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# Analytic treatment of neutrino oscillation and decay in matter

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We analyze invisible decay of neutrinos in the presence of oscillation and matter effects. The inclusion of decay can be accommodated by a non-Hermitian effective Hamiltonian, with the Hermitian component giving rise to oscillations, and the anti-Hermitian component leading to the invisible decay of neutrinos. We consider the possibility that the oscillation and decay matrix may not commute; in fact, in matter, they will invariably become non-commuting. This would lead to a mismatch between the effective mass eigenstates and the decay eigenstates. Employing a resummation of the Zassenhaus expansion, we develop a formalism for calculating the neutrino oscillation probabilities in the two-flavor scenario. This technique can easily be extended to three flavors.

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## 1. Introduction

It has been unequivocally established from neutrino oscillation experiments that neutrinos have masses. This leads to neutrino flavor mixing during their propagation. However, the possibility of new physics effects at a subleading level, such as neutrinos decaying to lighter invisible states [1], remains.

The effective Hamiltonian for neutrino decay may be represented by a non-Hermitian matrix, with its anti-Hermitian component corresponding to decay and the Hermitian component corresponding to oscillations. These Hermitian and anti-Hermitian components may not commute, leading to a mismatch in the mass and decay eigenstates. Even when these eigenstates coincide in vacuum, matter effects make this mismatch inevitable.

In this Proceeding, we discuss the inevitability of the mismatch and present a novel technique for computing the survival and conversion probabilities of neutrinos, for the scenario with oscillation and invisible decay in constant matter density. We calculate the probabilities in the two-flavor limit, although the technique can be extended to three flavors in a straightforward manner.

## 2. Formalism

The effective Hamiltonian in the basis of neutrino mass eigenstates in matter is

$$\mathcal{H}_m = \begin{pmatrix} a_1 - ib_1 & -\frac{1}{2}i\gamma e^{i\chi} \\ -\frac{1}{2}i\gamma e^{-i\chi} & a_2 - ib_2 \end{pmatrix}, \tag{1}$$

where $a_i, b_i, \gamma, \chi$ are real, with '$m$' denoting the matter basis. Since the decay matrix $\Gamma_m$ needs to be positive semidefinite, $b_i \geq 0$ and $\gamma^2 \leq 4b_1b_2$. The sign of $\gamma$ is always taken to be positive.

As ensured by our choice of basis, the Hermitian part of this Hamiltonian is diagonal. The anti-Hermitian part on the other hand is composed of both diagonal and off-diagonal components, regulated by $b_i$ and $\gamma$ respectively. Note that $b_i = [\Gamma_m]_{ii}/2$ is simply the diagonal contribution to the decay, and $\gamma e^{i\chi} = [\Gamma_m]_{12} = [\Gamma_m]^*_{21}$ regulates the amount of mismatch between the oscillation and decay eigenstates.

Even for the simplified scenario where only the mass eigenstate $\nu_2$ in vacuum decays (with lifetime $\tau_2$), matter effects lead to the identification:

$$a_{1,2} = \frac{\tilde{m}^2_{1,2}}{2E} \quad , \qquad b_{1,2} = \frac{\alpha_2}{4E}[1 \mp \cos[2(\theta - \theta_m)]] \; , \tag{2}$$

$$\chi = 0 \quad , \qquad \gamma = \frac{\alpha_2}{2E}\sin[2(\theta - \theta_m)] \; , \tag{3}$$

where, $\tilde{m}_i(m_i)$ and $\theta_m(\theta)$ are the mass eigenvalues and mixing angle in matter (vacuum), and $\alpha_2 \equiv m_2/\tau_2$. Note how all the elements of the $\Gamma_m$ matrix are non-zero ($b_1,\ b_2,\ \gamma \neq 0$) due to matter effects, leading to both the neutrino mass eigenstates showing decaying behavior. This shows that the mismatch between the decay and oscillation eigenstates is inevitable in the presence of matter. This simple mapping, previously not explicitly given in the literature, gives the correct analytic probability expressions for decaying neutrinos in matter. In the vacuum limit ($\theta_m \to \theta$, $\tilde{m}_i \to m_i$, $b_1 \to 0$ and $\gamma \to 0$) one obtains the standard probabilities in vacuum [2].



