Taming a resurgent ultra-violet renormalon

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Perturbative expansions in quantum field theory diverge for at least two reasons: the number of Feynman diagrams increases dramatically with the loop number and the process of renormalization may make the contribution of some diagrams large. We give an example of the second problem, from an ultra-violent renormalon of ϕ^3 theory in 6 dimensions, where we can compute to very high loop-order. Taming this renormalon involves recent work on resurgence. This challenge is much more demanding than the corresponding problem for Yukawa theory in 4 dimensions.
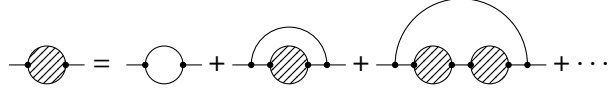
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1. Dyson-Schwinger equation and asymptotic expansion

Consider the perturbation expansion generated by the iteration of a single divergent diagram, via the non-linear Dyson-Schwinger equation [1–4],

contributing to the self-energy term Σ in the inverse propagator $q^2(1 - \Sigma)$, for a massless scalar particle with a ϕ^3 interaction, in the critical space-time dimension $D = 6$, for which the coupling constant is dimensionless.

The dependence of Σ on the external momentum q comes solely from renormalization. At n loops, we get a contribution that is a polynomial of degree n in $\log(q^2/\mu^2)$, multiplied by a^n where $a = \lambda^2/(4\pi)^3$, λ is the coupling constant and μ is the renormalization scale.

If we use momentum-space subtraction, so that Σ vanishes at $q^2 = \mu^2$, the dependence on momentum is completely determined by the anomalous dimension, with

$$\gamma(a) = - \left. q^2 \frac{\mathrm{d}\Sigma}{\mathrm{d}q^2} \right|_{q^2=\mu^2} \quad \text{giving} \quad \left. \frac{\mathrm{d}\log(1 - \Sigma)}{\mathrm{d}\log q^2} \right|_{q^2=\mu^2} = \gamma \left(\frac{a}{(1 - \Sigma)^2} \right). \tag{1}$$

The number of distinct diagrams at n loops is the number T_n of rooted trees with n nodes, which gives the sequence [5],

$$1, 1, 2, 4, 9, 20, 48, 115, 286, 719, 1842, 4766, 12486, 32973, 87811, 235381, \ldots$$

up to 16 loops. The iterated structure: tree = root + branches, with every branch being itself a tree, gives the asymptotic growth

