\lambda C^\alpha_{\varphi,\beta} + O(\lambda^2)$. At this order λ can be replaced by renormalized coupling constant g since $g(\lambda) = \lambda + O(\lambda^2)$ (notice that the normalization scale was chosen to be 1). So, finally we get Zamolodchikov's formula

$$\Gamma_{\alpha\beta} = \Delta_\alpha\delta_{\alpha\beta} + \pi g C^\alpha_{\varphi,\beta} + O(g^2) \tag{41}$$

Let us derive the matrix of anomalous dimension Γ for the first two sets. The firs set contains only the field $\phi = \phi\left[\begin{smallmatrix} n & n \\ n' & n'\end{smallmatrix}\right]$. Its Virasoro dimension (17) for small ϵ is given by

$$\Delta\left[\begin{smallmatrix} n & n \\ n' & n'\end{smallmatrix}\right] = \tfrac{1}{27}\epsilon^2\left(n^2+nn'+n'^2-3\right) + \tfrac{1}{81}\epsilon^3\left(n^2+nn'+n'^2-3\right) + O\left(\epsilon^4\right) \tag{42}$$

We will use (41) to find Γ. In [18] we got

$$C^\phi_{\varphi;\phi} = \frac{\epsilon^2\left(n^2+nn'+n'^2-3\right)}{27\sqrt{2}} + O\left(\epsilon^3\right) \tag{43}$$

At fixed point for small ϵ we have $g = g^* = \dfrac{\left(2\sqrt{2}\epsilon\right)}{3\pi} + O(\epsilon^2)$ so

$$\Gamma_{\phi\phi} = \tfrac{1}{27}\epsilon^2\left(n^2+nn'+n'^2-3\right) + \tfrac{1}{27}\epsilon^3\left(n^2+nn'+n'^2-3\right) + O\left(\epsilon^4\right) \tag{44}$$

From (17) it is straightforward to see that this just coincides with the conformal dimension $\Delta\left[\begin{smallmatrix} n & n \\ n' & n'\end{smallmatrix}\right]|_{p-1}$. Thus we conclude that ϕ in the $A_2^{(p)}$ theory flows to ϕ in $A_2^{(p-1)}$.

The second set contains tree fields:

$$\phi_1 = \phi\left[\begin{smallmatrix} n & n+1 \\ n' & n'\end{smallmatrix}\right]; \quad \phi_2 = \phi\left[\begin{smallmatrix} n & n \\ n' & n'-1\end{smallmatrix}\right]; \quad \phi_3 = \phi\left[\begin{smallmatrix} n & n-1 \\ n' & n'+1\end{smallmatrix}\right] \tag{45}$$

Their conformal dimensions (17) are

$$\Delta\left[\begin{smallmatrix} n & n+1 \\ n' & n'\end{smallmatrix}\right] = \frac{1}{3} + \frac{\epsilon}{9}(-2n-n'-1) + O\left(\epsilon^2\right) \tag{46}$$

$$\Delta\left[\begin{smallmatrix} n & n \\ n' & n'-1\end{smallmatrix}\right] = \frac{1}{3} + \frac{\epsilon}{9}(n+2n'-1) + O\left(\epsilon^2\right) \tag{47}$$

$$\Delta\left[\begin{smallmatrix} n & n-1 \\ n' & n'+1\end{smallmatrix}\right] = \frac{1}{3} + \frac{\epsilon}{9}(n-n'-1) + O\left(\epsilon^2\right) \tag{48}$$

To derive the anomalous dimensions with (41) we will need the appropriate structure constants. The structure constants for small ϵ are [our work]:

$$C^1_{\varphi,1} = \frac{2n(n+n'+3)+3(n'+1)}{6\sqrt{2}n(n+n')}, \quad C^2_{\varphi,2} = \frac{n(2n'-3)+2n'^2-6n'+3}{6\sqrt{2}n'(n+n')} \tag{49}$$

$$C^3_{\varphi,3} = \frac{2nn'+3n-3n'-3}{6\sqrt{2}nn'}, \quad C^2_{\varphi,1} = \frac{1}{n+n'}\sqrt{\frac{(n+1)(n'-1)\left((n+n')^2-1\right)}{8nn'}} \tag{50}$$

$$C^3_{\varphi,1} = \frac{1}{n}\sqrt{\frac{\left(n^2-1\right),(n'+1)(n+n'+1)}{8n'(n+n')}} \quad C^3_{\varphi,2} = \frac{1}{n'}\sqrt{\frac{(n-1)\left(n'^2-1\right)(n+n'-1)}{8n(n+n')}} \tag{51}$$