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For our analysis, we define the parameter $d_i \equiv a_i - ib_i$, the differences $\Delta_a \equiv a_2 - a_1$, $\Delta_b \equiv b_2 - b_1$, $\Delta_d \equiv d_2 - d_1$, and the dimensionless quantities

$$\bar{\gamma} \equiv \gamma/|\Delta_d| \,, \qquad \bar{\Delta}_a \equiv \Delta_a/|\Delta_d| \,, \qquad \bar{\Delta}_b \equiv \Delta_b/|\Delta_d| \,. \tag{4}$$

We also express the term $-i\mathcal{H}_m t = -\frac{it}{2}\,(d_1 + d_2)\,\mathbb{I} + \mathbb{X} + \mathbb{Y}$, in terms of the identity matrix $\mathbb{I}$ and

$$\mathbb{X} \equiv -\frac{i\,\Delta_d\,t}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \mathbb{Y} \equiv -\frac{\gamma t}{2}\begin{pmatrix} 0 & e^{i\chi} \\ e^{-i\chi} & 0 \end{pmatrix}. \tag{5}$$

## 3. Zassenhaus Expansion

To calculate the evolution matrix $\exp(-i\mathcal{H}_m t)$ in the matter basis, we first need to calculate the quantity $e^{\mathbb{X}+\mathbb{Y}}$. Since $\mathbb{X}$ and $\mathbb{Y}$ do not commute in general, this may be expressed in terms of the Zassenhaus (inverse BCH) expansion [3] as an infinite series:

$$e^{\mathbb{X}+\mathbb{Y}} = e^{\mathbb{X}}\,e^{\mathbb{Y}}\,e^{-\frac{1}{2}[\mathbb{X},\mathbb{Y}]}\,e^{\frac{1}{6}(2[\mathbb{Y},[\mathbb{X},\mathbb{Y}]]+[\mathbb{X},[\mathbb{X},\mathbb{Y}]])}\cdots \tag{6}$$

Note that $|\mathbb{Y}| \sim \bar{\gamma}|\mathbb{X}|$ and $\mathcal{L}_{\mathbb{X}}\mathbb{Y} \sim \bar{\gamma}|\mathbb{X}|^2$. Here the absolute sign '$(|\cdot|)$' represents the value of a typical element in the matrix. Therefore, for higher-order commutators, $\mathcal{L}_{\mathbb{X}}^{k-1}\mathbb{Y} \sim \bar{\gamma}|\mathbb{X}|^k$. This implies that it would be erroneous to truncate the expansion in eq. (6) after a particular fixed order of commutators of our choice. We need to collect all $O(\bar{\gamma}^k)$ terms for all values of '$k$' from commutators of all orders by performing a resummation procedure over the infinite series. This is achieved by employing the expression for Zassenhaus expansion in terms of a resummed series [4]:

$$e^{\mathbb{X}+\mathbb{Y}} = \Big(1 + \sum_{p=1}^{\infty}\sum_{n_1,\ldots,n_p=1}^{\infty}\frac{n_p\ldots n_1}{n_p(n_p+n_{p-1})\ldots(n_p+\ldots+n_1)}\mathcal{Y}_{n_p}\ldots\mathcal{Y}_{n_1}\Big)e^{\mathbb{X}}\,, \tag{7}$$

where $\mathcal{Y}_n = \frac{1}{n!}\mathcal{L}_{\mathbb{X}}^{n-1}\mathbb{Y}$. For calculating the expansion up to $O(\bar{\gamma})$ and $O(\bar{\gamma}^2)$, we can truncate the summation at $p = 1$ and $p = 2$, respectively. For an accuracy of $O(\bar{\gamma}^2)$, we have

$$e^{\mathbb{X}+\mathbb{Y}} \approx \Big(1 + \sum_{n_1=1}^{\infty}\mathcal{Y}_{n_1} + \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\frac{n_1}{(n_1+n_2)}\mathcal{Y}_{n_2}\mathcal{Y}_{n_1}\Big)e^{\mathbb{X}}, \tag{8}$$

with the double summation term not required for $O(\bar{\gamma})$ accuracy.

## 4. Two-Flavor Oscillation Probabilities

The neutrino mixing probability $P_{\beta\alpha} \equiv P(\nu_\beta \to \nu_\alpha)$ may be obtained by calculating the flavor conversion amplitude $[\mathcal{A}_f]_{\alpha\beta}$, with the probability given by $P_{\beta\alpha} = |[\mathcal{A}_f]_{\alpha\beta}|^2$. One may express

$$\mathcal{A}_f = \Big[U_m(\theta_m)\;e^{-i\mathcal{H}_m t}\;U_m^\dagger(\theta_m)\Big]_{\alpha\beta} = \begin{pmatrix} g_-(t)\,A(\chi) + g_+(t) & g_-(t)\,B(\chi) \\ g_-(t)\,B(-\chi) & -g_-(t)\,A(\chi) + g_+(t) \end{pmatrix}, \tag{9}$$

where $g_\pm(t) = \frac{1}{2}(e^{-id_2 t} \pm e^{-id_1 t})$ and $\theta_m$ is the mixing angle in matter in the no-decay limit, with the corresponding unitary matrix $U_m$. The coefficients $A(\chi)$ and $B(\chi)$ are given in Table 1.

