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An exceptional $G(2)$ extension of the Standard Model from an algebraic conjecture: implications for the strong sector and dark matter

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A new criterion to extend the Standard Model of particle physics is proposed: the symmetries of physical microscopic forces originate from the automorphism groups of main Cayley–Dickson algebras, from complex numbers to octonions and sedenions. This correspondence leads to a natural and minimal enlargement of the color sector, from a $SU(3)$ gauge group to an exceptional Higgs-broken $G(2)$ group. In this picture, an additional ensemble of massive $G(2)$-gluons emerges, which is separated from the particle dynamics of the SM and might play the role of dark matter. A fully Lagrangian approach is provided, along with the description of the breaking mechanism, the $G(2)$ particle spectrum, the possible composite DM states and their stability examination. Moreover, $G(2)$ gauge theory could guarantee peculiar manifestations in astrophysical compact objects, which can be observed in the future studying gravitational waves.

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1. Introduction

Despite its great success and prediction capability, the Standard Model (SM) of particle physics is afflicted by several problems: above all, dark matter (DM) is probably the most compelling and very long-standing conflict of modern physics. The most convincing particle candidates, the weakly interacting massive particles (WIMPS), have not been discovered yet: direct, indirect and collider searches show no evidence of new particles approximately up to the 1 TeV scale [3, 22, 27, 30, 31]. This is a strong hint that the *Naturalness* criterion for the Higgs sector and the so-called *WIMP Miracle* [38] could not be a prerogative of *Nature*. We propose an approach based on a division algebras conjecture capable of selecting a unique extension of the SM, which introduces a branch of *exceptional* matter particles from a simple and minimally high symmetry [23]: fundamental interactions are identified with the automorphism groups of Cayley-Dickson algebras. From the automorphism of octonions (and sedenions) algebra, the promising exceptional symmetry group $G(2)$ can be pinpointed to solve the DM problem. We will demonstrate that, once broken through a Higgs-like mechanism, $G(2)$ represents the optimal gauge group to describe strong interaction and dark matter at the same time.

2. Fundamental forces from division algebras automorphisms

In the last decades many attempts to connect the Standard Model of elementary particles with division algebras have been made, showing it is worthwhile establishing relations between algebraic structures and symmetry groups [10, 12, 15, 16, 18, 20, 21]. Following the Cayley–Dickson construction process [12, 16], one can build up a sequence of larger and larger algebras, adding new imaginary units: $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ are the so-called division algebras, which are real numbers, complex numbers, quaternions and octonions, respectively. During the construction process, the algebras lose some peculiar properties, one at a time. The process does not terminate with octonions: greater 2^n-dimensional algebras can be constructed, for any positive integer n. The link between unitary groups and division algebras has been diffusely studied [7, 13, 32]. Unitary groups are the fundamental bricks to build the particle Standard Model, because each fundamental force can be described by a unitary or special unitary group [6, 24, 33, 34, 35], being $G = SU(3) \times SU(2) \times U(1)$ the SM group of strong $SU(3)$, weak $SU(2)$ and hypercharge $U(1)$ interactions [35]. Besides its symmetry, the SM includes three fermions families. Starting from the most simple complex algebra and SM symmetry group, it is easy to find a direct connection between the electromagnetism (or Quantum Electrodynamics) $U(1)$ formalism and the complex number field $\mathbb{C}$: in fact the group $U(1)$, the smallest compact real Lie group, corresponds to the circle group S^1, consisting of all complex numbers with absolute value 1 under multiplication, which is isomorphic to the $SO(2)$ group of rotation [14]. For $n \geq 1$, one can also consider for the comparison the n-torus T_n, that is defined to be $\mathbb{R}^n/\mathbb{Z}^n \cong U(n) \cong SO(2)^n \cong (S^1)^n$, which shows off the deep connection between $U(1)$ gauge symmetry and other representations strictly connected to complex numbers [14, 29]. It is also true that the $n \times n$ complex matrices which leave the scalar product $\langle , \rangle$ invariant form the group $U(n) = Aut(\mathbb{C}^n, \langle , \rangle)$, *i.e.* the group of automorphisms of $\mathbb{C}^n$ as a Hilbert space [2]. Even $SU(2)$ weak isospin can be connected with quaternions, [20]: $SU(2)$ naturally embeds into $\mathbb{H}$ as the group of quaternion elements of norm 1, with a perfect analogy with respect to $U(1)$ and complex numbers. Indeed, since unit quaternions can be used to represent rotations in 3-dimensional space, there is a surjective homomorphism from $SU(2)$ to the rotation group $SO(3)$

Charge (n_g)	Group	Force	Algebra	Dim	Commutative	Associative	Alternative	Normed	Flexible
$Q(1)$	$U(1)$	EM	$\mathbb{C}$	2	Yes	Yes	Yes	Yes	Yes
$T(3)$	$SU(2)$	Weak	$\mathbb{H}$	4	No	Yes	Yes	Yes	Yes
$C(8)$	$SU(3)$	Strong	$\mathbb{O}$ or $\mathbb{S}$	8/16	No	No	Yes	Yes	Yes
$EC(6)$	broken-$G(2)$	Exceptional Strong	$\mathbb{O}$ or $\mathbb{S}$	8/16	No	No	No	No	Yes

Table 1 Schematic correspondence between forces, groups and algebras. See [23] for details.

