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Dispersive analysis of the Primakoff reaction $\gamma K \to K\pi$

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We provide a dispersion-theoretical representation of the reaction amplitudes $\gamma K \to K\pi$ in all charge channels, based on modern pion–kaon P-wave phase shift input. Crossed-channel singularities are fixed from phenomenology as far as possible. We demonstrate how the subtraction constants can be matched to a low-energy theorem and radiative couplings of the $K^*(892)$ resonances, thereby providing a model-independent framework for future analyses of high-precision kaon Primakoff data.†

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1. Introduction

The Wess–Zumino–Witten anomaly [2, 3] provides QCD predictions for processes of odd intrinsic parity at low energies. Low-energy theorems for three-pseudoscalar–photon processes [4] such as $\gamma\pi \to \pi\pi$ or $\eta \to \pi\pi\gamma$ provide parameter-free predictions in terms of e and F_π, e.g.

$$F_{3\pi} = \frac{e}{4\pi^2 F_\pi^3} \tag{1}$$

for $\gamma\pi \to \pi\pi$ [5–7]. This reaction can be investigated experimentally in a Primakoff reaction [8], with a charged-pion beam scattered off the Coulomb field of a heavy nucleus. Currently, the OKA experiment [9, 10] analyzes data on charged-kaon Primakoff reactions. In the future, high-precision data is expected from the upgrade to a kaon beam at COMPASS++/AMBER [11–13]. It was realized in Refs. [14, 15] for the anomalous photon–pion reaction that both aspects, low-energy theorem and chiral anomaly on the one hand, and radiative resonance couplings on the other, are intimately related to each other. Unitarity implies a close link between the amplitude $\gamma\pi \to \pi\pi$, at zero energy and in the chiral limit, and its behavior in the resonance peak region of the $\rho(770)$. This has the practical consequence that the prediction due to the anomaly can be tested with much better statistics [14]. In addition, using a dispersion-theoretical representation, the radiative coupling $\rho \to \pi\gamma$ can be extracted in a model-independent way, from the residue of the pole on the second Riemann sheet [15].

In these proceedings, we describe a dispersion-theoretical representation for $\gamma K \to K\pi$ (in all possible charge configurations) that fulfills a similar feat [1]. The chiral anomaly predicts the amplitudes for π^0 production to have the exact same value in the chiral limit and at zero energy as the analogous photon–pion reaction, see Eq. (1); based on the fundamental principles of analyticity and unitarity, the anomaly can also here be related to the radiative couplings of $K^*(892) \to K\gamma$ [16–18]. In this manner, our analysis provides a consistent framework to analyze future data, from OKA or COMPASS++/AMBER, in a theoretically sound setting. Here we derive Khuri–Treiman-type equations [19] for all possible charge configurations and solve these self-consistently for the (crossing-symmetric) s- and u-channels, while t-channel singularities are fixed from data and symmetry arguments as much as possible. To guarantee an accurate description of the universal kaon–pion final-state interactions, we employ phase shift input from corresponding Roy–Steiner analyses [20–22].

2. Decomposition of the amplitude

We decompose the amplitude for the reaction $\gamma(q)K(p_1) \to K(p_2)\pi(p_0)$ in terms of a kinematic prefactor of odd intrinsic parity and the scalar amplitude $\mathcal{F}(s,t,u)$ according to

$$\mathcal{M} = i\varepsilon_{\mu\nu\alpha\beta}\epsilon^\mu p_1^\nu p_2^\alpha p_0^\beta \mathcal{F}(s,t,u)\,, \tag{2}$$

where the total cross section is given by

$$\sigma(s) = \frac{(s-M_K^2)\lambda^{3/2}(s,M_\pi^2,M_K^2)}{1024\pi s^2}\int_{-1}^{1} \mathrm{d}z_s(1-z_s^2)|\mathcal{F}(s,t,u)|^2\,. \tag{3}$$

