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Three-loop planar integrals for four-point one-mass processes

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In this contribution we will present analytic expressions for the two tennis-court integral families relevant to three-loop 2→2 scattering processes involving one massive external particle and massless propagators in terms of Goncharov polylogarithms of up to transcendental weight six. Additionally, we will also present analytic expressions for physical kinematics for the ladder-box family and the two tennis-court families in terms of real-valued polylogarithmic functions, making these results well-suited for phenomenological applications.

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1. Introduction

Current and future precision LHC measurements require theoretical predictions of equal precision, in order to fully exploit the discovery potential of the machine [1–3]. When considering QCD corrections for $2 \to 2$ scattering processes which are relevant for LHC phenomenology, these are well under control at the Next-to-Next-to-Leading-Order (NNLO) in perturbation theory [4].

It is estimated however, that percent level precision of theoretical predictions for LHC observables will be required, which makes it necessary to push our understanding of perturbative QCD to N3LO [5]. One key aspect of these precision calculations, is our ability to compute multiloop Feynman integrals, which for N3LO corrections are typically at the level of three loops.

Currently, the most successful method for the computation of multiloop Feynman integrals is the method of differential equations [6–9]. For a given process, one identifies one or several so-called integral families. For each family a minimal set of Feynman integrals, often called *master integrals* (MI), is identified using Integration-By-Parts (IBP) identities [10, 11]. One then proceeds by deriving differential equations for these master integrals with respect to kinematic invariants and, if present, internal masses. The result of the differentiation is then written in terms of MI of the same integral family. The final ingredient is the determination of appropriate boundary terms for the solution of the differential equations.

In recent years, important results have been obtained through the use of the so-called *canonical differential equations* [12]. The important observation is that by carefully choosing a so-called canonical, or *pure* [13], basis of MI, the differential equation that they will satisfy will have only logarithmic singularities and the dependence of the dimensional regulator will be factored out. The mathematical simplicity of this kind of differential equations allows us, usually in cases involving few kinematic scales, to directly express their solution in terms of a well-studied class of special functions known us Multiple or Goncharov Polylogarithms (GPLs) [14–16]. In the simplest cases, these GPLs will have as arguments rational functions of the kinematic variables, while in more complicated ones, they can also include algebraic functions of the kinematics. Finally, it has been established that GPLs are not enough to span the space of functions that are needed to describe all relevant multiloop Feynman integrals, leading to several studies of a new class of functions called Elliptic integrals [17].

Concerning three-loop MI for $2 \to 2$ scattering, all MI involving four massless external particles have been recently computed in terms of GPLs [18, 19]. Regarding processes that involve up to one massive external particle, a planar family of MI associated to the so-called ladder topology, depicted in figure 1, was computed in [20, 21]. In these proceedings we report on a recent computation involving the remaining planar topologies [22], known as tennis-courts and depicted in figure 1.

For our calculation, we constructed canonical bases for the two planar families of MI and employed a variant of the method of differential equations, known as the Simplified Differential Equations (SDE) approach [23], to analytically compute them in terms of GPLs. We used the method of Expansion-by-Regions [24] to obtain all relevant boundary terms and then we analytically continued our solutions to physical regions of phase space using well-known properties of GPLs.

The remaining of this contribution is structured as follows. In section 2, we define the integral families under consideration and their kinematics. In section 3, we describe the main steps of the calculation and the validation of our results. We conclude in section 4. For more details, we refer the interested reader to [22].

