

## Structure-dependent electromagnetic finite-size effects

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We present a model-independent and relativistic approach to analytically derive electromagnetic finite-size effects beyond the point-like approximation. The key element is the use of electromagnetic Ward identities to constrain vertex functions, and structure-dependence appears via physical form-factors and their derivatives. We apply our general method to study the leading finite-size structure-dependence in the pseudoscalar mass (at order  $1/L^3$ ) as well as in the leptonic decay amplitudes of pions and kaons (at order  $1/L^2$ ). Knowledge of the latter is essential for Standard Model precision tests in the flavour physics sector from lattice simulations.

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## 1. Introduction

Lattice quantum chromodynamics (QCD) allows for systematically improvable Standard Model (SM) precision tests from numerical simulations performed in a finite-volume (FV), discretised Euclidean spacetime. In order to reach (sub-)percent precision in lattice predictions, also strong and electromagnetic isospin breaking corrections have to be included. The latter are encoded via quantum electrodynamics (QED), but the inclusion of QED in a FV spacetime is complicated because of Gauss' law [1]. This problem is related to zero-momentum modes of photons and the absence of a QED mass-gap. Several prescriptions of how to include QED in a finite volume have been formulated and the one used here is QED<sub>L</sub> where the spatial zero-modes are removed on each time-slice. The long-range nature of QED in addition enhances the FV effects (FVEs), which typically leads to power-law FVEs that are larger than the exponentially suppressed ones for single-particle matrix elements in QCD alone.

The FVEs for a QCD+QED process depend on properties of the involved particles, including masses and charges, but also structure-dependent quantities such as electromagnetic form-factors and their derivatives. In order to analytically capture the finite-volume scaling fully, one cannot neglect hadron structure, and in the following we develop a relativistic and model-independent method to go beyond the point-like approximation at order  $e^2$  in QED<sub>L</sub>.

We consider a space-time with periodic spatial extents  $L$  but with infinite time-extent. To exemplify the method, we first consider the pseudoscalar mass in Sec. 2, and then proceed to leptonic decays in Sec. 3. The discussion is based on the results in Ref. [2], and the reader is referred there for further technical details.

## 2. Pseudoscalar Mass

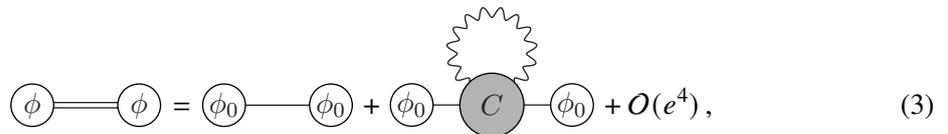
To study the finite-size scaling in the mass  $m_P(L)$  of a charged hadronic spin-0 particle  $P$ , we first define the full QCD+QED infinite-volume two-point Euclidean correlation function

$$C_2^\infty(p) = \int d^4x \langle 0 | T[\phi(x)\phi^\dagger(0)] | 0 \rangle e^{-ipx}. \quad (1)$$

Here  $\phi$  is an interpolating operator coupling to  $P$ , and  $p = (p_0, \mathbf{p})$  is the momentum. We denote the finite-volume counterpart of this correlator  $C_2^L(p)$ , but for the moment only consider  $C_2^\infty(p)$ . This can be diagrammatically represented as

$$C_2^\infty(p) = \text{---}\phi\text{---}\phi\text{---} = Z_P \cdot D(p) \cdot Z_P, \quad D(p) = \frac{Z(p^2)}{p^2 + m_P^2}, \quad Z_P = \langle 0 | \phi(0) | P, \mathbf{p} \rangle, \quad (2)$$

where the double-line represents the QCD+QED propagator  $D(p)$ , the  $\phi$ -blob is the overlap between  $\phi$  and  $P$  and  $Z(p^2) = 1 + O(p^2 + m_P^2)$  is the residue of the propagator. Expanding  $C_2^\infty(p)$  in (2) around  $e = 0$  yields

$$\text{---}\phi\text{---}\phi\text{---} = \text{---}\phi_0\text{---}\phi_0\text{---} + \text{---}\phi_0\text{---}\text{---}C\text{---}\phi_0\text{---} + O(e^4), \quad (3)$$




where the  $c_j$  are finite-volume coefficients specific to QED<sub>L</sub> arising from the sum-integral difference in (5). These are discussed in detail in Ref. [2]. Here we see the charge radius  $\langle r_p^2 \rangle$  appearing at order  $1/L^3$  and its coefficient agrees with that derived within non-relativistic scalar QED [6]. However, there is an additional structure-dependent term  $C$  related to the branch-cut of the forward, on-shell Compton amplitude. This contribution can be found, in other forms, also in Refs. [3, 7], and only arises because of the QED<sub>L</sub> prescription with the subtracted zero-mode. Its value is currently unknown but one can show  $C > 0$  [2], meaning that it cannot cancel the charge radius contribution. Note that all unphysical off-shell contributions from the form-factors  $F$  and  $G$  have vanished.

