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From quantum to classical theories: the origin of the ODE/IM correspondence

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We exemplify our procedure on how to derive an Ordinary Differential Equation from a quantum system for the case of the quantum sine-Gordon model. In specific, we move from the functional relations and properties satisfied by Q-functions and derive a Lax pair, thus reversing the usual construction in the Ordinary Differential Equation/Integrable Model correspondence. The link between quantum and classical theory is provided by a Marchenko-like equation, which descends from the quantum TQ-system and is the basis for deriving the Lax pair. We discuss a special case of our construction, in which the classical problem is related to a Painlevé equation, and the massless (conformal) limit. Ostensibly, the whole construction can be modified by introducing moduli and further generalisations.

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1. Introduction

The history of the Ordinary Differential Equation/Integrable Model (ODE/IM) correspondence started as an intriguing duality which identifies connection coefficients between different pairs of solution of second (or higher) order ODEs with eigenvalues of Baxter T and Q operators for Conformal Field Theories (or better Minimal Models) [1, 2]. Later on, the correspondence was generalised to massive quantum integrable models [3, 4] by considering first order differential 2×2 matrix operators

$$D = \frac{\partial}{\partial w} + \frac{1}{2}\frac{\partial \hat{\eta}}{\partial w}\sigma^3 - e^{\theta+\hat{\eta}}\sigma^+ - e^{\theta-\hat{\eta}}\sigma^- , \quad \bar{D} = \frac{\partial}{\partial \bar{w}} - \frac{1}{2}\frac{\partial \hat{\eta}}{\partial \bar{w}}\sigma^3 - e^{-\theta+\hat{\eta}}\sigma^- - e^{-\theta-\hat{\eta}}\sigma^+ , \quad (1)$$

both annihilating a single wave function (Ψ)

$$D\Psi = 0 , \quad \bar{D}\Psi = 0 , \quad (2)$$

so that $\hat{\eta}$ must solve the classical sinh-Gordon equation $[D, \bar{D}] = 0$: the connection coefficients (Wronskians) between Jost solutions of (2) and those solutions with power-like behaviour around $w = 0$ coincide with vacuum eigenvalues of Baxter's Q-operators of quantum sine-Gordon model.

In this contribution we exemplify in a simple model how to revert the arrow of the correspondence. In other words, we start from a particular state of a 'simple' quantum integrable model and associate to it a couple of differential operators. In detail, we choose the (integrable) quantum sine-Gordon field theory on a cylinder with the circumference R, defined by the Lagrangian

$$\mathcal{L} = \frac{1}{16\pi}\left[(\partial_t\varphi)^2 - (\partial_x\varphi)^2\right] + 2\mu\cos\beta\varphi , \quad \varphi(x+R,t) = \varphi(x,t) .$$

The Hilbert space of the theory is labelled by $k \in [0,1]$, which dictates the response of a state $\Psi_k\rangle \longrightarrow e^{2\pi i k}|\Psi_k\rangle$ under the symmetry $\varphi \longrightarrow \varphi + 2\pi/\beta$. The infinite many conserved charges $I_n, \bar{I}_n$ of the model appear in the asymptotic expansion at $\theta \longrightarrow \pm\infty$ of commuting operators $\hat{Q}_\pm(\theta)$ ($\pm$ is the sign of k). We denote $Q_\pm(\theta)$ their eigenvalue on the k-vacuum state. These Q-functions enjoy important functional relations and properties. Here we list the ones which are important for our construction.

They are entire, quasi-periodic functions: $Q_\pm(\theta + i\tau) = e^{\pm i\pi(l+\frac{1}{2})}Q_\pm(\theta)$, $l = 2|k| - 1/2$, with quasi-period $\tau = \pi/(1-\beta^2)$. They satisfy the so-called TQ-system

$$T(\theta)Q_\pm(\theta) = e^{\mp i\pi(l+\frac{1}{2})}Q_\pm(\theta + i\pi) + e^{\pm i\pi(l+\frac{1}{2})}Q_\pm(\theta - i\pi) , \quad (3)$$

where the transfer matrix eigenvalue $T(\theta)$ is a quadratic construct in terms of fundamental Q-functions:

$$T(\theta) = \frac{i}{2\cos\pi l}\left[e^{-2i\pi l}Q_+(\theta+i\pi)Q_-(\theta-i\pi) - e^{2i\pi l}Q_+(\theta-i\pi)Q_-(\theta+i\pi)\right] .$$