$$T_n = \frac{b}{n^{3/2}} c^n (1 + O(1/n)) \tag{2}$$

$$b = 0.4399240125710253040409033914345447647980854079401 \ldots \tag{3}$$

$$c = 2.9557652856519949747148175241231945883754923046635 9 \ldots \tag{4}$$

At 250 loops, the number of Feynman diagrams is

```
T_250=51776375575461331089789949639841237225690858998065731 6
271041790137801884375338813698141647334732891545098109934676.
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This is not the main source of the problem. If the contribution of each diagram was bounded, there would be a finite radius of convergence for the perturbation expansion. The divergence of the series comes from renormalization, which makes the n-loop term grow factorially. This is called a *renormalon* singularity [6].

At 4 loops, we have a rainbow, a chain and two more interesting diagrams:

 (5)
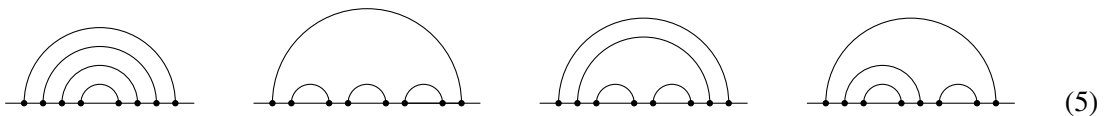
The sum of rainbows converges. Chains can be summed by Borel transformation [1].

$$\gamma_{\text{rainbow}} = \frac{3 - \sqrt{5 + 4\sqrt{1+a}}}{2} = -\frac{a}{6} + 11\frac{a^2}{6^3} - 206\frac{a^3}{6^5} + 4711\frac{a^4}{6^7} + O(a^5) \tag{6}$$

$$\gamma_{\text{chain}} = -\int_0^\infty \frac{6\exp(-6z/a)\mathrm{d}z}{(z+1)(z+2)(z+3)} = -\frac{a}{6} + 11\frac{a^2}{6^3} - 170\frac{a^3}{6^5} + 3450\frac{a^4}{6^7} + O(a^5) \tag{7}$$

$$\gamma = \sum_{n>0} G_n \frac{(-a)^n}{6^{2n-1}} = -\frac{a}{6} + 11\frac{a^2}{6^3} - 376\frac{a^3}{6^5} + 20241\frac{a^4}{6^7} + O(a^5) \tag{8}$$

with large integers G_n in the *alternating* asymptotic series for γ. Note that $G_4 = 20241 > 4711 + 3450$, because of two further diagrams in (5). In one we have a chain inside a double rainbow. In the other, a double rainbow is chained with the primitive divergence. This interplay is coded by rooted trees.

At 500 loops, the integer coefficient G_{500} has 1675 decimal digits. It was determined in work of the second author with Dirk Kreimer that resulted in a third-order differential equation [2],

$$8a^3\gamma\left\{\gamma^2\gamma''' + 4\gamma\gamma'\gamma'' + (\gamma')^3\right\} + 4a^2\gamma\left\{2\gamma(\gamma-3)\gamma'' + (\gamma-6)(\gamma')^2\right\}$$
$$+ 2a\gamma(2\gamma^2 + 6\gamma + 11)\gamma' - \gamma(\gamma+1)(\gamma+2)(\gamma+3) = a \tag{9}$$

with quartic non-linearity.

Interest in this problem came from Kreimer's discovery of the Hopf algebra of the iterated subtraction of subdivergences [7], whose utility was illustrated in this example, with a single primitive divergence leading to undecorated rooted trees.

The corresponding diagrams in Yukawa theory, in its critical dimension $D = 4$, give a first-order equation with merely quadratic non-linearity, which was solved using the complementary error function [2], thereby achieving explicit all-orders results for both the anomalous dimension and the self-energy. The expansion coefficients in this simpler case enumerate connected chord diagrams and an all-order resurgence analysis is possible [8].

The $D = 4$ and $D = 6$ examples were also investigated in the more cumbersome minimal subtraction scheme, where one retains finite parts of Σ at $q^2 = \mu^2$. Here one encounters unwieldy products of zeta values with weights that increase linearly with the loop-number. Recently, Paul-Hermann Balduf has shown how to absorb these into a rescaling of μ that can be expanded in the coupling a [9].

2. Padé-Borel summation with alternating signs

Broadhurst and Kreimer resummed the factorially divergent alternating series by an Ansatz [1]

$$\gamma(a) = -\frac{a}{6\Gamma(\beta)}\int_0^\infty P(ax/3)\exp(-x)x^{\beta-1}\mathrm{d}x, \quad P(z) = \frac{N(z)}{D(z)}. \tag{10}$$

The expansion coefficients of $P(z) = 1 + O(z)$ are obtained from those those of $\gamma(a)/a$ by dividing the latter by factorially increasing factors, producing a function P which was expected to have a finite radius of convergence in the Borel variable z, with singularities on the negative z-axis, as for the sum of chains.