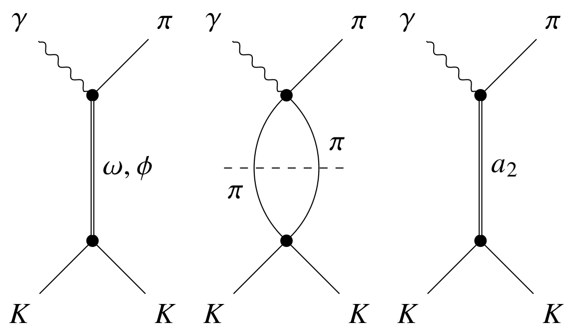

Figure 1: t-channel contributions to $\gamma K \to K\pi$; see main text for the individual terms.

In terms of isospin, the reaction $\gamma K \to K\pi$ is equivalent to pion photoproduction off a nucleon $\gamma N \to N\pi$ studied, e.g., in Refs. [23, 24]. For dispersion-theoretical analyses of scattering or (three-body) decay amplitudes, it is highly advantageous to decompose these in terms of single-variable amplitudes (SVAs). Decompositions of such a kind are commonly referred to as reconstruction theorems [25–27]. With one exception, we neglect discontinuities of partial waves with $\ell \geq 2$, resulting in the reconstruction theorems for $\mathcal{F}$ in the four different charge channels [1].

Despite the (potentially) very high accuracy of the representation at low energies, the range of applicability towards higher energies is clearly limited. One of the main limiting factors for a description of cross-section data in the direct or s-channel is the appearance of a resonant $K\pi$ D-wave around the $K_2^*(1430)$. Given a width of $\Gamma_{K_2^*(1430)} = 100(2)$ MeV, this suggests our representation to be applicable up to well below $\sqrt{s} = 1.3$ GeV [28]. Furthermore, for the dominant $K\pi$ P-wave, we will stick to the implementation of elastic unitarity with $K\pi$ intermediate states only, which will break down around the $K^*(1410)$ resonance ($\Gamma_{K^*(1410)} = 232(21)$ MeV) with its large inelastic coupling mainly to $K\pi\pi$. For this reason, the dispersive amplitude representation we aim for is supposed to be valid to good approximation up to $\sqrt{s} = 1.2$ GeV.

3. Singularities in the t-channel

The usual approach to analyzing Khuri–Treiman-type systems is to solve the unitarity relations for the single-variable amplitudes in all three channels fully self-consistently. This is an obvious strategy for perfectly crossing-symmetric systems such as $\gamma\pi \to \pi\pi$ [14, 29], related three-pion decays [30], or even pion–pion scattering [31], but has also been followed for less symmetric processes such as $\eta \to \pi^+\pi^-\pi^0$ [32], $\eta' \to \eta\pi\pi$ [33], or $D \to \bar{K}\pi\pi$ [34, 35]. We do not pursue this approach here as far as the t-channel is concerned, for the following reasons: the t-channel singularities in $\gamma K \to K\pi$ are either dominantly inelastic (ρ, a_2), or consist of very narrow poles (ϕ), or both (ω); see Fig. 1. For this reason, in our analysis we approximate these by fixed t-channel contributions, similar in spirit e.g. to various analyses of $\gamma\gamma \to \pi\pi$ [36, 37] or the description of left-hand cuts in $\eta^{(\prime)} \to \pi^+\pi^-\gamma$ [38].

The $\omega \to K\bar{K}$ coupling required for the dominant ω-exchange contribution cannot be determined from a direct decay. One option is to simply fix it using $SU(3)$ symmetry, with plausible guesses at best of the uncertainty attached. A somewhat more data-driven access to this coupling can be obtained by relying on a vector-meson-dominance (VMD) model fitted to time-like kaon form factor data from $e^+e^- \to K^+K^-$, K_SK_L, and $\tau^- \to K^-K_S\nu_\tau$, see Model II in Ref. [39]. Together with the ω–photon coupling from $\omega \to e^+e^-$ [15] we obtain $g_\omega = 7.1(0.8)$. The error is dominated by the fit value from Ref. [39]; within uncertainties, g_ω is indeed compatible with $SU(3)$ symmetry. Furthermore, a combined analysis of the space- and timelike kaon form factors can be used to determine $g_{\omega_n} = 5.7(1.9)$ and $g_{\omega_c} = 8.1(1.9)$ in the neutral and charged case, hence compatible, but with much larger uncertainties [40].