2. Integral families

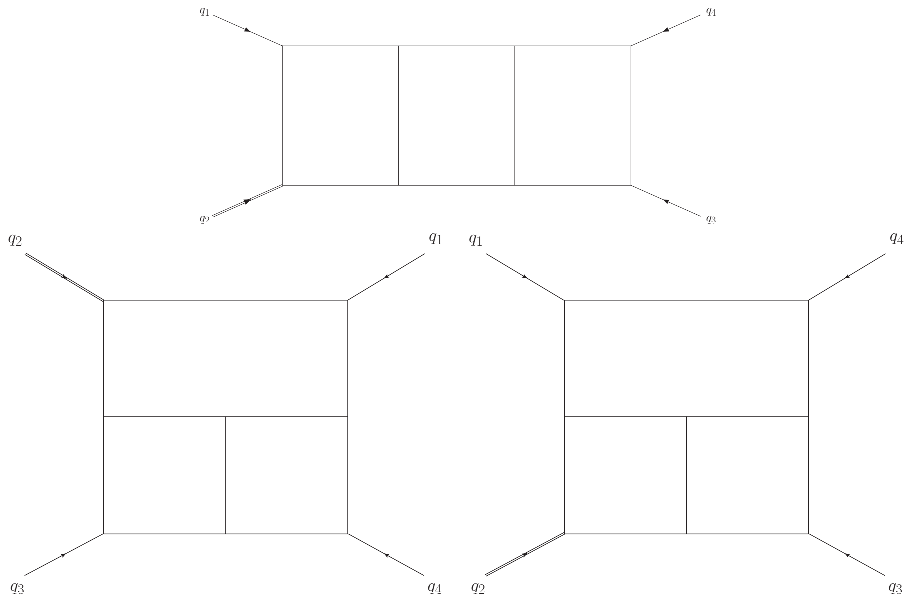

Figure 1: The F1 (top), F2 (bottom left) and F3 (bottom right) top-sector diagrams. The double line represents the massive particle and all external momenta are taken to be incoming.

We consider the calculation of families F2 and F3, whose top sector diagram can be seen in figure 1. Their integral representation is defined as follows[1]:

$$G^{F2}_{a_1\cdots a_{15}} := \int \left(\prod_{l=1}^{3} e^{\gamma_E \epsilon} \frac{d^d k_l}{i\pi^{d/2}}\right) \frac{(k_1+q_{123})^{-2a_{11}}\, k_2^{-2a_{12}}}{(k_1+q_{12})^{2a_1}\,(k_2+q_{12})^{2a_2}\,(k_2+q_{123})^{2a_3}} \times \frac{(k_2+q_1)^{-2a_{13}}\,(k_3+q_1)^{-2a_{14}}\,(k_3+q_{12})^{-2a_{15}}}{(k_3+q_{123})^{2a_4}\, k_3^{2a_5}\, k_1^{2a_6}\,(k_1+q_1)^{2a_7}\,(k_1-k_2)^{2a_8}\,(k_1-k_3)^{2a_9}\,(k_3-k_2)^{2a_{10}}} \tag{1}$$

$$G^{F3}_{a_1\cdots a_{15}} := \int \left(\prod_{l=1}^{3} e^{\gamma_E \epsilon} \frac{d^d k_l}{i\pi^{d/2}}\right) \frac{(k_1+q_{12})^{-2a_{11}}\, k_2^{-2a_{12}}}{(k_1+q_1)^{2a_1}\,(k_2+q_1)^{2a_2}\,(k_2+q_{12})^{2a_3}\,(k_3+q_{12})^{2a_4}} \times \frac{(k_2+q_{123})^{-2a_{13}}\, k_3^{-2a_{14}}\,(k_3+q_1)^{-2a_{15}}}{(k_3+q_{123})^{2a_5}\,(k_1+q_{123})^{2a_6}\, k_1^{2a_7}\,(k_1-k_2)^{2a_8}\,(k_1-k_3)^{2a_9}\,(k_3-k_2)^{2a_{10}}} \tag{2}$$

with a_i being integers and $a_i \leq 0$ for $i = 11, \ldots, 15$. The external momenta of the considered families obey the following kinematics: $\sum_{i=1}^{4} q_i = 0$, $q_2^2 = m^2$, $q_i^2 = 0$ for $i = 1, 3, 4$ and $S_{12} = (q_1+q_2)^2$, $S_{23} = (q_2+q_3)^2$, $S_{13} = m^2 - S_{12} - S_{23}$. When considering $2 \to 2$ processes with

[1] Where we use the abbreviation $q_{12} = q_1 + q_2$ and $q_{123} = q_1 + q_2 + q_3$.

one massive particle, we may identify four relevant phase space regions, one unphysical, Euclidean, region and 3 physical regions. These are,

$$\text{Euclidean}: m^2 < 0, \quad S_{12} < 0, \quad S_{23} < 0 \tag{3}$$
$$\text{s-channel}: m^2 > 0, \quad S_{12} \geq m^2, \quad S_{23} \leq 0, \quad S_{13} \leq 0 \tag{4}$$
$$\text{t-channel}: m^2 > 0, \quad S_{12} \leq 0, \quad S_{23} \geq m^2, \quad S_{13} \leq 0 \tag{5}$$
$$\text{u-channel}: m^2 > 0, \quad S_{12} \leq 0, \quad S_{23} \leq 0, \quad S_{13} \geq m^2. \tag{6}$$

Using automated IBP tools such as `Kira2` [25] and `FIRE6` [26] we identified 117 MI for family F2 and 166 MI for family F3.