The Padé trick is to convert the expansion of P, up to n loops, into a ratio N/D of polynomials of degrees close to $n/2$. Then one can check how well this method reproduces G_{n+1}. It was found that this works rather well with $\beta \approx 3$. For example, fitting the first 29 values of G_n with a ratio of polynomials of degree 14 gave a pole, coming from the denominator $D(z)$, at $z = -0.994$. The other 13 poles occurred further to the left, with $\Re z < -1$. Moreover the numerator $N(z)$ gave no zero with $\Re z > 0$. Then this method reproduced the first 15 decimal digits of G_{30}.

The first author, Gerald Dunne and Max Meynig have recently shown that this method works even better with $\beta = \frac{35}{12}$ [3], for reasons that we now explain.

3. Trans-series and resurgent hyperasymptotics

There is an old and rather loose argument, going back to Freeman Dyson in 1952 [10], that we should *not* expect realistic field theories to give convergent expansions in the square of a coupling constant. If they did, we could get sensible answers for a non-unitary theory with an imaginary coupling constant, such as an electrodynamics in which electrons repel positrons.

There is an amusing converse of this suggestion. If you find an expansion that is Borel summable, then study it at imaginary coupling. In the case of ϕ^3 theory the resulting non-unitary theory relates to the Yang-Lee edge singularity in condensed matter physics [11].

So now we recast the Broadhurst-Kreimer problem, in the manner of Borinksy, Dunne and Meynig [3], by setting $g(x) = \gamma(-3x)/x$, to obtain an ODE that is economically written as

$$(g(x)P - 1)(g(x)P - 2)(g(x)P - 3)g(x) = -3, \quad P = x\left(2x\frac{\mathrm{d}}{\mathrm{d}x} + 1\right), \tag{11}$$

and has an unsummable formal perturbative solution

$$g_0(x) \sim \sum_{n=0}^{\infty} A_n x^n = \frac{1}{2} + \frac{11}{24}x + \frac{47}{36}x^2 + \frac{2249}{384}x^3 + \frac{356789}{10368}x^4 + \frac{60819625}{248832}x^5 + O(x^6). \tag{12}$$

The expansion coefficients behave as

$$A_n = S_1 \Gamma\left(n + \frac{35}{12}\right)\left(1 - \frac{97}{48}\left(\frac{1}{n}\right) + O\left(\frac{1}{n^2}\right)\right), \tag{13}$$

at large n, with a Stokes constant

$$S_1 = 0.087595552909179124483795447421262990627388017406822\ldots \tag{14}$$

that can be determined, empirically, by considering a solution

$$g(x) = g_0(x) + \sigma_1 x^{-\beta} \exp(-1/x) h_1(x) + O(\sigma_1^2) \tag{15}$$

and retaining terms linear in σ_1 in the non-linear ODE. This yields a linear homogeneous ODE for $h_1(x)$, which permits a solution that is finite and regular at $x = 0$ if and only if $\beta = \frac{35}{12}$. Normalizing σ_1 by setting $h_1(0) = -1$, we obtain the expansion of

$$h_1(x) \sim \sum_{k=0}^{\infty} B_k x^k = -1 + \frac{97}{48}x + \frac{53917}{13824}x^2 + \frac{3026443}{221184}x^3 + \frac{32035763261}{382205952}x^4 + O(x^5) \tag{16}$$

which gives the first-instanton correction to the perturbative solution, suppressed by $\exp(-1/x)$.

By developing the series A_n and B_k, we were able to determine 3000 digits of S_1 [4] in

$$A_n \sim -S_1 \sum_{k \geq 0} \Gamma\left(n + \frac{35}{12} - k\right) B_k. \tag{17}$$

This is an example of *resurgence* [12]: information about A_n resurges in B_k, and vice versa, because both $A(x) = g_0(x)$ and $B(x) = h_1(x)$ know about the same physics.

Hyperasymptotic expansions concern the study of how B_n behaves at large n, which involves another set of numbers C_k, at small k, and so on, *ad infinitum*. They involve terms suppressed by $\exp(-m/x)$, with action $m > 1$. For this third-order ODE, there are 3 solutions to the linearized problem [3], namely

$$g(x) = g_0(x) + \sigma_m \left(x^{-\frac{35}{12}} e^{-\frac{1}{x}}\right)^m h_m(x) + O(\sigma_m^2), \quad m \in \{1, 2, 3\}, \tag{18}$$

with $h_2/x^5 = C$ and $h_3/x^5 = D$ finite and regular near the origin.

Then we use a linearized ODE to develop the expansions

$$C(x) = h_2(x)/x^5 = -1 + \frac{151}{24}x - \frac{63727}{3456}x^2 + \frac{7112963}{82944}x^3 - \frac{7975908763x}{23887872}x^4 + O(x^5), \tag{19}$$

$$D(x) = h_3(x)/x^5 = -1 + \frac{227}{48}x + \frac{1399}{4608}x^2 + \frac{814211}{73728}x^3 + \frac{3444654437}{42467328}x^4 + O(x^5). \tag{20}$$