Thus we have all ingredients to derive the Γ from (41). The result is

$$\Gamma_{11} = \frac{1}{3} - \frac{\epsilon}{9}\left(1 + 2n + n' - \frac{2n(n+n'+3)+3(n'+1)}{n(n+n')}\right), \tag{52}$$

$$\Gamma_{22} = \frac{1}{3} - \frac{\epsilon}{9}\left(1 - n - 2n' - \frac{n(2n'-3)+2(n'-3)n'+3}{n'(n+n')}\right), \tag{53}$$

$$\Gamma_{33} = \frac{1}{3} + \frac{\epsilon}{9}\left(1 + n - n' - \frac{3(n'+1)}{nn'} + \frac{3}{n'}\right), \tag{54}$$

$$\Gamma_{12} = \Gamma_{21} = \frac{\epsilon}{3(n+n')}\sqrt{\frac{(n+1)(n'-1)\left((n+n')^2-1\right)}{nn'}}, \tag{55}$$

$$\Gamma_{13} = \Gamma_{31} = \frac{\epsilon}{3n}\sqrt{\frac{\left(n^2-1\right)(n'+1)(n+n'+1)}{n'(n+n')}}, \tag{56}$$

$$\Gamma_{23} = \Gamma_{32} = \frac{\epsilon}{3n'}\sqrt{\frac{(n-1)\left(n'^2-1\right)(n+n'-1)}{n(n+n')}} \tag{57}$$

The eigenvalues of this matrix are:

$$\Gamma_{diag} = \left\{\tfrac{1}{9}(3+\epsilon(n-n'+1)), \tfrac{1}{9}(3-\epsilon(2n+n'-1)), \tfrac{1}{9}(3+\epsilon(n+2n'+1))\right\} \tag{58}$$

Using (17) we see that the fields ϕ_1, ϕ_2 and ϕ_3 in $A_2^{(p)}$ flow to the fields $\phi\left[\begin{smallmatrix} n+1 & n \\ n'-1 & n' \end{smallmatrix}\right]$ $\phi\left[\begin{smallmatrix} n-1 & n \\ n' & n' \end{smallmatrix}\right]$ $\phi\left[\begin{smallmatrix} n & n \\ n'+1 & n' \end{smallmatrix}\right]$ in $A_2^{(p-1)}$.

4. Matrix of anomalous w-weights

In perturbed theory W is no longer holomorphic. Indeed consider

$$\begin{gathered} \bar\partial\langle W(z)e^{-\int\lambda\varphi d^2x}\rangle = \textstyle\sum_{n=0}^{\infty}\sum_{i=1}^{n}\frac{(-\lambda)^n}{n!} \\ \int\bar\partial\langle\varphi(x_1)\ldots\left(\tfrac{1}{(z-x_i)^2}+\tfrac{2}{(\Delta+1)(z-x_i)}\tfrac{\partial}{\partial x_i}\right)W_{-1}\varphi(x_i)\ldots\varphi(x_n)\rangle d^2x_1\ldots d^2x_2 \end{gathered} \tag{59}$$

where the Ward identities together with $w_\varphi = 0$ and $W_{-2}\varphi(z_i) = \frac{2}{\Delta+1}\partial_{z_i}W_{-1}\varphi(z_i)$ were used. From (29) with simple manipulations we get

$$\bar\partial\langle W(z)e^{-\int\lambda\varphi d^2x}\rangle = -\lambda\pi\tfrac{1-\Delta}{1+\Delta}\langle(\partial W_{-1}\varphi(z))e^{-\int\lambda\varphi d^2x}\rangle \tag{60}$$

so we obtain the conservation low

$$\bar\partial W_{zzz}(z,\bar z) + \pi\lambda\tfrac{1-\Delta}{\Delta+1}\partial W_{-1}\varphi(z,\bar z) = 0 \tag{61}$$