3. Calculation

The first step of our calculation is to construct canonical bases for the two integral families under consideration. To do so, we relied on several different methods that allowed us to obtain canonical candidates. For several low sectors, involving up to seven propagators, we used the approach based on Magnus series expansions [27], which was already successfully applied for the construction of a pure basis for family F1 in [20]. For higher sectors, involving up to nine propagators, we used the `Mathematica` package `DlogBasis` [19] to identify appropriate candidates as pure basis elements. The last but most extensively used approach is a heuristic method based on [28], working loop-by-loop and using already known one, two- and three-loop pure basis elements [20, 29, 30].

The ultimate test that a canonical basis has been obtained is whether it satisfies a canonical DE. In this work we employed the SDE approach. More specifically, we parametrise the external momenta by introducing a dimensionless parameter x in the following manner

$$q_1 = xp_1, \quad q_2 = p_1 + p_2 - xp_1, \quad q_3 = p_3, \quad q_4 = p_4 \tag{7}$$

where the new momenta p_i are all massless. This parametrisation produces the following mapping for the kinematic invariants between the two momentum configurations

$$S_{12} = s_{12}, \quad S_{23} = s_{23}x, \quad m^2 = s_{12}(1-x) \tag{8}$$

with $s_{12} = (p_1+p_2)^2$, $s_{23} = (p_2+p_3)^2$. Since we will use the SDE approach for the solution of (1) and (2), we would like to have the corresponding limits for each region of phase-space expressed in terms of the $x,\ s_{12},\ s_{23}$ variables. The mapping of (8) allows us to do so, although for reasons that will become clear at a later stage, we define the ratio $y = \frac{s_{23}}{s_{12}}$ and use the variables $x,\ y,\ s_{12}$. Our approach therefore will be to compute all MI in terms of real-valued GPLs in the Euclidean region

$$0 < x < 1, \quad s_{12} < 0, \quad 0 < y < 1 \tag{9}$$

and then, using tools such as `HyperInt`[31] and `PolyLogTools`[32], analytically continue our solutions in the physical regions

$$\text{s-channel}: 0 < x < 1, \quad s_{12} > 0, \quad -1 \leq y \leq 0 \tag{10}$$
$$\text{t-channel}: 1 < x, \quad s_{12} < 0, \quad y \leq -1 \tag{11}$$
$$\text{u-channel}: 1 < x, \quad s_{12} < 0, \quad y \geq 0. \tag{12}$$

Having introduced the SDE parametrisation (7), the MI are now dependent on x through the external momenta. By differentiating with respect to x we were able to obtain the following SDE in canonical form for families F2 and F3,

$$\partial_x \mathbf{g} = \epsilon \left(\sum_{i=1}^{4} \frac{\mathbf{M}_i}{x - l_i} \right) \mathbf{g} \tag{13}$$

where $\mathbf{g}$ is the pure basis and $\mathbf{M}_i$ are the residue matrices corresponding to each pole l_i. All kinematic dependence is included in the poles l_i, leaving the matrices $\mathbf{M}_i$ to consist solely of rational numbers. We have found an alphabet consisting of the four following letters

$$\left\{ x,\ x-1,\ x-\frac{1}{1+y},\ x+\frac{1}{y} \right\}. \tag{14}$$

It is interesting to note here that the same letters were found in the case of the family F1 [21]. The form of (13) allows us to write down a general solution in terms of GPLs:

