PROCEEDINGS OF SCIENCE

A large family of IRF solvable lattice models based on WZW models

Vladimir Belavin,[a] **Doron Gepner**[b] **and J. Ramos Cabezas**[a,*]

[a] *Department of Physics, Ariel University, Ariel 40700, Israel.*
[b] *Department of Particle Physics and Astrophysics, Weizmann Institute, Rehovot 76100, Israel.*
E-mail: vladimirbe@ariel.ac.il, Doron.gepner@weizmann.ac.il, juanjose.ramoscab@msmail.ariel.ac.il

We consider the lattice models based on affine algebras described by Jimbo et al., for the algebras $ABCD$ and by Kuniba et al. for G_2. We find a general formula for the crossing multipliers of these models, with the result that these crossing multipliers are given by the principally specialized characters of the model in question. Therefore we conjecture that the crossing multipliers in a large class of solvable interaction round the face lattice models are given by the characters of the conformal field theory on which they are based. We also elaborate on some details of the computation of the Local state probability of these models, conjecturing an expression for it.

41st International Conference on High Energy physics - ICHEP2022
6-13 July, 2022
Bologna, Italy

PoS(ICHEP2022)432

*Speaker

1. Introduction

Solvable lattice models in two dimensions are a fruitful ground to test phase transitions, universality and two-dimensional condensed matter systems. For a review, see [1]. In ref. [2] an approach to solvable Interaction Round the Face (IRF) lattice models was presented where the lattice model itself is built out of the data of some conformal field theory and two primary fields in this conformal field theory. On each vertex of the model sits a primary field in the theory and the admissibility condition is given by the fusion rules. For a review of conformal field theory (CFT), see, e.g., [3].

The solution of ref. [2] is based on the Baxterization of the braiding matrix and is a trigonometric solution to the Yang Baxter equation (YBE). Our aim here is to extend this solution to the elliptic (thermalized) case. In ref. [4], the inversion relations of the general elliptical IRF model were conjectured and based on this, the free energy was calculated in the four main regimes. Our aim here is to conjecture the crossing relation in the elliptic case and in particular the crossing multiplier.

In ref. [2], the crossing multiplier for the trigonometric solution was proposed to be given by the modular matrix. We extend this result to the elliptic case by conjecturing that the crossing multipliers are given, in general, by the characters.

We check this statement in models where the crossing multiplier is known explicitly. These are the WZW A_n models [5], the BCD models [6] and the G_2 model [7]. We find a general formula for the crossing multiplier when the CFT is given by a WZW model (for a review of WZW models, see [3, 8]). This formula asserts that the crossing multiplier is given by the principally specialized character. This agrees with our general conjecture for the crossing multiplier. Then, we turn to the computation of the Local state probability (LSP). There we use the expression of the crossing multiplier given by the character and two conjectured inversion relations (specified below) to argue that the LSP could be expressed in terms of the branching function of a coset algebra.

2. The IRF models and their crossing relations

We define the IRF lattice models based on some rational conformal field theory (RCFT), O, and a pair of primary fields in this RCFT denoted by h and v [2]. The model is denoted accordingly by IRF(O, h, v). For simplicity, we assume that $h = v$. We define the models on a square lattice, where on each vertex sits some primary field. We assume that the face Boltzmann weight vanishes unless the admissibility condition is obeyed, which is,

$$f^b_{a,h} > 0, \quad f^d_{c,h} > 0, \quad f^c_{a,h} > 0, \quad f^d_{b,h} > 0, \tag{1}$$

where a, b, c, d are the four primary fields sitting on a face and $f^z_{x,y}$ is the fusion coefficient in the RCFT O. For an explanation of these notions see e.g. [2, 3]. The partition function of the model is

$$Z = \sum_{\text{configurations}} \prod_{\text{faces}} \omega \begin{pmatrix} a & b \\ c & d \end{pmatrix}\!\!\left| u \right), \tag{2}$$

where ω is the Boltzmann weight, and u is the spectral parameter.