Using this it is straightforward to see that

$$\bar\partial(z^2W) + \pi\lambda\tfrac{1-\Delta}{1+\Delta}\partial(z^2W_{-1}\varphi) = 2\pi\lambda\tfrac{1-\Delta}{1+\Delta}zW_{-1}\varphi \tag{62}$$

The left hand side is the definition of the conserved current (in perturbation theory no longer conserved) connected to W zero mode. So the corresponding (unconserved) charge is

$$\hat Q_W = \int_{\partial\Lambda}\left(z^2W(z)\tfrac{dz}{2\pi i} - \pi\lambda\tfrac{1-\Delta}{1+\Delta}z^2W_{-1}\varphi(z)\tfrac{d\bar z}{2\pi i}\right) \tag{63}$$

Similar to (34) by using (62) we can show that the following combination of integrals do not depend on the choice of Λ

$$\begin{gathered} -\int_{\partial\Lambda}\left(z^2W(z)\phi_\beta(0)\tfrac{dz}{2\pi i} - \pi\lambda\tfrac{1-\Delta}{1+\Delta}z^2W_{-1}\varphi(z)\phi_\beta(0)\tfrac{d\bar z}{2\pi i}\right) + \\ +2\pi\lambda\tfrac{1-\Delta}{1+\Delta}\int_\Lambda zW_{-1}\varphi(z)\phi_\beta(0)\tfrac{d\bar z dz}{2\pi i} = -w_\beta\phi_\beta(0) \end{gathered} \tag{64}$$

where we have used the fact that for small Λ only the first term in the first integral contributes. But again the initial right hand side of this equation is independent of Λ so this equality holds for arbitrary Λ. Finally we obtain

$$\hat Q_W(\phi_\beta) = w_\beta\phi_\beta(0) + 2\pi\lambda\tfrac{\epsilon}{2-\epsilon}\int_\Lambda zW_{-1}\varphi(z)\phi_\beta(0)\tfrac{d\bar z dz}{2\pi i} \tag{65}$$

From general arguments for the anomalous w-weight matrix up to $O(g^2)$ we may write

$$\mathfrak{W}_{\alpha\beta} = w_\beta \delta_{\alpha\beta} + \pi g b_{\alpha\beta} C^{\alpha}_{\varphi;\beta} + O(g^2) \tag{66}$$

with some unknown coefficients $b_{\alpha\beta}$. For the case of second set of fields already discussed we can say more. Interestingly the first order in ϵ terms of Δ_α and w_α are proportional to each other

$$\lim_{\epsilon\to 0} \frac{w_i - \frac{\sqrt{\frac{2}{3}}}{9}}{\Delta_i - \frac{1}{3}} = \frac{1}{\sqrt{6}}, \quad \text{for} \quad i = 1,2,3 \tag{67}$$

For self consistency at IR fixed $g = g_*$ Γ and $\mathfrak{W}$ should commute to be simultaneously diagonalizable. It is easily seen that this is possible only if $b_{\alpha\beta} = \frac{1}{\sqrt{6}}$ for $\alpha, \beta = 1,2,3$.

Thus the second set of fields (45) the matrix elements of $\mathfrak{W}$ we get (cf. (52))

$$\mathfrak{W}_{11} = \frac{\sqrt{2/3}}{9} + \frac{\epsilon}{9\sqrt{6}}\left(\frac{6}{n+n'} + \frac{3(n'+1)}{n(n+n')} - 2n - n' + 1\right), \tag{68}$$

$$\mathfrak{W}_{22} = \frac{\sqrt{2/3}}{9} + \frac{\epsilon\left(n^2 n' + n\left(3n'^2 + n' - 3\right) + n'\left(2n'^2 + n' - 6\right) + 3\right)}{9\sqrt{6}n'(n+n')}, \tag{69}$$

$$\mathfrak{W}_{33} = \frac{\sqrt{2/3}}{9} + \frac{\epsilon}{9\sqrt{6}nn'}\left(n(n'(n-n'+1)+3) - 3(n'+1)\right), \tag{70}$$

$$\mathfrak{W}_{12} = \mathfrak{W}_{21} = \frac{\epsilon}{3\sqrt{6}(n+n')}\sqrt{\frac{(n+1)(n'-1)\left((n+n')^2-1\right)}{nn'}}, \tag{71}$$

$$\mathfrak{W}_{13} = \mathfrak{W}_{31} = \frac{\epsilon}{3\sqrt{6}n}\sqrt{\frac{\left(n^2-1\right)(n'+1)(n+n'+1)}{n'(n+n')}}, \tag{72}$$

$$\mathfrak{W}_{23} = \mathfrak{W}_{32} = \frac{\epsilon}{3\sqrt{6}n'}\sqrt{\frac{(n-1)\left(n'^2-1\right)(n+n'-1)}{n(n+n')}} \tag{73}$$