$$\begin{aligned}
\mathbf{g} = {} & \epsilon^0 \mathbf{b}_0^{(0)} + \epsilon \left(\sum \mathcal{G}_i \mathbf{M}_i \mathbf{b}_0^{(0)} + \mathbf{b}_0^{(1)} \right) + \epsilon^2 \left(\sum \mathcal{G}_{ij} \mathbf{M}_i \mathbf{M}_j \mathbf{b}_0^{(0)} + \sum \mathcal{G}_i \mathbf{M}_i \mathbf{b}_0^{(1)} + \mathbf{b}_0^{(2)} \right) + \ldots \\
& + \epsilon^6 \left(\mathbf{b}_0^{(6)} + \sum \mathcal{G}_{ijklmn} \mathbf{M}_i \mathbf{M}_j \mathbf{M}_k \mathbf{M}_l \mathbf{M}_m \mathbf{M}_n \mathbf{b}_0^{(0)} + \sum \mathcal{G}_{ijklm} \mathbf{M}_i \mathbf{M}_j \mathbf{M}_k \mathbf{M}_l \mathbf{M}_m \mathbf{b}_0^{(1)} \right. \\
& \left. + \sum \mathcal{G}_{ijkl} \mathbf{M}_i \mathbf{M}_j \mathbf{M}_k \mathbf{M}_l \mathbf{b}_0^{(2)} + \sum \mathcal{G}_{ijk} \mathbf{M}_i \mathbf{M}_j \mathbf{M}_k \mathbf{b}_0^{(3)} + \sum \mathcal{G}_{ij} \mathbf{M}_i \mathbf{M}_j \mathbf{b}_0^{(4)} + \sum \mathcal{G}_i \mathbf{M}_i \mathbf{b}_0^{(5)} \right)
\end{aligned} \tag{15}$$

where $\mathcal{G}_{ab\ldots} := \mathcal{G}(l_a, l_b, \ldots; x)$ represent the GPLs. The $\mathbf{b}_0^{(i)}$ terms represent the boundary terms that need to be determined, with i indicating the corresponding weight, and consist of Zeta functions $\zeta(i)$ and logarithms $\{\log(-s_{12}),\ \log(y)\}$ of weight i. Our results are presented in such a way that each coefficient of ϵ^i has transcendental weight i. If we assign weight -1 to ϵ, then (15) has uniform weight zero. We consider results up to weight six, which is the usual practise for three-loop computations.

In order to determine the necessary boundary terms $\mathbf{b}_0^{(i)}$ in (15) we will employ techniques developed in [21, 33], which rely on exploiting information from the canonical DE itself, as well as using the method of Expansion-by-Regions. We are interested in taking the limit $x \to 0$ limit as a boundary condition. The master equation that allows us in principle to compute all boundary terms can be written as

$$\mathbf{R}\mathbf{b} = \lim_{x \to 0} \mathbf{T}\mathbf{G} \Big|_{\mathcal{O}\left(x^{0+a_j\epsilon}\right)} \tag{16}$$

where on the left-hand-side we have information on the $x \to 0$ limit coming from the canonical DE (13) through the definition of the resummation matrix $\mathbf{R} = \mathbf{S} e^{\epsilon \mathbf{D} \log(x)} \mathbf{S}^{-1}$, with $\mathbf{M}_1 = \mathbf{S}\mathbf{D}\mathbf{S}^{-1}$. The right-hand-side of (16) comes from IBP reducing the canonical basis for each family in terms of individual Feynman integrals, $\mathbf{g} = \mathbf{T}\mathbf{G}$, and then using Expansion-by-Regions in order to obtain their asymptotic behaviour at $x \to 0$, $G_i \underset{x \to 0}{=} \sum_j x^{b_j + a_j \epsilon} G_i^{(b_j + a_j \epsilon)}$. Finally, apart from the terms $x^{a_i \epsilon}$, we expand around $x = 0$, keeping only terms of order x^0.

The above approach allows us to determine all top-sector boundaries in terms of lower-sector ones. For family F3 for example we have for boundary term b_{166},