[20]. For the quaternionic basis these maps $\mathbf{1} \mapsto I$, $\mathbf{i} \mapsto -i\sigma_1$, $\mathbf{j} \mapsto -i\sigma_2$, $\mathbf{k} \mapsto -i\sigma_3$ stand, where $\sigma_{1,2,3}$ are the three Pauli matrices and I the identity matrix. So the correspondence between the automorphism of quaternion algebra and the SM symmetry group of weak force can be clearly shown: for quaternions $Aut(\mathbb{H}) = SO(3)$, where $SO(3)$ is homomorphic to $SU(2)$ in turn, and the universal cover of $SO(3)$ is the spin group $Spin(3)$, which is isomorphic to $SU(2)$. So $SU(2)$ and $SO(3)$ algebraic structures are equivalent and $su(2) = span(i\sigma_1, i\sigma_2, i\sigma_3)$. It seems logical to revise the next division algebra, the octonion algebra $\mathbb{O}$ for a possible description of $SU(3)$ [9, 28], but the result is less straightforward: the group of automorphisms of the octonion algebra corresponds to the exceptional Lie algebra $G(2)$, the smallest among the known exceptional Lie algebras: $Aut(\mathbb{O}) = G(2)$ [37]. The exceptional $G(2)$ group is certainly bigger than SM $SU(3)$, as it includes $SU(3)$ and is equipped with six additional generators [17]. Summarizing, for non real division algebras it turns out that $\mathrm{Aut}(\mathbb{C}) \cong U(1)$, $\mathrm{Aut}(\mathbb{H}) \cong SU(2)$, $\mathrm{Aut}(\mathbb{O}) \equiv G(2)$. For the sedenion algebra, the fifth Cayley-Dickson algebra, there is an important relation demonstrated by Brown in [5]: $\mathrm{Aut}(\mathbb{S}) = \mathrm{Aut}(\mathbb{O}) \times S_3$, where S_3 is the permutation group of degree three. So the inner symmetries of this non-division algebra can be again extracted from the automorphism group of octonions and the only difference is a factor of the permutation group S_3: sedenion algebra could constitute the searched simplicity criterion to select the full symmetry of a three generations Standard Model strong force. The gauge groups $U(1), SU(2), SU(3)$, describing the three fundamental forces, find mathematical correspondence into the division algebras $\mathbb{C}, \mathbb{H}, \mathbb{O}$ respectively: Table 1 summarizes this correspondence. However, the octonion and sedenion automorphism relations point towards a different group, which is manifestly larger than the usual 8-dimensional $SU(3)$ color group of the Standard Model. Therefore

$$\mathrm{Aut}(\mathbb{C}) \times \mathrm{Aut}(\mathbb{H}) \times \mathrm{Aut}(\mathbb{S}) = \mathrm{Aut}(\mathbb{C}) \times \mathrm{Aut}(\mathbb{H}) \times \mathrm{Aut}(\mathbb{O}) \times S_3 \cong U(1) \times SU(2) \times G(2) \times S_3 \tag{1}$$

could give the overall unbroken Standard Model symmetry. Here the automorphism selection is invoked to predict something beyond current SM and it works as a guideline to replace $SU(3)$ color itself with the smallest exceptional group: fundamental forces must be isomorphic to the automorphisms groups of the division algebras built up through the Cayley–Dickson construction. No more physics is needed nor predicted, except for the six additional degrees of freedom, *i.e.* boson fields, which represent the discrepancy between $G(2)$ and $SU(3)$ generators. In this scenario, the strong force acquires a more complex structure, which includes the usual color sector and an enlarged strong exceptional dynamics: to recover standard $SU(3)$ color strong force description, the new $G(2)$ color sector should be broken by a Higgs-like mechanism and separated into two parts, one visible and the other excluded from the dynamics due to its peculiar properties.

