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Quasinormal modes of black holes from supersymmetric gauge theory and integrability

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We explain how integrable models can be connected to both $\mathcal{N} = 2$ supersymmetric gauge theory and black hole perturbation theory. This important fact relies on the analysis of the connection coefficients of the differential equations which all this theories happen to share. In particular we prove that quasinormal modes of black holes are given by quantisation conditions on gauge periods or, equivalently, integrability Q functions. Besides, this allows us to find a new efficient method to characterise both analytically and numerically the quasinormal modes: analytically, we shed light on the gauge theory application; numerically, we compare it with other methods. For simplicity and limits of space we restrict the discussion to the simplest case of Liouville integrable model/pure $SU(2)$ gauge theory/D3 brane gravitation background triad.

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1. Introduction

In the last few years, gravitational waves detections have opened the doors for new gravitational phenomenology. A black hole (BH) collision can be divided in three phases: inspiral, merger and ringdown. The quasinormal modes (QNMs) are responsible for the damped oscillations appearing in the ringdown phase. General relativity (GR) BHs present fundamental theoretical problems (for instance, the information paradox). Also to solve such problems, theoretical models of Exotic Compact Objects (ECOs) in alternative theories of gravity have been developed. They differ from GR black holes in the sense that they have horizon scale structure. For subtype of ECOs, called Clean Photosphere Objects (ClePhOs), the later stage ringdown signal shows a peculiar train of echoes, with significant deviations from GR. An example of ClePhoS are fuzzballs in String Theory, with neither horizon nor central singularity and which may solve also the information paradox. Thus, thanks to gravitational wave astronomy, we can fully scientifically investigate whether real astrophysical black holes (BHs) show deviations from GR. [1–3].

To achieve this, also theoretical developments are highly desirable. Especially the research fields of integrability and extended supersymmetric gauge theory could yield a new non-perturbative understanding of some aspects of black hole physics. Indeed, computing QNMs numerically has been until now typically quite laborious and this has been also due to the difficulties in developing exact analytic characterizations of QNMs. In this direction, a significant improvement has been realized very recently as QNMs have been identified by exact quantization conditions on *dual* periods of some $\mathcal{N}=2$ supersymmetric gauge, *i.e.* deformed Seiberg-Witten (SW), theories [4–6]. On the latter we have in fact some exact control and then this SW-QNM correspondence has given some new theoretical and computational results for BHs and other spacetime geometries [7].

In section 2 we show how to construct a triple correspondence - *a triality* - among integrable models, $\mathcal{N}=2$ supersymmetric gauge theory and black hole perturbation theory. We do this by analysing closely the Ordinary Differential Equations (ODEs) describing the perturbations in gravitational physics, through an elegant and effective extension of the original ODE/IM correspondence between ODEs and Integrable Models (IM) [8, 9]. For limits of space, we only deal with the simplest case, the Liouville integrable model/pure $SU(2)$ gauge theory/D3 brane gravitation background triad. So we prove that QNMs are nothing but the zeros of the Q function (Bethe roots) and then find an entirely new set of functional equations for them (in particular quantization conditions). This yields also an explanation of the SW-QNM correspondence. Then from this integrability set up in section 3, we find a non-linear integral equation, the Thermodynamic Bethe Ansatz (TBA) one, which turns out to be a very simple and powerful way to compute QNMs and which we compare with other methods. For further details, explanations, extensions and general validity of the method, we refer to our recent papers [10–14].

2. Integrability, supersymmetric gauge theory and black holes physics

The D3 brance gravitational background has line element

$$ds^2 = H(r)^{-\frac{1}{2}}(-dt^2 + d\mathbf{x}^2) + H(r)^{\frac{1}{2}}(dr^2 + r^2 d\Omega_5^2)\,, \tag{1}$$

where $\mathbf{x}$ are the longitudinal coordinates, $H(r) = 1 + L^4/r^4$and $d\Omega_5^2$ denotes the metric of the transverse round S^5-sphere [2]. The ODE which governs the scalar field perturbation of the D3 brane is [2]

$$\frac{d^2}{dr^2}\phi(r) + \left[\omega^2\left(1+\frac{L^4}{r^4}\right) - \frac{(l+2)^2-\frac{1}{4}}{r^2}\right]\phi(r) = 0\,. \tag{2}$$

where ω is the QNM frequency and $l \in \mathbb{N}$. Upon the change of variables

$$r = Le^{\frac{y}{2}} \qquad \omega L = -2ie^{\theta} \qquad P = \frac{1}{2}(l+2) \qquad \phi(r) = e^{\frac{y}{4}}\psi(y)\,, \tag{3}$$

the equation reduces to the generalized Mathieu equation [2]

$$-\frac{d^2}{dy^2}\psi(y) + \left[2e^{2\theta}\cosh y + P^2\right]\psi(y) = 0\,. \tag{4}$$