Finally, the Q-functions have the asymptotic behaviour $\ln Q_\pm(\theta) \simeq -w_0 e^{\theta} - \bar{w}_0 e^{-\theta}$, $\quad w_0 = -\frac{MR}{4\cos\frac{\pi\beta^2}{2(1-\beta^2)}}$ when $\mathrm{Re}\theta \longrightarrow \pm\infty$, with M the soliton mass.

The aim of this paper is to associate a couple of differential operators to functions $Q_\pm$ satisfying these properties.

2. A Marchenko-like equation

The first step is to prove that a couple of functions satisfying previously mentioned quasi-periodicity, TQ-system and asymptotics coincide with the (unique) solutions of the integral equations

$$Q_\pm(\theta+i\tau/2)=q_\pm(\theta)\pm\int_{-\infty}^{+\infty}\frac{d\theta'}{4\pi}\tanh\frac{\theta-\theta'}{2}T\left(\theta'+i\frac{\tau}{2}\right)e^{-w_0(e^\theta+e^{\theta'})-\bar{w}_0(e^{-\theta}+e^{-\theta'})}\cdot$$
$$\cdot e^{\pm(\theta-\theta')l}Q_\pm\left(\theta'+i\frac{\tau}{2}\right),\quad q_\pm(\theta)=Ce^{\pm\frac{i\pi}{4}\pm\left(\theta+\frac{i\pi}{2}\right)l}e^{-w_0e^\theta-\bar{w}_0e^{-\theta}}. \tag{4}$$

The TQ-system holds due to the property (of the kernel):

$$\lim_{\epsilon\to0^+}\left[\tanh\left(x+\frac{i\pi}{2}-i\epsilon\right)-\tanh\left(x-\frac{i\pi}{2}+i\epsilon\right)\right]=2\pi i\delta(x),\quad x\in\mathbb{R},$$

the asymptotic is assured by the driving term $q_\pm(\theta)$ and the quasi-periodicity of the solutions is provided by comparing (4) with their complex conjugates.

Next step is to define the functions $X_\pm(\theta)=Q_\pm(\theta+i\tau/2)/q_\pm(\theta$ which satisfy an integral equation with 1 as a driving term. The crucial move now is to promote the constants $w_0,\bar{w}_0$ appearing in the exponents of (4) to dynamical variables: $w_0\to-iw'$, $\bar{w}_0\to i\bar{w}'$. Importantly, the transfer matrix T, which also depends (in a complicated way) on w_0, $\bar{w}_0$ is left unscathed. After these operations we are led to consider functions $X_\pm(w',\bar{w}'|\theta)$, which satisfy

$$X_\pm(w',\bar{w}'|\theta)=1\pm\int_0^{+\infty}\frac{d\lambda'}{4\pi\lambda'}\frac{\lambda-\lambda'}{\lambda+\lambda'}T(\lambda'e^{\frac{i\tau}{2}})e^{-2iw'\lambda'+2i\frac{\bar{w}'}{\lambda'}}X_\pm(w',\bar{w}'|\theta'),\quad\lambda=e^\theta. \tag{5}$$

In order to solve integral equation (5), we define the Fourier transform of $X_\pm-1$

$$K_\pm(w',\xi;\bar{w}')=\int_{-\infty-i\epsilon}^{+\infty-i\epsilon}d\lambda e^{i(\xi-w')\lambda}[X_\pm(w',\bar{w}'|\theta)-1]. \tag{6}$$

Because of (5), $X_\pm-1$ has a pole in λ on the real axis: then (6) is different from zero only if $\xi>w'$. Its inverse is

$$X_\pm(w',\bar{w}'|\theta)-1=\int_{-\infty}^{+\infty}\frac{d\xi}{2\pi}e^{-i(\xi-w')\lambda}K_\pm(w',\xi;\bar{w}'),\quad\lambda=e^\theta. \tag{7}$$