**Table 1:** The quantities appearing in $\mathcal{A}_f$ in eq. (9), calculated up to $O(\bar{\gamma})$.

| Term | Expression |
|---|---|
| $A(\chi) \equiv A^{(0)} + \gamma A^{(1)}$ | $-\cos 2\theta_m - i\dfrac{\gamma}{\Delta_d}\sin 2\theta_m \cos\chi$ |
| $B(\chi) \equiv B^{(0)} + \gamma B^{(1)}$ | $\sin 2\theta_m - i\dfrac{\gamma}{\Delta_d}\Big(\cos 2\theta_m \cos\chi + i\sin\chi\Big)$ |

**Table 2:** The quantities in the analytic probability expressions in eqs. (10) and (11), calculated up to $O(\bar{\gamma})$.

| Term | Expression | Term | Expression |
|---|---|---|---|
| $\mathrm{Re}(A)$ | $-\cos 2\theta_m + \bar{\gamma}\bar{\Delta}_b \sin 2\theta_m \cos\chi$ | $\lvert A\rvert^2$ | $\cos^2 2\theta_m - 2\bar{\gamma}\bar{\Delta}_b \sin 2\theta_m \cos 2\theta_m \cos\chi$ |
| $\mathrm{Im}(A)$ | $-\bar{\gamma}\bar{\Delta}_a \sin 2\theta_m \cos\chi$ | $\lvert B\rvert^2$ | $\sin^2 2\theta_m + 2\bar{\gamma}\sin 2\theta_m\Big(\bar{\Delta}_a \sin\chi + \bar{\Delta}_b \cos 2\theta_m \cos\chi\Big)$ |

The survival probability for $\nu_\alpha \to \nu_\alpha$ is given by [5]

$$P_{\alpha\alpha} = \frac{e^{-(b_1+b_2)t}}{2}\Big[(1+|A|^2)\cosh(\Delta_b t) + (1-|A|^2)\cos(\Delta_a t) - 2\mathrm{Re}(A)\sinh(\Delta_b t) + 2\mathrm{Im}(A)\sin(\Delta_a t)\Big], \tag{10}$$

with $P_{\beta\beta} = P_{\alpha\alpha}(A \to -A)$. The probability for $\nu_\beta \to \nu_\alpha$ conversion is

$$P_{\beta\alpha} = \frac{e^{-(b_1+b_2)t}}{2}|B(\chi)|^2\left[\cosh(\Delta_b t) - \cos(\Delta_a t)\right], \tag{11}$$

with $P_{\alpha\beta} = P_{\beta\alpha}(\chi \to -\chi)$. The functional form of the explicit expressions used in eqs. (10) and (11) are given in Table 2. Note that, within the two-flavor approximation in the absence of neutrino decay, i.e. $b_1 = b_2 = \gamma = 0$, we have $P_{\alpha\alpha} = P_{\beta\beta}$ and $P_{\beta\alpha} = P_{\alpha\beta}$. These equalities are no longer true in the scenario with invisible decay of neutrinos.

Following a similar technique as before and considering the contribution from the double summation term, we may also calculate the neutrino flavor conversion probabilities up to $O(\bar{\gamma}^2)$. Expressing the difference of the exact eigenvalues as $\Delta_D = \sqrt{\Delta_d^2 - \gamma^2}$, the 2-flavor probabilities at $O(\bar{\gamma}^2)$ can be written in the same form as in eqs. (10) and (11) with the replacements

$$\Delta_a \to \mathrm{Re}(\Delta_D)\,, \quad \Delta_b \to -\mathrm{Im}(\Delta_D)\,. \tag{12}$$

Further, we also need to replace the entries in Table 1 by

$$A(\chi) \to A^{(0)} + \gamma A^{(1)} - \gamma^2 \cos 2\theta_m/(2\Delta_d^2)\,, \quad B(\chi) \to A^{(0)} + \gamma A^{(1)} + \gamma^2 \sin 2\theta_m/(2\Delta_d^2)\,, \tag{13}$$

with the relevant quantities in Table 2 calculated using the replacement rule in eq. (13).

## 5. Numerical Comparison

Let us explore the accuracy of our analytic expressions towards the exact neutrino oscillation probabilities within the 2-flavor formalism, when higher and higher order contributions from $\bar{\gamma}$ are included in the analytic expansion.