As IM it corresponds to the conformal self-dual Liouville theory (central charge $c = 25$) with momentum P and rapidity θ, besides as gauge theory it corresponds to $\mathcal{N} = 2$ $SU(2)$ pure ($N_f = 0$) theory with Omega background $\epsilon_2 \longrightarrow 0$, $\epsilon_1 = \hbar$ (Nekrasov-Shatashvili limit [6]): for the latter this change of parameters is needed $\omega L = -2i\frac{\Lambda_0}{\hbar}$, $\frac{1}{8}(l+2)^2 = \frac{u}{\hbar^2}$, where u is the Coulomb branch modulus and Λ_0 the instanton coupling, so that [10]

$$-\frac{\hbar^2}{2}\frac{d^2}{dy^2}\psi(y) + [\Lambda_0^2\cosh y + u]\psi(y) = 0\ . \tag{5}$$

Now, we develop the ODE/IM procedure for this simple model summarizing the procedure of [10]. The regular solutions at $y \longrightarrow \pm\infty$ are determined by

$$\psi_{\pm,0}(y) \simeq 2^{-\frac{1}{2}}e^{-\frac{1}{2}\theta\mp\frac{1}{4}y}e^{-e^{\theta\pm y/2}}\,, \qquad y \longrightarrow \pm\infty\,. \tag{6}$$

Equation (4) enjoys the discrete symmetries

$$\Omega_{\pm}: \quad y \longrightarrow y \pm i\pi\ , \quad \theta \longrightarrow \theta + i\pi/2\ , \quad P \longrightarrow P\ . \tag{7}$$

Thanks to these symmetries, one can define other independent solutions $\psi_{-,k} = \Omega_-^k\psi_{-,0}$, $\psi_{+,k} = \Omega_+^k\psi_{+,0}$, and has these invariance properties $\Omega_+^k\psi_{-,0} = \psi_{-,0}$, $\Omega_-^k\psi_{+,0} = \psi_{+,0}$. These solutions are normalized so that their wronskians are $W[\psi_{-,k+1},\psi_{-,k}] = -i$, $W[\psi_{+,k+1},\psi_{+,k}] = i$.

In general in the ODE/IM method the Q function is defined as the wronskian of two regular solutions $\psi_{+,0},\psi_{-,0}$ at different singular points $y \longrightarrow \pm\infty$:

$$Q(\theta,P) = W[\psi_{+,0},\psi_{-,0}](\theta,P)\,. \tag{8}$$

By the properties of wronskians, we can write the linear central connection relations as

$$i\psi_{-,0} = Q(\theta+i\pi/2)\psi_{+,0} - Q(\theta)\psi_{+,1} \qquad i\psi_{-,1} = Q(\theta+i\pi)\psi_{+,0} - Q(\theta+i\pi/2)\psi_{+,1} \tag{9}$$

and taking their wronskian we obtain the QQ system

$$Q(\theta+i\pi/2)Q(\theta-i\pi/2) = 1 + Q(\theta)^2\,, \tag{10}$$

from which we can derive all the theory. Here and in the following we can omit the dependence on P as it stays fixed. As crucially noted in [13], the QNMs are defined as the zeroes of the same Wronksian (8) [15], namely the Bethe roots

$$Q(\theta_n) = 0\,. \tag{11}$$

The ODE/IM construction goes further because the presence of the irregular singularity of (4) at $y \to \pm\infty$ allow us to define T function (Stokes coefficient) as

$$T(\theta) = -iW[\psi_{-,-1}, \psi_{-,1}](\theta, P)\,. \tag{12}$$

Then from the lateral connection relation expressing $\psi_{\pm,1}$ it terms of of $\psi_{\pm,0}$ and $\psi_{\pm,-1}$ we find