The eigenvalues of it are

$$\mathfrak{W}_{diag} = \left\{\frac{2+\epsilon(n-n'+1)}{9\sqrt{6}}, \frac{2-\epsilon(2n+n'-1)}{9\sqrt{6}}, \frac{2+\epsilon(n+2n'+1)}{9\sqrt{6}}\right\} \tag{74}$$

These coincides with the w dimension of the fields $\phi\left[\begin{smallmatrix} n+1 & n \\ n'-1 & n'\end{smallmatrix}\right]$ $\phi\left[\begin{smallmatrix} n-1 & n \\ n' & n'\end{smallmatrix}\right]$ $\phi\left[\begin{smallmatrix} n & n \\ n'+1 & n'\end{smallmatrix}\right]$ in $A_2^{(p-1)}$ as expected.

A. How to pass to ϕ^λ basis

Consider a basis of primary fields satisfying the condition $\langle\phi^\lambda_\alpha(1)\phi^\lambda_\beta(0)\rangle_\lambda = \delta_{\alpha\beta}$. Let us try to find the matrix which connects our initial fields to this new bases ϕ^λ. So $\phi^\lambda_\beta = B_{\beta n}\phi_n$ where $B_{\beta n} = \delta_{\beta n} + \lambda B^{(1)}_{\beta n}$. To derive $B^{(1)}$ let us first consider

$$\begin{aligned}\langle\phi_\alpha(1)\phi_\beta(0)\rangle_\lambda &= \langle\phi_\alpha(1)\phi_\beta(0)\rangle - \lambda\int\langle\phi_\alpha(1)\varphi(x)\phi_\beta(0)\rangle d^2x = \\ &= \delta_{\alpha\beta} - \lambda\pi C^{\alpha}_{\varphi,\beta}\frac{\gamma(\Delta_{\alpha\beta}+\epsilon)\gamma(\Delta_{\beta\alpha}+\epsilon)}{\gamma(2\epsilon)} = \delta_{\alpha\beta} - \lambda\pi C^{\alpha}_{\varphi,\beta}\frac{2\epsilon}{\epsilon^2-\Delta^2_{\alpha\beta}}\end{aligned} \tag{75}$$

where in the last line we have used the fact that $\Delta_{\alpha\beta}$ is small (of order ϵ). Now let us consider

$$\langle\phi_\alpha(1)\phi_\beta(0)\rangle_\lambda = \langle\phi^\lambda_\alpha\phi^\lambda_\beta\rangle - \lambda B^{(1)}_{\alpha n}\langle\phi^\lambda_n\phi^\lambda_\beta\rangle - \lambda B^{(1)}_{\beta n}\langle\phi^\lambda_\alpha\phi^\lambda_n\rangle = \delta_{\alpha\beta} - \lambda B^{(1)}_{\alpha\beta} - \lambda B^{(1)}_{\beta\alpha} \tag{76}$$

By compering this (75) we obtain

$$B^{(1)}_{\alpha\beta} + B^{(1)}_{\beta\alpha} = \pi C^{\alpha}_{\varphi,\beta}\frac{2\epsilon}{(\epsilon+\Delta_{\alpha\beta})(\epsilon+\Delta_{\beta\alpha})} = \pi C^{\alpha}_{\varphi,\beta}\left(\frac{1}{(\epsilon+\Delta_{\beta\alpha})} + \frac{1}{(\epsilon+\Delta_{\alpha\beta})}\right) \tag{77}$$

Since $C^{\alpha}_{\varphi,\beta}$ is symmetric the natural solution $B^{(1)}$ is $B^{(1)}_{\alpha\beta} = \frac{\pi C^{\alpha}_{\varphi,\beta}}{(\epsilon+\Delta_{\beta\alpha})}$. So we recovered the result of (38).

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