We wish to define the Boltzmann weight ω in such a way that the model will be solvable. Namely, that the transfer matrices will commute for different spectral parameters. This is guaranteed by the Yang–Baxter equation (YBE), see, e.g., [1]. It is simpler to define this equation in operator form. For this, we define the operator,

$$< a_1, a_2, \ldots, a_n | R_i(u) | a'_1, a'_2, \ldots, a'_n >= \omega \begin{pmatrix} a_{i-1} & a_i \\ a'_i & a_{i+1} \end{pmatrix} u \Bigg) \prod_{\substack{m=1 \\ m \neq i}}^{n} \delta_{a_m, a'_m}. \tag{3}$$

Then the YBE assumes the form,

$$R_{i+1}(u) R_i(u+v) R_{i+1}(v) = R_i(v) R_{i+1}(u+v) R_i(u). \tag{4}$$

We utilize a trigonometric solution of the YBE, conjecturally, for any RCFT O and for any h and v, provided that the fusion coefficients of h and v are zero or one [2]. This solution is obtained by a Baxterization of the braiding matrix of h with v; we shall not need here the explicit solution. For details of the braiding matrix refer to [9]. Our focus will be on the crossing relation. This is given by

$$R^{h,\bar{h}} \begin{pmatrix} d & c \\ a & b \end{pmatrix} (u) = \left(\frac{\psi_a \psi_c}{\psi_b \psi_d} \right)^{1/2} R^{h,h} \begin{pmatrix} a & d \\ b & c \end{pmatrix} (l-u), \tag{5}$$

where l is the crossing parameter given by

$$l = \pi \Delta_{\mathrm{adjoint}}/2, \tag{6}$$

where $\Delta_{\mathrm{adjoint}}$ is the conformal dimension of the adjoint representation (assuming a WZW model or similar, see e.g. [3].) Generally, it is the conformal dimension of the lowest field in the fusion product of h and $\bar{h}$. We denoted by $R^{h,\bar{h}}$ and $R^{h,h}$ the trigonometric solution of the YBE based on the braiding of h with $\bar{h}$, or h with h, respectively.

The ψ_a in eq. (5) are called the crossing multipliers. These are given, conjecturally, by [2]

$$\psi_a = \frac{S_{a,0}}{S_{0,0}}, \tag{7}$$

where $S_{a,b}$ is the matrix of modular transformation for the primary fields a and b, and 0 denotes the unit primary field. For an explanation of these notions, see, e.g., [3].

We wish to describe the crossing relation for the elliptic solution of the YBE. Roughly, this is given by replacing $\sin u$ in the trigonometric solution with the theta function

$$\theta_1(u,q) = 2q^{1/8} \sin u \prod_{n=1}^{\infty} (1 - 2q^n \cos 2u + q^{2n})(1 - q^n), \tag{8}$$

where q is some parameter $0 < q < 1$, called the elliptic modulus. We call this, a thermalization of the IRF model.

Our wish is to conjecture the thermalization of the crossing relation. The thermal crossing relation remains the same as in eq. (5), except that we need to change the crossing multiplier. It is given by

$$\psi_a^t = \chi_a((q')^\alpha) / \chi_0((q')^\alpha), \tag{9}$$

where χ_a is the character in the RCFT O of the primary field a, defined as

$$\chi_a(q) = \sum_{\mathcal{H}_a} q^{\Delta - c/24}, \tag{10}$$

where $\mathcal{H}_a$ is the representation with the highest weight a and Δ is the dimension of the fields in this representation and c is the central charge. Since we will be considering ratios of characters, we can ignore the factor of c [1]. We define $q = \exp(2\pi i\tau)$ and its modular transformation $q' = \exp(-2\pi i/\tau)$. Here α is some exponent, which we will specify later.

We wish to show that in the critical limit $q \to 0^+$, the thermalized crossing multiplier becomes the critical crossing multiplier, ψ_a, eq. (7). In this limit, it is clear that $q' \to 1^-$. Then, using a modular transformation,

$$\chi_a(1) = \sum_b S_{a,b}\chi_b(0) = S_{a,0}, \tag{11}$$

since $\chi_b(0) = \delta_{a,b}$. Thus we find

$$\lim_{q\to 0} \psi^t_a(q) = S_{a,0}/S_{0,0} = \psi_a, \tag{12}$$

which is the desired relation.