$$T(\theta)Q(\theta) = Q(\theta - i\pi/2) + Q(\theta + i\pi/2)\,, \tag{13}$$

that is called TQ relation. Now, within our set-up of functional and integral equations for entire functions in θ (integrability), we can find other quantization conditions on the roots θ_n (QNMs). the TQ relation (13) means $Q(\theta_n - i\pi/2) + Q(\theta_n + i\pi/2) = 0$. This and the QQ relation (10) actually fixes $Q(\theta_n + i\pi/2)Q(\theta_n - i\pi/2) = 1$ and then it is also fixed

$$Q(\theta_n \pm i\pi/2) = \pm i\,. \tag{14}$$

Now we are going to prove that condition (14) is equivalent to the quantization condition of the dual gauge period a_D

$$a_D(\theta_n + i\pi/2, -u, \Lambda_0) = \frac{1}{2}\left(n + \frac{1}{2}\right)\,, \qquad n \in \mathbb{N}\,. \tag{15}$$

as found heuristically in [4]. Now (15) follows directly from (14) and the identification [10]

$$Q(\theta, P) = \exp\{2\pi i a_D(\theta, u, \Lambda_0)\}\,. \tag{16}$$

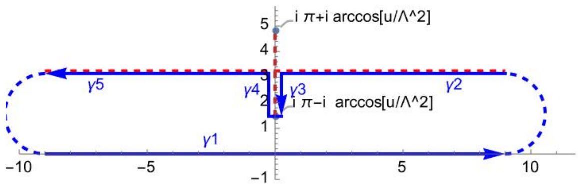

Figure 1: The y complex plane, where in blue we show the contour of integration of regularized SW differential $\mathcal{P}_{reg,-1}$ to prove the relation between $a_D^{(0)}$ and $\ln Q^{(0)}$. In red are shown the branch cuts of $\mathcal{P}_{reg,-1}$.

We give here an idea of how to prove this relation by considering the leading $\theta \to +\infty$ (or $\hbar \to 0$, Seiberg-Witten) order. Now as explained in [10] on one hand $a_D^{(0)}$ is the following integral for the leading order $\mathcal{P}_{-1} = -i\Lambda_0\sqrt{2\cosh y' + 2\frac{u}{\Lambda_0^2}}$ of the quantum momentum, called SW differential

$$a_D^{(0)}(u, \Lambda_0) = \frac{1}{2\pi}\int_{i\pi - i\arccos(u/\Lambda_0^2)+0^+}^{i\pi + i\arccos(u/\Lambda_0^2)+0^+} i\mathcal{P}_{-1}(y; u, \Lambda_0)\, dy\,. \tag{17}$$

On the other hand the leading order of $\ln Q$ is an integral of the same object for y from $-\infty$ to $+\infty$, but with a regularization. Since, in the limits $y \to \pm\infty$, we have $\mathcal{P}_{-1} = -i\frac{\Lambda_0}{\hbar}e^{\pm y/2} + O(e^{\mp y/2})$, it follows that this is (cf. [10])

$$\ln Q^{(0)}(u, \Lambda_0) = \int_{-\infty}^{\infty} i\mathcal{P}_{reg,-1}(y)\, dy = \Lambda_0 \int_{-\infty}^{\infty}\left[\sqrt{2\cosh y + 2\frac{u}{\Lambda_0^2}} - 2\cosh\frac{y}{2}\right] dy\,. \tag{18}$$

Assuming $u < \Lambda_0^2$, let us consider the integral of $i\mathcal{P}_{reg,-1}(y)$ on the (oriented) closed curve $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4 \cup \gamma_5$ which runs along the real axis, slightly below the cut and closes laterally, as depicted in 1. It turns out that the integrals on γ_2 and γ_5 cancel each other (cf. [10]). In the integrals on γ_3 and γ_4 there is no contribution from the regularizing part, which has no cut. Besides, by antisymmetry symmetry of $\mathcal{P}_{-1}(y)$ its integral on $\gamma_4 = (i\pi - i\arccos(u/\Lambda_0^2) + 0^-, i\pi + 0^-)$ is equal to the integral on the integral on $(i\pi + 0^+, i\pi + i \arccos{(u/\Lambda_0^2)} + 0^+)$. So by Cauchy theorem the integral on all γ is zero and we find the relation