Then, we take the Fourier transform of (5). We get

$$K_\pm(w',\xi;\bar{w}')\pm F(w'+\xi;\bar{w}')\pm\int_{w'}^{+\infty}\frac{d\xi'}{2\pi}K_\pm(w',\xi';\bar{w}')F(\xi'+\xi;\bar{w}')=0,\quad\xi>w', \tag{8}$$

with $F(x;\bar{w}')=i\int_0^{+\infty}d\lambda'e^{-ix\lambda'+2i\frac{\bar{w}'}{\lambda'}}T(\lambda'e^{i\frac{\tau}{2}})$. Equation (8) has the structure of a Marchenko equation [5]. Marchenko equation is used [6] to derive from the knowledge of scattering data and bound states the potential of a Schrödinger equation. However, in the usual Marchenko equation [5] the driving term has the structure $F(x)=\int_{-\infty}^{+\infty}d\lambda e^{-ix\lambda}(S(\lambda)-1)+\sum_n S(\lambda_n)$, with S the S-matrix and λ_n the bound states. In contrast, in our construction scattering data and bound states are compactly encoded in T, the vacuum eigenvalue of the transfer matrix of sine-Gordon model. This is why we address to equation (8) as a Marchenko-like equation.

3. From Marchenko to Schrödinger

From the Marchenko-like equation (8) it is easy to derive a Schrödinger problem. We define the wave function $\psi_\pm(w', \bar{w}'|\theta) = e^{-iw'\lambda + i\frac{\bar{w}'}{\lambda}} X_\pm(w', \bar{w}'|\theta)$, where $X_\pm$ is given by (7); then, we differentiate twice $\psi_\pm$ w.r.t. w' and use the Marchenko-like equation (8). We end up with the Schrödinger equation

$$\frac{\partial^2}{\partial w'^2}\psi_\pm(w', \bar{w}'|\theta) + e^{2\theta}\psi_\pm(w', \bar{w}'|\theta) = u_\pm(w'; \bar{w}')\psi_\pm(w', \bar{w}'|\theta)\,, \tag{9}$$

with potential

$$u_\pm(w'; \bar{w}') = -2\frac{d}{dw'}\frac{K_\pm(w', w'; \bar{w}')}{2\pi}\,,$$

depending on the solution of (8) on the diagonal ($w' = \xi$). Equation (8) can be solved by Fourier transform and its solution gives both the potential and the wave function (through (7)). For the potential, we get the expression $u_\pm(w'; \bar{w}') = \mp\partial_{w'^2}\hat{\eta} + (\partial_{w'}\hat{\eta})^2$, where the real field $\hat{\eta}$ is expressed in terms of (logarithms) of Fredholm determinants:

$$\hat{\eta} = \ln\det(1 + \hat{V}) - \ln\det(1 - \hat{V})\,, \quad V(\theta, \theta') = \frac{T\left(\theta + i\frac{\tau}{2}\right)}{4\pi}\frac{e^{-2iw'e^{\theta} + 2i\bar{w}'e^{-\theta}}}{\cosh\frac{\theta - \theta'}{2}}\,.$$

For what concerns the wave function $\psi_\pm(w', \bar{w}'|\theta) = X_\pm(w', \bar{w}'|\theta)e^{-iw'\lambda + i\frac{\bar{w}'}{\lambda}}$, the information we get from the Marchenko equation is that it can be found from the solution of an integral equation:

$$X_\pm(w', \bar{w}'|\theta) = -2 \mp \int \frac{d\theta'}{4\pi} e^{\frac{\theta - \theta'}{2}} V(\theta, \theta') X_\pm(w', \bar{w}'|\theta')\,. \tag{10}$$

Using the shift property $\lim_{\epsilon\to 0^+}\left[\cosh^{-1}\left(x + \frac{i\pi}{2} - i\epsilon\right) - \cosh^{-1}\left(x - \frac{i\pi}{2} + i\epsilon\right)\right] = 2\pi\delta(x)\,, \quad x \in \mathbb{R}$ on (10), we find that the wave function satisfy a functional relation, the '$T\psi$-system'

$$T\left(\theta + i\frac{\tau}{2}\right)\psi_\pm(w', \bar{w}'|\theta) = \mp i\psi_\pm(w', \bar{w}'|\theta + i\pi) \pm i\psi_\pm(w', \bar{w}'|\theta - i\pi)\,,$$

which is an extension by the variables $w', \bar{w}'$ of the TQ-system (3).