The thermalized crossing relation, eqs. (5, 9) was established before in explicit IRF models, such as, the A_n height models of Jimbo et al. [5], the BCD height models [6] and the G_2 models by Kuniba et al. [7, 12, 13]. These models correspond in our language to IRF(O, h, h) where the RCFT O is a WZW model based on the algebras A_n, B_n, C_n, D_n, and G_2, respectively and the primary field h is the fundamental for A_n, the vector for B_n, C_n, D_n and the 7 representation for G_2. In all these examples, the crossing multiplier can be summarized neatly by the formula

$$\psi^t_a(q) = C \prod_{\alpha\in\overset{o}{\Delta}_+} \theta_1\left(\frac{\pi(\overset{o}{\lambda}_a + \overset{o}{\rho}, \alpha)}{k+g}, q\right), \tag{13}$$

where $\overset{o}{\lambda}_a$ is the finite counterpart of the highest weight of the representation a, $\overset{o}{\rho}$ is half the sum of finite positive roots (also known as the finite counterpart of the Weyl vector ρ), $\overset{o}{\Delta}_+$ are the finite positive roots of the algebra and the product $(.,.)$ denotes the product of two weights. k is the level of the WZW model, and g is the dual Coxeter number. C is an irrelevant constant. We conjecture that this formula, eq. (13), holds for all the WZW models IRF's even though it has not been worked out in detail for F_4 and E_n algebras.

We wish to make a connection between the formula for the crossing multiplier for WZW models for the group G, eq. (13) and the general formula eq. (9). We prove that (for details of the proof see [10]) $\psi^t_a(q)$ is given by

$$\psi^t_a(q) = \prod_{\alpha\in\overset{o}{\Delta}_+} \theta_1\left(\frac{\pi(\overset{o}{\lambda}_a + \overset{o}{\rho}, \alpha)}{k+g}, q\right) = \chi_a((q')^{g/(k+g)}), \tag{14}$$

[1] Actually, as we will see, we need to take the principal gradation in affine models, rather than the basic gradation as in eq. (10). This does not change the subsequent discussion. For the exact definition of the character in the principal gradation (or principally specialized character) see [10] or chapter 10 of [11].

where $\chi_a(q)$ is the principally graded character (also called principally specialized character) of the affine algebra $\hat{G}$ with the highest weight λ_a (its finite part is $\overset{o}{\lambda}_a$). Thus, we prove for WZW models eq. (9) with the exponent[2] $\alpha = \Delta_{\text{adjoint}} = g/(k+g)$.

3. Local state probability

We now turn to the calculation of the Local state probability. This is the probability $P(a|b,c)$ to find at the origin of the lattice (central vertex) some primary field a, given some boundary values b, c (which are taken to be in the ground state values), i.e.,

$$P(a|b,c) = \frac{Z_a}{Z}, \tag{15}$$

where Z_a is the partition function (2) that takes into account only those configurations which have at the origin of the lattice the value a. Baxter's corner transfer matrices method (described in [1]) is used to calculate the LSP. However, to use this method, it is required that the braiding matrices of the model (described in [2, 9]) satisfy the following two inversion relations

$$\begin{aligned} R_i^{h,h}(u) R_i^{h,h}(-u) &= \rho(u)\rho(-u) 1_i, \\ R_i^{h,\bar{h}}(u) R_i^{h,\bar{h}}(-u) &= \tilde{\rho}(u)\tilde{\rho}(-u) 1_i, \end{aligned} \tag{16}$$

where $\rho(u) = \prod_{r=0}^{n-2} \theta_1(\zeta_r - u, q)/\theta_1(\zeta_r, q)$, $\tilde{\rho}(u) = \prod_{r=0}^{n-2} \theta_1(\tilde{\zeta}_r - u, q)/\theta_1(\zeta_r, q)$, the theta function was defined in eq. (8), 1_i is the identity operator and $\zeta_r = \frac{\pi}{2}(\Delta_{r+1} - \Delta_r)$, $\tilde{\zeta}_r = \frac{\pi}{2}(\tilde{\Delta}_{r+1} - \tilde{\Delta}_r)$. Here Δ_r, $\tilde{\Delta}_r$ are the dimensions of the fields ψ_r, $\tilde{\psi}_r$ respectively appearing in the fusion products $h \cdot h = \sum_{r=0}^{n-1} \psi_r$, $h \cdot \bar{h} = \sum_{r=0}^{n-1} \tilde{\psi}_r$, and $r = 0, 1, \ldots, n-2$. Thus, if the crossing multipliers of the model are given by the character (14), and (16) holds, then, one can give arguments to conjecture that (see [10] for the details of these arguments), in the regime III of the model which is defined by $0 < q < 1$ and $0 < u < l$ and in the limit $q' \to 0$, $(q')^u \to fixed$, the LSP is given by