Adding the ω and ϕ tree-level contributions, we obtain the SVA

$$\mathcal{G}^{(+)}(t) = e\left[\frac{g_\omega d_\omega}{M_\omega^2 - t} - \frac{\sqrt{2}g_\phi^c \bar{d}_\phi}{M_\phi^2 - t}\right]. \tag{4}$$

We use zero-width propagators for the vector mesons as t is negative in $\gamma K \to K\pi$.

The isospin $I = 1$ $\gamma\pi \to K\bar{K}$ P-wave $\mathcal{G}^{(0)}(t)$ is dominated by the rather broad $\rho(770)$. Since the ρ is a $\pi\pi$ P-wave resonance, we can employ a more sophisticated approach than the VMD approximation and compute this SVA dispersively, taking into account intermediate $\pi\pi$ states; cf. Fig. 1. The corresponding unitarity relation reads

$$\operatorname{disc}\mathcal{G}^{(0)}(t) = -i\frac{t}{2\sqrt{2}}\sigma_\pi^3(t)\left[g_1^1(t)\right]^* h_1^1(t)\,, \tag{5}$$

with the isospin $I = 1$ P-waves $h_1^1(t)$ for $\gamma\pi \to \pi\pi$ [14] and $g_1^1(t)$ for $\pi\pi \to K\bar{K}$ [41].

For the resonant t-channel D-wave contribution we follow the approach of Ref. [38]. We compute the SVA $\mathcal{H}^{(-)}$ via the tree-level diagram with an intermediate $a_2(1320)$ tensor meson, see Fig. 1. For the interaction vertices we use the formalism presented in Ref. [42].

4. Dispersive representations and Khuri–Treiman solutions

We now discuss the main part of the dispersive representation of the $\gamma K \to K\pi$ amplitudes, the reconstruction of the s- and u-channel partial waves or SVAs. This consistently incorporates $K\pi$ P-wave rescattering in the elastic approximation. From the reconstruction theorems [1], we can obtain the relevant partial waves, i.e., the P-waves of different isospins with the result

$$f_1^{(i)}(s) = \mathcal{F}^{(i)}(s) + \widehat{\mathcal{F}}^{(i)}(s)\,, \quad i = 0,\ 1/2,\ 3/2\,. \tag{6}$$

In the approximation of elastic unitarity, a right-hand cut in the amplitude is induced by intermediate $K\pi$ states. Here, the partial waves for $i = 0,\ 1/2$ are both associated with $I = 1/2$, while $i = 3/2$ requires $I = 3/2$. The unitarity relation implies Watson's final-state theorem, which states that the phase of $f_1^{(i)}(s)$ coincides with $\delta_1^I(s)$ [43]. Remembering that the $\widehat{\mathcal{F}}^{(i)}(s)$ are free of right-hand-cut discontinuities, we find a unitarity relation for the SVAs,

$$\operatorname{Im}\mathcal{F}^{(i)}(s) = \left(\mathcal{F}^{(i)}(s) + \widehat{\mathcal{F}}^{(i)}(s)\right)e^{-i\delta_1^I(s)}\sin\delta_1^I(s)\,. \tag{7}$$