To construct a Lax pair and get information on $\hat{\eta}$, we proceed through the following steps. First, we introduce a first order matrix equation (first Lax) $\mathbf{D}\Psi = 0$, where $\mathbf{D} = \begin{pmatrix} D_{\hat{\eta}} & 0 \\ 0 & D_{-\hat{\eta}} \end{pmatrix}$, with

$$D_{\hat{\eta}} = \partial_w + \frac{1}{2}\partial_w\hat{\eta}\,\sigma^3 - e^{\theta + \hat{\eta}}\sigma^+ - e^{\theta - \hat{\eta}}\sigma^-\,, \quad \Psi = \begin{pmatrix} e^{\frac{\theta + \hat{\eta}}{2}}\psi_+ \\ e^{-\frac{\theta + \hat{\eta}}{2}}(\partial_w + \partial_w\hat{\eta})\psi_+ \\ e^{\frac{\theta - \hat{\eta}}{2}}\psi_- \\ e^{-\frac{\theta - \hat{\eta}}{2}}(\partial_w - \partial_w\hat{\eta})\psi_- \end{pmatrix}$$

The first order matrix equation $\mathbf{D}\Psi = 0$ is equivalent to Schrödinger equations (9) in w'.

The second step is to start again from integral equation (5) and to change definition of Fourier transform

$$K_\pm^{bis}(w', \xi; \bar{w}') = \int_{-\infty - i\epsilon}^{+\infty - i\epsilon} d\lambda^{-1} e^{i(\xi + \bar{w}')\lambda^{-1}}\left[X_\pm(w', \bar{w}'|\theta) - 1\right]\,, \tag{11}$$

by exchanging the roles of $-w'$ and $\bar{w}'$. We get a Marchenko-like equation for $K_{\pm}^{bis}$, from which we derive a differential equation in $\bar{w}'$ for the wave function $\psi_{\pm}^{bis}(w', \bar{w}'|\theta) = e^{-iw'\lambda + i\bar{w}'\lambda^{-1}} X_{\pm}(w', \bar{w}'|\theta)$:

$$\frac{\partial^2}{\partial \bar{w}'^2}\psi_{\pm}^{bis}(w', \bar{w}'|\theta) + \lambda^{-2}\psi_{\pm}^{bis}(w', \bar{w}'|\theta) = \bar{u}_{\mp}(w', \bar{w}')\psi_{\pm}^{bis}(w', \bar{w}'|\theta) \,. \tag{12}$$

In an equivalent way we can write the second order differential problem (12) as (second Lax) $\bar{\mathbf{D}}\Psi^{bis} = 0$, where $\bar{\mathbf{D}} = \begin{pmatrix} \bar{D}_{\hat{\eta}} & 0 \\ 0 & \bar{D}_{-\hat{\eta}} \end{pmatrix}$, with

$$\bar{D}_{\hat{\eta}} = \partial_{\bar{w}} - \frac{1}{2}\partial_{\bar{w}}\hat{\eta}\,\sigma^3 - e^{-\theta+\hat{\eta}}\sigma^- - e^{-\theta-\hat{\eta}}\sigma^+ \,, \qquad \Psi^{bis} = \begin{pmatrix} e^{\frac{\theta-\hat{\eta}}{2}}(\partial_{\bar{w}} + \partial_{\bar{w}}\hat{\eta})\psi_{-}^{bis} \\ e^{-\frac{\theta-\hat{\eta}}{2}}\psi_{-}^{bis} \\ e^{\frac{\theta+\hat{\eta}}{2}}(\partial_{\bar{w}} - \partial_{\bar{w}}\hat{\eta})\psi_{+}^{bis} \\ e^{-\frac{\theta+\hat{\eta}}{2}}\psi_{+}^{bis} \end{pmatrix} .$$