$$\begin{aligned} \int_{-\infty}^{+\infty} i\mathcal{P}_{reg,-1}(y)\, dy &= \frac{1}{2\pi} \int_{i\pi - i \arccos{(u/\Lambda_0^2)}+0^+}^{i\pi + i \arccos{(u/\Lambda_0^2)}+0^+} i\mathcal{P}_{-1}(y; u, \Lambda_0)\, dy\,, \\ \ln Q^{(0)}(u, \Lambda_0) &= 2\pi i a_D^{(0)}(u, \Lambda_0)\,, \end{aligned} \tag{19}$$

which is the leading order version (16). The exact proof is similar and its details can be found in [10] and that proves (15) (notice $u \to -u$ as $\theta \to \theta + i\pi/2$ since P is fixed) [13, 14].

3. A new method of computation of quasinormal modes

Now we can use our connection among integrability, gauge and black hole theories to derive a new numerical method of computation of QNMs. To this end, let us now define the Y function as $Y(\theta, P) = Q^2(\theta, P)$ and derive from (10) the Y-system

$$Y(\theta + i\pi/2)Y(\theta - i\pi/2) = \Big(1 + Y(\theta)\Big)^2\,. \tag{20}$$

Eventually, we solve it explicitly (*up to quadratures*) via a Thermodynamic Bethe Ansatz (TBA) integral equation for the pseudoenergy $\varepsilon(\theta) = -\ln Y(\theta)$:

$$\varepsilon(\theta) = \frac{16\sqrt{\pi^3}}{\Gamma(\frac{1}{4})^2} e^{\theta} - 2\int_{-\infty}^{\infty} \frac{\ln\left[1 + \exp\{-\varepsilon(\theta')\}\right]}{\cosh(\theta - \theta')} \frac{d\theta'}{2\pi}\,. \tag{21}$$

In this P does not appear explicitly, but fixes the solution by its asymptotic linear behaviour $\varepsilon(\theta, P) \simeq +8P\theta$, $P > 0$, at $\theta \to -\infty$. Eventually, the QQ system (10) characterizes the QNMs as $Y(\theta_n - i\pi/2) = -1$, *i.e.* the *TBA quantization condition*

$$\varepsilon(\theta_n - i\pi/2) = -i\pi(2n+1)\,, \qquad n \in \mathbb{Z} \tag{22}$$

which can be easily implemented by using the TBA (21) as table 1 shows. These values match very well with those obtained by the standard method of continued fractions by Leaver [13, 16].

n	l	TBA	Leaver
0	0	$1.36912 - 0.504048i$	$1.36972 - 0.504311i$
0	1	$2.09118 - 0.501788i$	$2.09176 - 0.501811i$
0	2	$2.8057 - 0.501009i$	$2.80629 - 0.501000i$
0	3	$3.51723 - 0.500649i$	$3.51783 - 0.500634i$
0	4	$4.22728 - 0.500453i$	$4.22790 - 0.500438i$

Table 1: Comparison of QNMs of the D3 brane from TBA (21) (through (22) with $n = 0$), Leaver method (with $L = 1$).

Let us conclude with a comparison of computation methods for QNMs. The standard analytic method is the one with the continued fractions by Leaver and is typically quite laborious. We found it to be not always applicable in its original form even if there is a further development, the so-called matrix Leaver method, which is still applicable [17]. Application of $\mathcal{N} = 2$ gauge theory is a new interesting analytic characterization of QNMs. In particular in this method the computation of a_D in [4] relies on the expansion of the prepotential $\mathcal{F}$ in powers (number of instantons) of Λ_0^4 [18]: the period a is related to the moduli parameter u through the Matone's relation [19, 20]. In this respect, only the first instanton contributions are easily accessible and summing them up is accurate

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as long as $|\Lambda_0|/\hbar \ll 1$. Thus, this makes hard to access QNMs values $|\Lambda_{0,n}|/\hbar \gg 1$ as in table 1. On the contrary, our new method through TBA automatically resums all instantons. It is naturally derived by extending the ODE/IM correspondence, and it seems so far extendible to more general cases [13, 14].

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