Due to Eq. (7), the functions $\widehat{\mathcal{F}}^{(i)}(s)$ are usually referred to as inhomogeneities, as they constitute the inhomogeneous contributions to the unitarity relations for $\mathcal{F}^{(i)}(s)$. The solution of the full, inhomogeneous unitarity relation (7) for the single-variable amplitudes is subsequently obtained using a separation ansatz with the Omnès function. The input for the latter is taken from Ref. [21]. The results are the Khuri–Treiman equations [19] for the SVAs

$$\mathcal{F}^{(0,1/2)}(s) = \Omega(s)\left(P^{(0,1/2)}_{n-1}(s) + \frac{s^n}{\pi}\int_{s_{\text{th}}}^{\infty}\frac{\mathrm{d}s'}{s'^n}\frac{\widehat{\mathcal{F}}^{(0,1/2)}(s')\sin\delta_1^{1/2}(s')}{|\Omega(s')|(s'-s)}\right), \quad \mathcal{F}^{(3/2)}(s) = P^{(3/2)}_{n'-1}(s)\,. \tag{8}$$

We can solve for the SVAs by inserting the fixed t-channel contributions from Sec. 3 into the inhomogeneities and then solving Eq. (8) iteratively. The system is linear in the subtraction constants, so that it is possible to construct basis functions. The calculation of the latter converges very quickly, such that they remain practically unchanged after at most five iterations.

5. Matching

Next, we discuss how to fix the free parameters of the dispersive representation, the subtraction constants, by matching them to the chiral anomaly on the one hand, and the radiative couplings of the $K^*(892)$ resonances on the other.

5.1 Chiral anomaly

The Wess–Zumino–Witten anomaly [2, 3] yields low-energy theorems for the different $\gamma K \to K\pi$ amplitudes in the limit of vanishing energies ($s = t = u = 0$) and vanishing (light as well as strange) quark masses. It contributes to the neutral-pion-production amplitudes, but not to the charge-exchange processes:

$$\mathcal{F}^{-0/00}(0,0,0) = F_{KK\pi}\,, \quad \mathcal{F}^{0-/-+}(0,0,0) = 0\,, \quad \text{where } F_{KK\pi} = \frac{e}{4\pi^2F_\pi^3} = 9.8\,\text{GeV}^{-3}\ [44,\ 45] \tag{9}$$

is given in terms of the pion decay constant $F_\pi = 92.28(3)$ MeV and the electric charge e, and is actually identical to the similarly defined anomaly $F_{3\pi}$ for $\gamma\pi \to \pi\pi$ [5–7]. Since it is hard to estimate the correlations between all the higher-order corrections, we simply estimate a resulting uncertainty of 25% on $F_{KK\pi}$. A complete next-to-leading-order calculation of all $\gamma K \to K\pi$ channels in chiral perturbation theory would certainly be highly desirable (see Refs. [44, 45] for partial results).

Obviously, also the vanishing charge-exchange amplitude will be modified due to higher-order corrections. Since a relative error estimate is not meaningful here, we use the absolute uncertainty given for the anomaly also for the charge-exchange amplitudes. Our combined assumption on the different amplitude normalizations in the soft-meson limits is therefore

$$\mathcal{F}^{-0/00}(0,0,0) = 9.8(2.4)\,\text{GeV}^{-3}\,, \quad \mathcal{F}^{0-/-+}(0,0,0) = 0.0(2.4)\,\text{GeV}^{-3}\,. \tag{10}$$

5.2 Radiative couplings of the $K^*(892)$

In the narrow-width approximation, the radiative widths of the $K^*(892)$ vector mesons are given by

$$\frac{1}{4}\Gamma_{K^{*0}\to K^0\gamma} = \Gamma_{K^{*\pm}\to K^\pm\gamma} = \frac{e^2 d_{K^*}^2}{864\pi}\left(\frac{M_{K^*}^2 - M_K^2}{M_{K^*}}\right)^3\,. \tag{11}$$