$$P(a|b,c) = \frac{\chi_a(x) B_{a,b,c}(x)}{\chi_b(x)\chi_c(x)}, \tag{17}$$

where $x = (q')^{g/(k+g)}$, and $B_{a,b,c}(x)$ is the branching function of a coset algebra that involves the initial affine Lie algebra $\hat{G}$. By using the definition of the branching function $\chi_b(x)\chi_c(x) = \sum_a \chi_a(x) B_{a,b,c}(x)$, one can see that (17) satisfies $\sum_a P(a|b,c) = 1$.

4. Conclusions

In this work, we have shown that the principally specialized character of integrable highest-weight representations of the untwisted affine Lie algebras is given by a formula that contains a product of n theta functions (where n is the numbers of positive roots of the corresponding finite Lie algebra). In turn, we have found that this character is the crossing multiplier described by Jimbo et al. [5, 6] for the algebras $ABCD$ and by Kuniba et al. [7] for G_2. We thus showed that the principally specialized character gives the crossing multiplier for affine theories.

[2]Do not confuse with the α in (14).

One of our main conjectures is that for the general IRF theory, built out of some conformal field theory data, the crossing symmetry holds, where the character of the CFT gives the crossing multiplier. This agrees with the affine case discussed above. We believe that this result is important in the study of lattice models and, in particular, for the computation of local state probabilities. In this respect, we also propose the conjecture (17), which relates the local state probability with the branching function of a coset algebra. The main examples for such models are WZW theories, see, [5, 6]. Other models based on other CFTs (coset of affine Lie algebras) are known in the literature, see [14]. However, for the most part, these models remain to be explored explicitly in the future.

References

[1] R. J. Baxter, *Exactly solved models in statistical mechanics*. Dover Publications, 1982.

[2] D. Gepner, *Foundations of Rational Quantum Field Theory, I*, hep-th/9211100.

[3] P. Di Francesco, P. Mathieu and D. Senechal, *Conformal Field Theory*. Graduate Texts in Contemporary Physics. Springer-Verlag, New York, 1997, 10.1007/978-1-4612-2256-9.

[4] D. Gepner, *On the free energy of solvable lattice models*, *Nucl. Phys. B* **971** (2021) 115532, [2011.14108].

[5] M. Jimbo, T. Miwa and M. Okado, *Local state probabilities of solvable lattice models: an* $A^{(1)}_{(n-1)}$ *family*, *Nucl. Phys. B* **300** (1988) 74–108.

[6] M. Jimbo, T. Miwa and M. Okado, *Solvable Lattice Models Related to the Vector Representation of Classical Simple Lie Algebras*, *Commun. Math. Phys.* **116** (1988) 507–525.

[7] A. Kuniba and J. Suzuki, *Exactly solvable G2(1) solid on solid model*, .

[8] E. Witten, *Nonabelian Bosonization in Two-Dimensions*, *Commun. Math. Phys.* **92** (1984) 455–472.

[9] G. W. Moore and N. Seiberg, *Polynomial Equations for Rational Conformal Field Theories*, *Phys. Lett. B* **212** (1988) 451–460.

[10] V. Belavin, D. Gepner and J. R. Cabezas, *The crossing multiplier for solvable lattice models*, *Journal of Physics Communications* **6** (2022) 085001.

[11] V. G. Kac, *Infinite-Dimensional Lie Algebras*. Cambridge University Press, 3 ed., 1990, 10.1017/CBO9780511626234.

[12] A. Kuniba, *Quantum R Matrix for G(2) and a Solvable 173 Vertex Model*, *J. Phys. A* **23** (1990) 1349.

[13] J. Suzuki, *Fusion U-q(G(2)(1)) vertex models and analytic Bethe ansatz*, *Phys. Lett. A* **195** (1994) 190, [hep-th/9405201].

[14] E. Baver, *New solvable lattice models from conformal field theory*, *Physics Letters B* **387** (1996) 502–512.