$$
\begin{aligned}
b_{166} =& -\frac{1531b_1}{4752} - \frac{128b_2}{297} + \frac{47b_4}{33} - \frac{1891b_5}{396} + \frac{74b_{10}}{9} + \frac{20b_{11}}{3} + \frac{7b_{12}}{3} - \frac{127b_{13}}{36} \\
& - \frac{415b_{15}}{264} + \frac{13b_{16}}{8} + \frac{10b_{17}}{3} - \frac{47b_{18}}{36} - 2b_{19} + \frac{5b_{20}}{6} - \frac{21b_{22}}{16} - \frac{11b_{23}}{6} + \frac{5b_{24}}{12} \\
& - \frac{35b_{25}}{132} - \frac{6b_{26}}{11} + \frac{16b_{29}}{3} + \frac{32b_{30}}{9} - \frac{10b_{31}}{3} + \frac{581b_{35}}{132} + \frac{29b_{36}}{18} - \frac{197b_{38}}{33} \\
& + \frac{3b_{43}}{2} - \frac{14b_{49}}{3} + 7b_{52} - 5b_{53} - \frac{89b_{54}}{12} + \frac{13b_{57}}{3} - \frac{8b_{60}}{3} - \frac{b_{61}}{6} + 2b_{62} - \frac{7b_{77}}{33} \\
& - \frac{b_{81}}{6} + 3b_{83} - \frac{b_{84}}{2} - \frac{13b_{87}}{6} + \frac{7b_{88}}{12} - \frac{2b_{89}}{3} + \frac{5b_{97}}{6} - \frac{b_{108}}{3} - \frac{2b_{123}}{3} - b_{130} \\
& + \frac{2b_{137}}{3} + 2b_{144} - \frac{4b_{152}}{3} - \frac{2b_{159}}{3}.
\end{aligned}
$$

We can fix in a similar manner 109 boundary terms for family F3. The remaining boundary terms, assuming that we have already solved families F1 and F2, are

$$
\left\{b_{108}, b_{123}, b_{135}, b_{144}, b_{157}, b_{159}\right\}. \tag{17}
$$

Boundary term b_{108} can be obtained from (16)

$$
\begin{aligned}
b_{108} =& -2b_{19} + \frac{3b_{21}}{4} + s_{12}^2\epsilon^5 G^{(-2\epsilon)}_{111101012000000} \\
& + 4s_{12}\epsilon^4 G^{(-\epsilon)}_{1022010110-10000} - 3s_{12}^2\epsilon^4 G^{(-\epsilon)}_{112201001000000} \\
& + 6s_{12}\epsilon^5 G^{(0)}_{011101012000000}
\end{aligned} \tag{18}
$$

through the direct integration of region integrals in the Feynman parameter representation appearing in (18). In this particular case, we have to consider integrals with up to seven Feynman parameters. This is in general a non-trivial task to perform, although in this particular case we were able to compute b_{108} fully analytically.

For the remaining five boundary terms, $\left\{b_{123}, b_{135}, b_{144}, b_{157}, b_{159}\right\}$, if we were to continue with direct integration over Feynman parameters, we would have to consider integrals with up to nine parameters. This is a highly non-trivial computation in general and we found it inefficient for the determination of the remaining boundary terms. To move forward, we exploited the $x \longrightarrow 1$ limit of our integral families as well as their solution, which going back to (7), yields the massless limit of the corresponding integral families. More specifically, the remaining boundaries can be determined by the following procedure

1. Construct solution using ansatz for the undetermined boundary terms, i.e. $b_i = \sum_{k=0}^{6} a(i,k)\epsilon^k$.

2. Take $x \longrightarrow 1$ limit of the solution: $\tilde{\mathbf{g}} = \tilde{\mathbf{R}}_0\, \mathbf{g}_{reg}|_{x=1}$ [21].

3. Map the $x \rightarrow 1$ limit of F3, i.e. the massless tennis-court, to the known solution of the same family from cite.

Having all necessary boundary terms at hand, we now present our results by showcasing the structure of the GPL functions that appear in our final solution for each region of phase space, as well as the number of GPL functions with specific transcendental weight. Through the analytic

Regions	**Indices**	**Argument**	**Indices**	**Argument**
Euclidean	$\{0, 1, -1/y, 1/(1+y)\}$	x	–	–
s-channel	$\{0, 1, -1/y, 1/(1+y)\}$	x	–	–
t-channel	$\{0, 1, -y, 1+y\}$	$1/x$	$\{0, 1\}$	$-1/y$
u-channel	$\{0, 1, -y, 1+y\}$	$1/x$	$\{0, -1\}$	y

Table 1: Structure of GPLs appearing in each of the 4 kinematic regions.

R	$W = 1$	$W = 2$	$W = 3$	$W = 4$	$W = 5$	$W = 6$	**Total**	**Timings (sec)**
E	4	14	50	124	367	734	1293	39.0225769
s	4	14	50	124	367	734	1293	39.2172529
t	6	18	58	155	419	603	1259	62.0567800
u	5	16	54	147	403	572	1197	55.1049640

Table 2: Number of GPLs per weight and region, and timings for the numerical evaluation of the total GPLs.

continuation of our results, using packages such as `HyperInt` and `PolyLogTools`, we obtain expressions in all physical regions of phase space involving real-valued GPLs, which makes their numerical computation through `GiNaC` fast and stable.