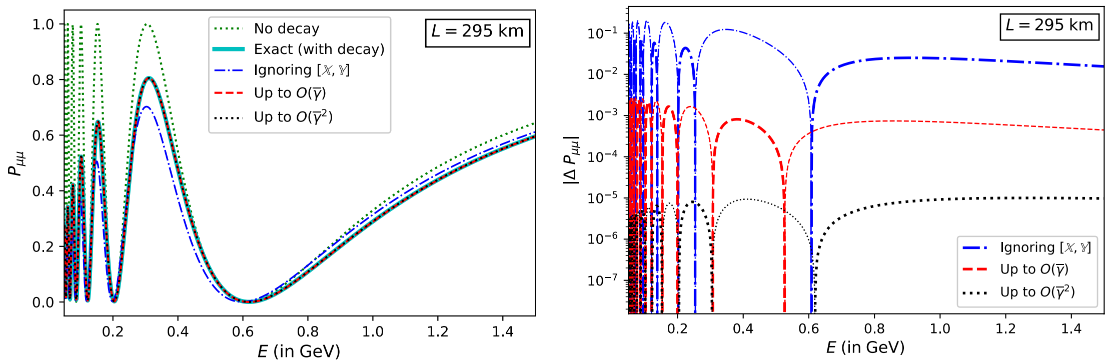


**Figure 1:** The survival probability $P_{\mu\mu}$ calculated exactly, and the analytic approximations in the text, in the presence and absence of neutrino decay [Left]. And the accuracy of our analytic expressions, i.e., differences between the analytic approximations and the exact numerical results in the presence of decay [Right]. The thick (thin) lines in the right panel correspond to $\Delta P_{\mu\mu} > 0$ ($\Delta P_{\mu\mu} < 0$). The sharp dips occur at energies where our analytic expressions and the exact probabilities match.

To show the effect of off-diagonal elements, we consider the survival probability $P(\nu_\mu \to \nu_\mu)$ with energy $E \sim$ GeV, for a baseline of 295 km, with $\theta_m = 45°$. Further, we take $\Delta_a = 2.56 \times 10^{-3}$ eV$^2/(2E)$, with the decay parameters $(b_1, b_2, \gamma) = (3, 6, 8) \times 10^{-5}$ eV$^2/(2E)$, $\chi = \pi/4$ to illustrate the importance of considering the off-diagonal 'mismatch' contribution. Note that we satisfy $b_1, b_2 \ll |\Delta_a|$, since the effects of decay should be subdominant to the effects of oscillations, and $\gamma^2 \leq 4b_1b_2$, since the decay matrix needs to be positive semi-definite.

In the left panel of Fig. 1, we show the probability $P_{\mu\mu}(E)$ without decay, the exact probability calculated numerically and analytic expressions up to $O(\bar{\gamma})$ and $O(\bar{\gamma}^2)$ in the presence of decay. The erroneous analytic approximation that neglects the commutator $[\mathbb{X}, \mathbb{Y}]$ is also shown. In the right panel of Fig. 1, the convergence towards the exact solution is very clearly demonstrated, where we plot the absolute accuracy defined as

$$\Delta P_{\mu\mu} \equiv P_{\mu\mu}(\text{analytic}) - P_{\mu\mu}(\text{exact}) \ . \tag{14}$$

The inclusion of $O(\bar{\gamma})$ and $O(\bar{\gamma}^2)$ terms is seen to be improving the accuracy by orders of magnitude.

## 6. Conclusion

The physics of neutrino decay and oscillation is characterized by a non-Hermitian effective Hamiltonian that cannot be diagonalized by a unitary transformation. Furthermore, even if the decay and mass eigenstates are the same in vacuum, matter effects invariably introduce mismatch between the two, warranting a more methodical treatment.

We perform our analysis in the basis of mass eigenstates in matter that one may obtain in the no-decay limit. This crucial step makes the Hermitian component of the Hamiltonian diagonal. We demonstrate the inevitability of the above mismatch in the presence of matter effects, even within the simple two-flavor approximation. This means that the Zassenhaus expansion can not be truncated to a fixed order of commutators. We resolve this issue by introducing a resummation in the Zassenhaus expansion, and compute the neutrino oscillation probabilities perturbatively in the mismatch parameter $\bar{\gamma}$. Such a formulation has been used to treat propagation of invisibly decaying neutrinos in matter for the first time in [5].

The framework described above can be extended to the three-flavor scenario. In the One Mass Scale Dominance (OMSD) approximation, the three-flavor oscillation probabilities can be written in terms of effective two-flavor oscillation probabilities by choosing an appropriate basis. Further, we can employ expansions in small parameters $\theta_{13}$, $\Delta m^2_{21}/\Delta m^2_{31}$ and the normalized decay widths to calculate the three-flavor oscillation probabilities more precisely [6].

The techniques described here can be employed for various other phenomena beyond the neutrino decay hypothesis, viz., axion-photon mixing in a semi-opaque medium, and the combined treatment of oscillations and absorption for high energy neutrinos.

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