3. A G(2) gauge theory for the strong sector and implications for dark matter

To explicitly construct the 14 7×7 real matrices in the fundamental representation, one can choose the first eight generators of $G(2)$ as [17, 25]:

$$\Lambda_a = \frac{1}{\sqrt{2}} \begin{pmatrix} \lambda_a & 0 & 0 \\ 0 & -\lambda_a^* & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2}$$

where λ_a (with $a \in \{1, 2, ..., 8\}$) are the Gell-Mann generators of $SU(3)$, which indeed is a subgroup of $G(2)$. Λ_3 and Λ_8 are diagonal and represent the Cartan generators w.r.t.$SU(3)$. The remaining six generators can be found studying the root and weight diagrams of the group [4, 8, 11], and can be written as:

$$\Lambda_9 = \frac{1}{\sqrt{6}} \begin{pmatrix} 0 & -i\lambda_2 & \sqrt{2}e_3 \\ i\lambda_2 & 0 & \sqrt{2}e_3 \\ \sqrt{2}e_3^T & \sqrt{2}e_3^T & 0 \end{pmatrix}, \Lambda_{10} = \frac{1}{\sqrt{6}} \begin{pmatrix} 0 & -\lambda_2 & i\sqrt{2}e_3 \\ -\lambda_2 & 0 & -i\sqrt{2}e_3 \\ -i\sqrt{2}e_3^T & i\sqrt{2}e_3^T & 0 \end{pmatrix}, \tag{3}$$

$$\Lambda_{11} = \frac{1}{\sqrt{6}} \begin{pmatrix} 0 & i\lambda_5 & \sqrt{2}e_2 \\ -i\lambda_5 & 0 & \sqrt{2}e_2 \\ \sqrt{2}e_2^T & \sqrt{2}e_2^T & 0 \end{pmatrix}, \Lambda_{12} = \frac{1}{\sqrt{6}} \begin{pmatrix} 0 & \lambda_5 & i\sqrt{2}e_2 \\ \lambda_5 & 0 & -i\sqrt{2}e_2 \\ -i\sqrt{2}e_2^T & i\sqrt{2}e_2^T & 0 \end{pmatrix}, \tag{4}$$

$$\Lambda_{13} = \frac{1}{\sqrt{6}} \begin{pmatrix} 0 & -i\lambda_7 & \sqrt{2}e_1 \\ i\lambda_7 & 0 & \sqrt{2}e_1 \\ \sqrt{2}e_1^T & \sqrt{2}e_1^T & 0 \end{pmatrix}, \Lambda_{14} = \frac{1}{\sqrt{6}} \begin{pmatrix} 0 & -\lambda_7 & i\sqrt{2}e_1 \\ -\lambda_7 & 0 & -i\sqrt{2}e_1 \\ -i\sqrt{2}e_1^T & i\sqrt{2}e_1^T & 0 \end{pmatrix}, \tag{5}$$

where e_i are unit vectors. Under $SU(3)$ subgroup transformations, the 7-dimensional representation decomposes into [17, 25] $\{7\} = \{3\} \oplus \{\bar{3}\} \oplus \{1\}$. Since all $G(2)$ representations are real, the $\{7\}$ representation is identical to its complex conjugate, so that $G(2)$ "quarks" and "anti-quarks" are conceptually indistinguishable. This representation describes a $SU(3)$ quark $\{3\}$, a $SU(3)$ anti-quark $\{\bar{3}\}$ and a $SU(3)$ singlet $\{1\}$. The generators transform under the 14-dimensional adjoint representation of $G(2)$ [17, 25], which decomposes into [17, 25, 26] $\{14\} = \{8\} \oplus \{3\} \oplus \{\bar{3}\}$. So the $G(2)$ "gluons" ensemble is made of $SU(3)$ gluons $\{8\}$ plus six additional "gluons" which have $SU(3)$ quark and anti-quark color quantum numbers. The product of two fundamental representations $\{7\} \otimes \{7\} = \{1\} \oplus \{7\} \oplus \{14\} \oplus \{27\}$, shows a singlet $\{1\}$: as a noteworthy implication, two $G(2)$ "quarks" can form a color-singlet, or a "diquark". Moreover, just as for $SU(3)$ color, three $G(2)$ "quarks" can form a color-singlet "baryon": $\{7\} \otimes \{7\} \otimes \{7\} = \{1\} \oplus 4\ \{7\} \oplus 2\ \{14\} \oplus 3\ \{27\} \oplus 2\ \{64\} \oplus \{77\}$. Also pentaquarks and hexaquarks are allowed. States with baryon number 0 and 3 are in common with QCD whereas $n_B = 1, 2$, of $J = 1/2$ color-singlet $qGGG$ hybrids and $J = 0, 1$ diquarks respectively, are $G(2)$ specific. Summarizing: a $G(2)$ gauge theory has colors, anticolors and color-singlet, and 14 generators, 8 of them transforming as ordinary gluons (as an octuplet of $SU(3)$), while the other 6 $G(2)$ gauge bosons separates into $\{3\}$ and $\{\bar{3}\}$, keeping the color quarks/antiquarks quantum numbers, but they are still vector bosons. $G(2)$ is not a proper gauge theory for a real Quantum Chromodynamics theory: adding a Higgs-like field, in order to break $G(2)$ down to $SU(3)$, 6 of the 14 $G(2)$ "gluons" gain a mass proportional to the vacuum expectation value (vev) w of the new Higgs-like field, the other 8 $SU(3)$ gluons remaining untouched and massless. The Lagrangian of such a $G(2)$-Higgs model can be written as [17, 19, 25, 26]:

$$\mathcal{L}_{G_2H}[A, \Phi] = \mathcal{L}_{YM}[A] + (D_\mu \Phi)^2 - V(\Phi) \tag{6}$$

where $\mathcal{L}_{YM}[A]$ is the proper Yang-Mills Lagrangian, $\Phi(x) = (\Phi^1(x), \Phi^2(x), ..., \Phi^7(x))$ is the real-valued Higgs-like field, $D_\mu\Phi = (\partial_\mu - ig_G A_\mu)\Phi$ is the covariant derivative and $V(\Phi) = \lambda(\Phi^2 - w^2)^2$ the quadratic scalar potential, with $\lambda > 0$. We can choose a simple vev like $\Phi_0 = \frac{1}{\sqrt{2}}(0, 0, 0, 0, 0, 0, w)$ to break $G(2)$ and re-obtain the familiar unbroken $SU(3)$ symmetry. This new scalar Φ, which acquires a typical mass of $M_H = \sqrt{2\lambda}w$ from the expansion of the potential about its minimum [35], should be a different Higgs field w.r.t. the SM one, with a much higher vev, in order to disjoin massive gluons dynamics from SM one, and a strong phenomenology. Such a strongly coupled massive field could be ruled out by future LHC and Future Circular Collider searches [1]. In this picture, 6 massless Goldstone bosons are eaten and become the longitudinal components of $G(2)$ vector gluons corresponding to the broken generators, which acquire the eigenvalue mass $M_G = g_G w$ through the Higgs mechanism. When the expectation value of the Higgs-like field is sent to infinity, the massive $G(2)$ “gluons" are completely removed from the dynamics and the usual $SU(3)$ string potential reappears. Only high w values (with w much greater that the SM Higgs vev) are considered in order to realize a consistent dark matter scenario: these bosons must be separated from the visible SM sector, without experimentally accessible electroweak interactions, unlike WIMPs. This could be due to the very high energy scale of the $G(2) - SU(3)$ phase transition, occurring at much greater energies than electroweak breaking scale. This could be the realization of a *beyond Naturalness* criterion. Indeed, $G(2)$ gluons, as $SU(3)$ ones, are electrically neutral and immune to interactions with light and weak W, Z bosons at tree level. In our case, the six *dark* gluons can form dark glueballs constituted by two or three (or multiples) $G(2)$ gluons, according to $\{14\} \otimes \{14\} = \{1\} \oplus ...$ and $\{14\} \otimes \{14\} \otimes \{14\} = \{1\} \oplus ...$ representations [36], with integer total angular momentum $J = 0, 2$ and $J = 1, 3$ for 2-gluons and 3-gluons balls respectively. In principle, exceptional-colored broken-$G(2)$ glueballs should not be stable and we have to introduce a stabilizing feature, such as an accidental symmetry, a global discrete Z_2 (or Z_N) symmetry or G–parity conservation for a generic Yang–Mills [23]. From a cosmological point of view, the usual WIMP-like scenario built via the freeze–out mechanism cannot be achieved, since these gluons/glueballs are never in thermal equilibrium with the baryon-photon fluid in the early Universe: we have to invoke a FIMP scenario and a SIMP Dark freez-out with number changing processes applied to an exotic Higgs-portal (see [23] for a complete dissertation). Possible manifestations of $G(2)$ gluons in the present Universe are massive boson stars with repulsive quartic self-interaction potential: this heavy $G(2)$ gluon dark matter is indeed capable of producing stellar objects, which could populate the dark halos [23].

4. Conclusions

Fundamental microscopic forces might be manifestation of the conceivable algebras that can be built via the Cayley-Dickson process. The automorphism correspondence higlights the mismatch between $SU(3)$ strong force and octonions: the automorphism group is the exceptional group $G(2)$, which is not exhausted by $SU(3)$. In their difference new physics lies, in the form of six additional massive bosons, potentially disconnected by SM dynamics: the exceptional-colored $G(2)$ gluons. When the Universe cooled down, reaching a proper far beyond TeV energy scale at which $G(2)$ becomes broken, usual $SU(3)$ QCD appeared, while an extra Higgs mechanism produced a secluded sector of cold bosonic states: the symmetry breaking mechanism is completely analogue w.r.t. the SM one for the electroweak sector. $G(2)$ could guarantee peculiar manifestations in extreme astrophysical compact objects, such as boson stars made of $G(2)$ glueballs, which can populate the dark halos and be observed in the future studying their gravitational waves and dynamics.

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