### 3. Leptonic Decays

Leptonic decay rates of light mesons are of the form  $P^- \rightarrow \ell^- \bar{\nu}_\ell$ , where  $P$  is a pion or kaon,  $\ell$  a lepton and  $\nu_\ell$  the corresponding neutrino. These are important for the extraction of the Cabibbo-Kobayashi-Maskawa matrix elements  $|V_{us}|$  and  $|V_{ud}|$  [8, 9]. The leading virtual electromagnetic correction to this process yields an infrared (IR) divergent decay rate  $\Gamma_0$ . One must therefore add the real radiative decay rate  $\Gamma_1(\Delta E)$  for  $P^- \rightarrow \ell^- \bar{\nu}_\ell \gamma$ , where the photon has energy below  $\Delta E$ , to cancel the IR-divergence in  $\Gamma_0$ . The IR-finite inclusive decay rate is thus  $\Gamma(P^- \rightarrow \ell^- \nu_\ell[\gamma])$ , and following the lattice procedure first laid out in Ref. [8] we may write

$$\Gamma_0 + \Gamma_1(\Delta E_\gamma) = \lim_{L \rightarrow \infty} [\Gamma_0(L) - \Gamma_0^{\text{uni}}(L)] + \lim_{L \rightarrow \infty} [\Gamma_0^{\text{uni}}(L) + \Gamma_1(L, \Delta E_\gamma)]. \quad (9)$$

Here, Ref. [8] chose to add and subtract the universal finite-volume decay rate  $\Gamma_0^{\text{uni}}(L)$ , calculated in point-like scalar QED in Ref. [4], to cancel separately the IR-divergences in  $\Gamma_0$  and  $\Gamma_1$ . In the following we are interested in only the first term in brackets. The subtracted term  $\Gamma_0^{\text{uni}}(L)$  cancels the FVEs in  $\Gamma_0(L)$  through order  $1/L$ , and hence  $\Gamma_0(L) - \Gamma_0^{\text{uni}}(L) \sim \mathcal{O}(1/L^2)$ . Structure-dependence enters at order  $1/L^2$ . With the goal of systematically improving the finite-volume scaling order by order including structure-dependence, we replace the universal contribution by

$$\Gamma_0^{\text{uni}}(L) \longrightarrow \Gamma_0^{(n)}(L) = \Gamma_0^{\text{uni}}(L) + \sum_{j=2}^n \Delta\Gamma_0^{(j)}(L), \quad (10)$$

where  $n \geq 2$  and  $\Delta\Gamma_0^{(j)}(L)$  contains the FVEs at order  $1/L^j$ . This means that the finite-volume residual instead scales as  $\Gamma_0(L) - \Gamma_0^{(n)}(L) \sim \mathcal{O}(1/L^{n+1})$ . We may parametrise  $\Gamma_0^{(n)}(L)$  in terms of a finite-volume function  $Y^{(n)}(L)$  according to

$$\Gamma_0^{(n)}(L) = \Gamma_0^{\text{tree}} \left[ 1 + 2 \frac{\alpha}{4\pi} Y^{(n)}(L) \right] + \mathcal{O}\left(\frac{1}{L^{n+1}}\right), \quad (11)$$

where  $\Gamma_0^{\text{tree}}$  is the tree-level decay rate.

Since we are interested in the leading structure-dependent contribution we consider  $Y^{(2)}(L)$ . In order to derive it, we define the QCD+QED correlation function

$$C_W^{rs}(p, p_\ell) = \int d^4z e^{ipz} \langle \ell^-, \mathbf{p}_\ell, r; \nu_\ell, \mathbf{p}_{\nu_\ell}, s | \text{T}[\mathcal{O}_W(0)\phi^\dagger(z)] | 0 \rangle, \quad (12)$$

where  $p_\ell = (p_\ell^0, \mathbf{p}_\ell)$  is the momentum of the on-shell lepton of mass  $m_\ell$ ,  $p_{\nu_\ell} = (p_{\nu_\ell}^0, \mathbf{p}_{\nu_\ell})$  is the momentum of the massless neutrino and  $\mathcal{O}_W(0)$  is the four-fermion operator of the decay in question. We may diagrammatically represent this in a similar way as for the mass according to

$$C_W^{rs}(p, p_\ell) = \text{diagram} = \text{diagram} + \text{diagram} + \text{diagram}. \quad (13)$$

The grey blob containing  $W$  is of order  $e^2$  and can be separated, just like the Compton amplitude, into several irreducible vertex functions. The exact definitions of these vertex functions are quite involved and can be found in Ref. [2], but several comments can be made. First of all, the vertex functions are related to various structure-dependent form-factors containing both on-shell and off-shell information. Again, the off-shellness must cancel. The vertex functions also contain physical structure-dependent information (similar to how  $\Gamma_1$  depends on the charge radius) and for  $Y^{(2)}(L)$  this is the axial-vector form-factor  $F_A(-m_P^2) = F_A^P$  from the real radiative decay  $P^- \rightarrow \ell^- \bar{\nu}_\ell \gamma$ .