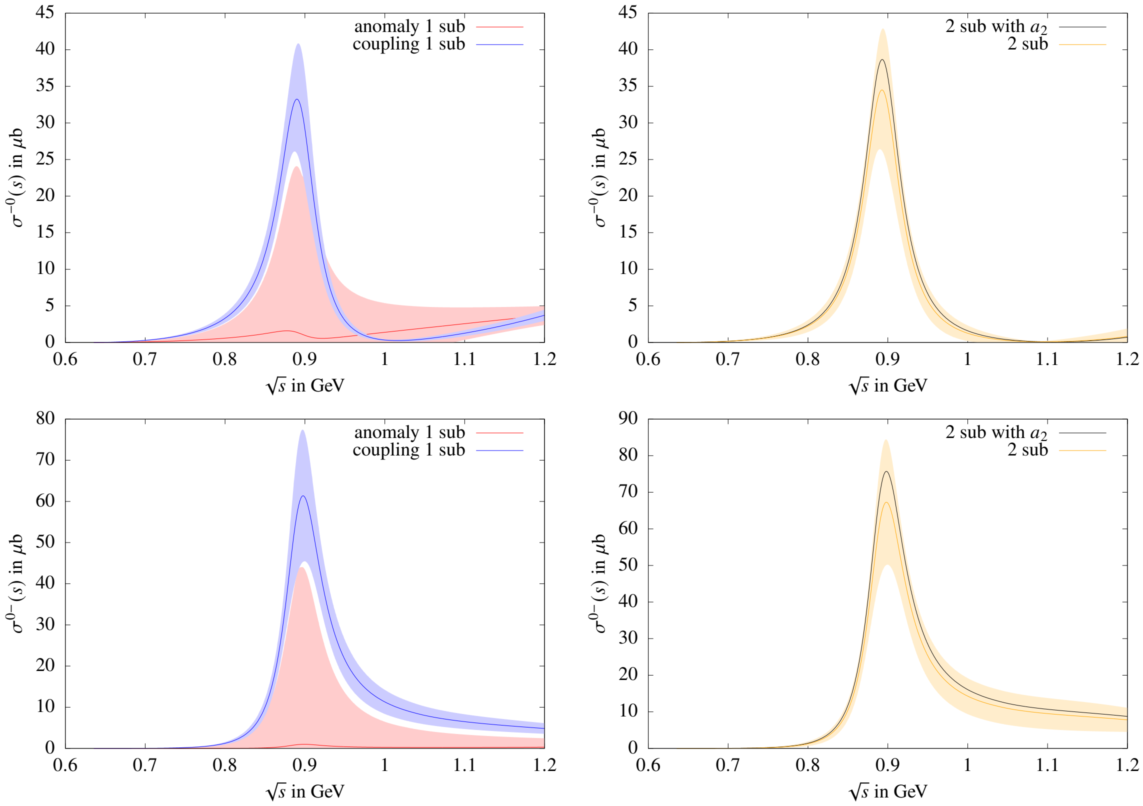

Figure 2: Cross section results for $\gamma K^- \to K^-\pi^0$ (top) and $\gamma K^- \to \bar{K}^0\pi^-$ (bottom) including P-wave amplitudes. Left panels: minimal subtraction scheme, matched to the chiral anomaly and the K^* radiative couplings separately. Right panels: twice subtracted scheme, matched to anomaly and radiative couplings simultaneously, with and without the a_2 contribution. The error bands correspond the propagated error of the real and imaginary parts.

The Particle Data Group [46] lists only three measurements from which these radiative widths have been extracted, one for $K^{*0} \to K^0\gamma$ [18] and two for $K^{*\pm} \to K^\pm\gamma$ [16, 17]. The extracted charged and neutral radiative couplings read $d^{\text{c}}_{K^*} = 2.50(12)\,\text{GeV}^{-1}$, $d^{\text{n}}_{K^*} = 1.93(8)\,\text{GeV}^{-1}$, and thus violate $SU(3)$ symmetry at the 20% level.