Jost solutions of (12) are the functions $\psi_{\pm}^{bis}(w', \bar{w}'|\theta) = \bar{\psi}_{\mp}(w', \bar{w}'|-\bar{\theta}) = \psi_{\pm}(w', \bar{w}'|\theta)e^{\pm\hat{\eta}(w,\bar{w})}$. For these solutions the two four-vectors Ψ and Ψ^{bis} are proportional: $\Psi = -e^{\theta}\Psi^{bis}$. Then, we have $\mathbf{D}\Psi = \bar{\mathbf{D}}\Psi = 0$, which means that $[\mathbf{D}, \bar{\mathbf{D}}] = 0$: this conditon implies that

$$\partial_w \partial_{\bar{w}}\hat{\eta} = 2\sinh 2\hat{\eta} \,, \tag{13}$$

i.e. that $\hat{\eta}$ - which enters the potential of the Schrödinger problem - satisfies the classical sinh-Gordon equation.

We have then completed our inverse construction. We started from a quantum model (sine-Gordon) and we got a classical problem, Schrödinger equations (9, 12) or Lax pair $\mathbf{D}\Psi = \bar{\mathbf{D}}\Psi = 0$, with potentials determined by solutions of the classical sinh-Gordon equation.

4. Special limits and cases

Potentials $u_{\pm}(w', \bar{w}')$ of Schrödinger equations are complicated functions. However, simplifications occur in some limits or special cases. In this context, of particular importance is the so-called conformal limit, when the mass $(w_0) \to 0$. If we want to concentrate on 'left-movers', we send $\bar{w}' \to 0$ and scale w' as

$$\frac{dw'}{dx} = \sqrt{p(x)}e^{-\theta} \quad \theta \to +\infty$$

with $p(x) = x^{2M} - E$, $M = 1/\beta^2 - 1$. Then, the new wave function $\psi^{cft}(x) = \psi_{+}(w')p(x)^{-\frac{1}{4}}$ is found to satisfy the ODE

$$-\frac{d^2}{dx^2}\psi^{cft}(x) + \left(p(x) + \frac{l(l+1)}{x^2}\right)\psi^{cft}(x) = 0$$

which is ODE considered in the historical papers [1].

Staying in the off-critical case, an important particular case is when $\beta^2 = 2/3, l = 0$: then, a careful use of the QQ-system and the definition of the transfer matrix, shows that $T = 1$. This means that $\hat{\eta} = \ln\det(1 + \hat{V}) - \ln\det(1 - \hat{V})$, with

$$V(\theta, \theta') = \frac{e^{-2iw'e^{\theta} + 2i\bar{w}'e^{-\theta}}}{4\pi\cosh\frac{\theta-\theta'}{2}} \,,$$

depends only on $t = 4\sqrt{w'\bar{w}'}$. Consequently, the sinh-Gordon equation (13) reduces to the Painlevé III_3 equation:

$$\frac{1}{t}\frac{d}{dt}\left(t\frac{d}{dt}\hat{\eta}(t)\right) = \frac{1}{2}\sinh 2\hat{\eta}(t) .$$

5. Conclusions and perspectives

We have given a possible explanation for the occurrence of the ODE/IM correspondence. The idea is that the Baxter's TQ-functional relation for a state of a quantum model can be extended in such a way that its Fourier transformation yields a Marchenko-like equation. From this we obtain Schrödinger equations by differentiation and Fourier anti-transform, so that the original Q is a simple limit of the wave function. We have carried out our construction in the case of vacuum eigenvalues of $\hat{T}, \hat{Q}$ for sine-Gordon model. However, as showed in [7] the extension to the vacuum of Homogeneous sine-Gordon model is also possible, in spite of the non orthodox functional relations of the model [8].

As a future application we may mention the construction of Schrödinger equations corresponding to excited states of sine-Gordon and Homogeneous sine-Gordon models. In full generality, the TQ-relations are functional relations (equivalent to Bethe Ansatz equation) which define any integrable model, so that our constructive procedure can be applied, in principle, everywhere. As an alternative example, our procedure allows us to derive Schrödinger equations corresponding to generic states of a spin chain, namely an extension of the ODE/IM correspondence to a case in which the quantum model is not a quantum field theory.

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