By performing the amputation on the external meson leg in (12) to obtain the matrix element needed for the decay rate in (11), one finds the finite-volume function  $Y^{(2)}(L)$  to be

$$Y^{(2)}(L) = \frac{3}{4} + 4 \log\left(\frac{m_\ell}{m_W}\right) + 2 \log\left(\frac{m_W L}{4\pi}\right) + \frac{c_3 - 2(c_3(\mathbf{v}_\ell) - B_1(\mathbf{v}_\ell))}{2\pi} - \quad (14)$$

$$- 2 A_1(\mathbf{v}_\ell) \left[ \log\left(\frac{m_P L}{2\pi}\right) + \log\left(\frac{m_\ell L}{2\pi}\right) - 1 \right] - \frac{1}{m_P L} \left[ \frac{(1+r_\ell^2)^2 c_2 - 4r_\ell^2 c_2(\mathbf{v}_\ell)}{1-r_\ell^4} \right] +$$

$$+ \frac{1}{(m_P L)^2} \left[ -\frac{F_A^P}{f_P} \frac{4\pi m_P [(1+r_\ell^2)^2 c_1 - 4r_\ell^2 c_1(\mathbf{v}_\ell)]}{1-r_\ell^4} + \frac{8\pi [(1+r_\ell^2) c_1 - 2c_1(\mathbf{v}_\ell)]}{(1-r_\ell^4)} \right].$$

Here,  $r_\ell = m_\ell/m_P$ ,  $\mathbf{v}_\ell = \mathbf{p}_\ell/E_\ell$  the lepton velocity in terms of the energy  $E_\ell$ , and  $m_W$  the  $W$ -boson mass. Also,  $c_k$ ,  $A_1(\mathbf{v}_\ell)$ ,  $B_1(\mathbf{v}_\ell)$  and  $c_j(\mathbf{v}_\ell)$  are finite-volume coefficients defined in Ref. [2]. Note that no unphysical quantities appear. At order  $1/L^2$ , there is one structure-dependent contribution proportional to  $F_A^P$  and the other term is purely point-like. This result is in perfect agreement with Ref. [4] for the universal terms up to  $\mathcal{O}(1/L)$ , which we derived in a completely different approach. The numerical impact of the  $1/L^2$ -corrections is studied in Ref. [2].

#### 4. Conclusions

We have presented a relativistic and model-independent method to derive electromagnetic FVEs beyond the point-like approximation. We are currently working to obtain the leading FVEs for semi-leptonic kaon decays, relevant for future precision tests in the SM flavour physics sector.

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## References

- [1] M. Hayakawa and S. Uno, *QED in finite volume and finite size scaling effect on electromagnetic properties of hadrons*, *Prog. Theor. Phys.* **120** (2008) 413 [0804.2044].
- [2] M. Di Carlo, M.T. Hansen, N. Hermansson-Truedsson and A. Portelli, *Relativistic, model-independent determination of electromagnetic finite-size effects beyond the point-like approximation*, 2109.05002.
- [3] S. Borsanyi et al., *Ab initio calculation of the neutron-proton mass difference*, *Science* **347** (2015) 1452 [1406.4088].
- [4] V. Lubicz, G. Martinelli, C.T. Sachrajda, F. Sanfilippo, S. Simula and N. Tantalo, *Finite-Volume QED Corrections to Decay Amplitudes in Lattice QCD*, *Phys. Rev.* **D95** (2017) 034504 [1611.08497].
- [5] P.D. Group, P.A. Zyla et al., *Review of Particle Physics*, *Progress of Theoretical and Experimental Physics* **2020** (2020) [<https://academic.oup.com/ptep/article-pdf/2020/8/083C01/33653179/ptaa104.pdf>].
- [6] Z. Davoudi and M.J. Savage, *Finite-Volume Electromagnetic Corrections to the Masses of Mesons, Baryons and Nuclei*, *Phys. Rev.* **D90** (2014) 054503 [1402.6741].
- [7] N. Tantalo, V. Lubicz, G. Martinelli, C.T. Sachrajda, F. Sanfilippo and S. Simula, *Electromagnetic corrections to leptonic decay rates of charged pseudoscalar mesons: finite-volume effects*, *PoS LATTICE2016* (2016) [1612.00199].
- [8] N. Carrasco, V. Lubicz, G. Martinelli, C. Sachrajda, N. Tantalo, C. Tarantino et al., *QED Corrections to Hadronic Processes in Lattice QCD*, *Phys. Rev. D* **91** (2015) 074506 [1502.00257].
- [9] M. Di Carlo, D. Giusti, V. Lubicz, G. Martinelli, C. Sachrajda, F. Sanfilippo et al., *Light-meson leptonic decay rates in lattice QCD+QED*, *Phys. Rev. D* **100** (2019) 034514 [1904.08731].