For a model-independent extraction of the radiative K^* coupling constants, we have to analytically continue the $\gamma K \to K\pi$ amplitudes onto the second Riemann sheet and connect them to the residues of the corresponding poles. The continuation to the second sheet can be found from the discontinuity,

$$f^{(i)}_{1,I}(s) - f^{(i)}_{1,II}(s) = -2\hat{\kappa}(s)t^{1/2}_{1,II}(s)f^{(i)}_{1,I}(s)\,, \tag{12}$$

where I (II) denotes the first (second) Riemann sheet.

6. Discussion and summary

We begin the discussion of numerical results with the minimal subtraction scheme, which contains two subtractions constants. According to the discussion of the previous section, we can choose to fix these in two different ways: via matching to the chiral anomaly or by reproducing the experimentally measured radiative K^* couplings. We start with the first option and match the subtraction constants to the low-energy theorems; see Fig. 2 (left). Obviously, the error bands are huge. This illustrates the very strong dependence of the partial waves, and in particular the $K^*(892)$ resonance signals, on the amplitudes in the low-energy limit. By reversing the argument, a concise measurement of the cross section around the resonance peak will help determine the anomaly and, potentially, its higher-order corrections very accurately if the minimal subtraction scheme can be validated experimentally to be sufficient. This is in strict analogy to the argument of Ref. [14] that the full resonance signal of the $\rho(770)$ can be employed to extract the chiral anomaly in $\gamma\pi \to \pi\pi$.

As the second approach, we fix the real subtraction constants in the minimal subtraction scheme using the radiative K^* couplings derived from experiment. We observe that the uncertainties are much smaller in this scheme. To obtain a larger degree of flexibility for the description of future high-precision cross section data, we can apply the twice subtracted version with four degrees of freedom. This allows us to include both constraints, low-energy theorems and resonance couplings, and combine them into a prediction for experiment. Furthermore, in the twice subtracted representation it is possible to include the a_2 t-channel contribution, which changes the isovector part of the photon only. The corresponding plots are also included in Fig. 2. Comparing the two solutions with and without a_2-exchange, we observe that this mechanism is very small below 1 GeV. We conclude that it is unnecessary to take D- and higher partial waves into account when considering the left-hand cuts at the current level of accuracy.

Using Eq. (3) and the respective partial-wave amplitudes, we can calculate the cross sections for all physical channels, see Fig. 2 and Ref. [1] for the $\gamma\bar{K}^0$ reactions. While the differences between the various channels at low energies, very discernible in the amplitudes, are hardly observable due to the phase space factors—the onset of the visible cross sections only seems to be deferred by about 50 MeV for the charge-exchange reactions with their suppressed near-threshold amplitudes—, we see a significant difference between the π^0 and the charge-exchange channels above the $K^*(892)$, where we predict a strong suppression of the π^0 production cross sections around 1.1 GeV. As we expect D-wave corrections to become important only above those energies [28], such a suppression should be realistically observable in experiments. With incoming neutral kaons, the cross sections are enhanced by about a factor of two compared to their charged-kaon counterparts, while the outgoing-neutral-pion channels are suppressed by again roughly a factor of two in the peak region in comparison to the charge-exchange reactions.

Options for future theoretical improvement comprise in particular the calculation of the next-to-leading-order, or $O(p^6)$, corrections to the chiral anomaly for this reaction. Furthermore, a reduction of the uncertainty in the $\omega \to K\bar{K}$ coupling, which affects our amplitude representation rather strongly, would be highly desirable.

Once high-precision, high-statistics experimental data is available, from COMPASS++/AMBER or elsewhere, a simultaneous fit to the two observable charge configurations in γK^- fixes the subtraction constants, from where it is possible to extract the physical quantities of interest. The dispersive representation therefore allows future experiments to determine precise information on the anomaly in a photon–kaon reaction as well as the radiative couplings of the $K^*(892)$ resonance from the complete measured energy range up to $\sqrt{s} \approx 1.2$ GeV.

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