The timings appearing in table 2 where obtained for the following points

$$\begin{aligned}
&\text{Euclidean}: \; s_{12} \to -7,\; y \to 3/7,\; x \to 1/4 \\
&\text{s-channel}: \; s_{12} \to 2,\; y \to -1/2,\; x \to 1/4 \\
&\text{t-channel}: \; s_{12} \to -2,\; y \to -3/2,\; x \to 5/3 \\
&\text{u-channel}: \; s_{12} \to -2,\; y \to 3/2,\; x \to 5/3.
\end{aligned}$$

For the same points we cross-checked our results against numerical results from `FIESTA4` [34] and `pySecDec` [35] and found excellent agreement. We also performed analytic checks at the limit $x \to 1$ against the results of [18] in the Euclidean region.

4. Conclusions

In these proceedings we reported on the recent calculation of the remaining three-loop planar topologies, known as tennis court topologies, for $2 \to 2$ scattering involving three massless and one massive external particle [22]. Our results are expressed in terms of real-valued GPLs for all physical regions of phase-space, which allows one to obtain fast and stable numerical evaluations for all MI. This is of key importance when considering the application of these solutions to phenomenological studies of scattering processes at particle colliders.

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References

[1] A. Huss, J. Huston, S. Jones and M. Pellen, *Les Houches 2021: Physics at TeV Colliders: Report on the Standard Model Precision Wishlist*, 2207.02122.

[2] S. Amoroso et al., *Les Houches 2019: Physics at TeV Colliders: Standard Model Working Group Report*, in *11th Les Houches Workshop on Physics at TeV Colliders: PhysTeV Les Houches*, 3, 2020 [2003.01700].

[3] G. Heinrich, *Collider Physics at the Precision Frontier*, *Phys. Rept.* **922** (2021) 1 [2009.00516].

[4] T. Gehrmann and B. Malaescu, *Precision QCD Physics at the LHC*, 2111.02319.

[5] F. Caola, W. Chen, C. Duhr, X. Liu, B. Mistlberger, F. Petriello et al., *The Path forward to N^3LO*, in *2022 Snowmass Summer Study*, 3, 2022 [2203.06730].

[6] A.V. Kotikov, *Differential equations method: New technique for massive Feynman diagrams calculation*, *Phys. Lett. B* **254** (1991) 158.

[7] A.V. Kotikov, *Differential equations method: The Calculation of vertex type Feynman diagrams*, *Phys. Lett. B* **259** (1991) 314.

[8] A.V. Kotikov, *Differential equation method: The Calculation of N point Feynman diagrams*, *Phys. Lett. B* **267** (1991) 123.

[9] T. Gehrmann and E. Remiddi, *Differential equations for two loop four point functions*, *Nucl. Phys. B* **580** (2000) 485 [hep-ph/9912329].

[10] K.G. Chetyrkin and F.V. Tkachov, *Integration by Parts: The Algorithm to Calculate beta Functions in 4 Loops*, *Nucl. Phys. B* **192** (1981) 159.

[11] S. Laporta, *High precision calculation of multiloop Feynman integrals by difference equations*, *Int. J. Mod. Phys. A* **15** (2000) 5087 [hep-ph/0102033].

[12] J.M. Henn, *Multiloop integrals in dimensional regularization made simple*, *Phys. Rev. Lett.* **110** (2013) 251601 [1304.1806].

[13] N. Arkani-Hamed, J.L. Bourjaily, F. Cachazo and J. Trnka, *Local Integrals for Planar Scattering Amplitudes*, *JHEP* **06** (2012) 125 [1012.6032].

[14] A.B. Goncharov, *Multiple polylogarithms, cyclotomy and modular complexes*, *Math. Res. Lett.* **5** (1998) 497 [1105.2076].

[15] C. Duhr, *Mathematical aspects of scattering amplitudes*, in *Theoretical Advanced Study Institute in Elementary Particle Physics: Journeys Through the Precision Frontier: Amplitudes for Colliders*, pp. 419–476, 2015, DOI [1411.7538].

[16] S. Abreu, R. Britto and C. Duhr, *The SAGEX Review on Scattering Amplitudes, Chapter 3: Mathematical structures in Feynman integrals*, 2203.13014.

[17] J.L. Bourjaily et al., *Functions Beyond Multiple Polylogarithms for Precision Collider Physics*, in *2022 Snowmass Summer Study*, 3, 2022 [2203.07088].

[18] J.M. Henn, A.V. Smirnov and V.A. Smirnov, *Analytic results for planar three-loop four-point integrals from a Knizhnik-Zamolodchikov equation*, *JHEP* **07** (2013) 128 [1306.2799].

[19] J. Henn, B. Mistlberger, V.A. Smirnov and P. Wasser, *Constructing d-log integrands and computing master integrals for three-loop four-particle scattering*, *JHEP* **04** (2020) 167 [2002.09492].

[20] S. Di Vita, P. Mastrolia, U. Schubert and V. Yundin, *Three-loop master integrals for ladder-box diagrams with one massive leg*, *JHEP* **09** (2014) 148 [1408.3107].

[21] D.D. Canko and N. Syrrakos, *Resummation methods for Master Integrals*, *JHEP* **02** (2021) 080 [2010.06947].

[22] D.D. Canko and N. Syrrakos, *Planar three-loop master integrals for $2 \to 2$ processes with one external massive particle*, *JHEP* **04** (2022) 134 [2112.14275].

[23] C.G. Papadopoulos, *Simplified differential equations approach for Master Integrals*, *JHEP* **07** (2014) 088 [1401.6057].

[24] B. Jantzen, A.V. Smirnov and V.A. Smirnov, *Expansion by regions: revealing potential and Glauber regions automatically*, *Eur. Phys. J. C* **72** (2012) 2139 [1206.0546].

[25] J. Klappert, F. Lange, P. Maierhöfer and J. Usovitsch, *Integral reduction with Kira 2.0 and finite field methods*, *Comput. Phys. Commun.* **266** (2021) 108024 [2008.06494].

[26] A.V. Smirnov and F.S. Chuharev, *FIRE6: Feynman Integral REduction with Modular Arithmetic*, *Comput. Phys. Commun.* **247** (2020) 106877 [1901.07808].

[27] M. Argeri, S. Di Vita, P. Mastrolia, E. Mirabella, J. Schlenk, U. Schubert et al., *Magnus and Dyson Series for Master Integrals*, *JHEP* **03** (2014) 082 [1401.2979].

[28] P. Wasser, *Analytic properties of Feynman integrals for scattering amplitudes*, Ph.D. thesis, Mainz U., 2018.

[29] J.M. Henn, K. Melnikov and V.A. Smirnov, *Two-loop planar master integrals for the production of off-shell vector bosons in hadron collisions*, *JHEP* **05** (2014) 090 [1402.7078].

[30] C. Dlapa, X. Li and Y. Zhang, *Leading singularities in Baikov representation and Feynman integrals with uniform transcendental weight*, *JHEP* **07** (2021) 227 [2103.04638].

[31] E. Panzer, *Algorithms for the symbolic integration of hyperlogarithms with applications to Feynman integrals*, *Comput. Phys. Commun.* **188** (2015) 148 [1403.3385].

[32] C. Duhr and F. Dulat, *PolyLogTools — polylogs for the masses*, *JHEP* **08** (2019) 135 [1904.07279].

[33] D.D. Canko, C.G. Papadopoulos and N. Syrrakos, *Analytic representation of all planar two-loop five-point Master Integrals with one off-shell leg*, *JHEP* **01** (2021) 199 [2009.13917].

[34] A.V. Smirnov, *FIESTA4: Optimized Feynman integral calculations with GPU support*, *Comput. Phys. Commun.* **204** (2016) 189 [1511.03614].

[35] S. Borowka, G. Heinrich, S. Jahn, S.P. Jones, M. Kerner, J. Schlenk et al., *pySecDec: a toolbox for the numerical evaluation of multi-scale integrals*, *Comput. Phys. Commun.* **222** (2018) 313 [1703.09692].

[36] D. Binosi, J. Collins, C. Kaufhold and L. Theussl, *JaxoDraw: A Graphical user interface for drawing Feynman diagrams. Version 2.0 release notes*, *Comput. Phys. Commun.* **180** (2009) 1709 [0811.4113].