This suggests to also study the higher order corrections in σ_1, σ_2 and σ_3, which leads to the *trans-series*. The trans-series organizes the higher order instanton corrections to the solution of the ODE and it neatly reflects the perpetuating low-order/large-order correspondence of the hyperasymptotic expansions. Before presenting the trans-series solution to (11), we remark on some of its general features.

1. The terms suppressed by $\exp(-2/x)$ involve σ_2 and σ_1^2. The former are given by C and the latter are determined by an inhomogeneous linear ODE, whose solution is ambiguous, up to a multiple of the homogeneous solution $h_2 = x^5 C$, since we can shift σ_2 by a multiple of σ_1^2.

2. In the terms suppressed by $\exp(-3/x)$ there a second ambiguity, since we can shift σ_3 by a multiple of σ_1^3.

3. Ambiguities of inhomogeneous solutions occur at places in expansions where logarithms first arise. This happens when the power of x in an expansion is a multiple of 5.

4. The highest power of $\log(x)$, in terms with action m, is $\lfloor m/2 \rfloor$.

The terms in the trans-series solution to (11) with action $m \leq 4$ are of the form [4],

$$g = \sum_{m \geq 0} g_m \left(x^{-\frac{35}{12}} e^{-\frac{1}{x}}\right)^m, \quad L = \frac{21265}{2304} x^5 \log(x), \tag{21}$$

$$g_0 = A, \quad g_1 = \sigma_1 B, \quad g_2 = \sigma_2 x^5 C + \sigma_1^2 (F + CL), \tag{22}$$

$$g_3 = \sigma_3 x^5 D + \sigma_1 \sigma_2 x^5 E + \sigma_1^3 (I + (D + E)L), \tag{23}$$

$$g_4 = \sigma_1 \sigma_3 x^5 G + \sigma_2^2 x^{10} H + \sigma_1^2 \sigma_2 x^5 (J + 2HL) + \sigma_1^4 (K + (G + J)L + HL^2). \tag{24}$$

Denoting the coefficients of x^n in functions by subscripts, we found that the choices

$$\frac{F_5}{2!} = \frac{I_5}{3!} = \frac{32642693907919}{36691771392} \tag{25}$$

greatly simplify our system of hyperasymptotic expansions. Then

$$B_n \sim -2S_1 \sum_{k \geq 0} F_k \Gamma(n + \tfrac{35}{12} - k)$$
$$+ 4S_1 \sum_{k \geq 0} C_k \Gamma(n - \tfrac{25}{12} - k) \left(\tfrac{21265}{4608} \psi(n - \tfrac{25}{12} - k) + d_1 \right), \tag{26}$$

$$d_1 = -43.332634728250755924500717390319380703460728022278\ldots \tag{27}$$

with $\psi(z) = \Gamma'(z)/\Gamma(z) = \log(z) + O(1/z)$, shows the $m = 1$ term, at large n, looking forward to $m = 2$ terms, at small k.

For the asymptotic expansion of the second-instanton coefficients, we found

$$C_n \sim -S_1 \sum_{k \geq 0} E_k \Gamma(n + \tfrac{35}{12} - k) + S_3 \sum_{k \geq 0} B_k (-1)^{n-k} \Gamma(n + \tfrac{25}{12} - k). \tag{28}$$

The first sum looks forwards to $m = 3$ in the trans-series, where coefficients of

$$E(x) = -4 + \tfrac{371}{12} x - \tfrac{111785}{1152} x^2 + \tfrac{8206067}{18432} x^3 - \tfrac{18251431003}{10616832} x^4 + O(x^5) \tag{29}$$

appear. It does not contain the coefficients D_k of the third instanton, which decouples from the asymptotic expansion for the second instanton.

The second sum in (28) has alternating signs, looks backwards to $m = 1$ and is suppressed by a factor of $1/n^{5/6}$. This can be understood using alien calculus. Likewise,

$$F_n \sim -3S_1 \sum_{k \geq 0} I_k \Gamma(n + \tfrac{35}{12} - k)$$
$$+ 2S_1 \sum_{k \geq 0} (3D_k + 2E_k) \Gamma(n - \tfrac{25}{12} - k) \left(\tfrac{21265}{4608} \psi(n - \tfrac{25}{12} - k) + d_1 \right)$$
$$- 2S_3 \sum_{k \geq 0} B_k (-1)^{n-k} \Gamma(n - \tfrac{35}{12} - k) \left(\tfrac{21265}{4608} \psi(n - \tfrac{35}{12} - k) + f_1 \right) \tag{30}$$

looks forwards to I_k, D_k and E_k, at $m = 3$, and backwards to B_k at, $m = 1$.

The new constants in (30) are

$$S_3 = 2.17178531405909902116086012279038923024794641930274\ldots \tag{31}$$
$$f_1 = -40.90369250922851500381447912690135478526366 9553014\ldots \tag{32}$$

Two more were discovered in the backward looking terms of

$$I_n \sim -4S_1 \sum_{k \geq 0} K_k \Gamma(n + \tfrac{35}{12} - k)$$

$$+2S_1 \sum_{k \geq 0} (3G_k + 2J_k)\Gamma(n - \tfrac{25}{12} - k) \left(\tfrac{21265}{4608}\psi(n - \tfrac{25}{12} - k) + d_1 \right)$$

$$-4S_3 \sum_{k \geq 0} F_k(-1)^{n-k}\Gamma(n - \tfrac{35}{12} - k) \left(\tfrac{21265}{4608}\psi(n - \tfrac{35}{12} - k) + f_1 \right)$$

$$-8S_3 \sum_{k \geq 0} C_k(-1)^{n-k}\Gamma(n - \tfrac{95}{12} - k)Q(n - \tfrac{95}{12} - k), \tag{33}$$

$$Q(z) = \left(\tfrac{21265}{4608} \right)^2 \left(\psi^2(z) + \psi'(z) \right) + 2c_1 \left(\tfrac{21265}{4608} \right)\psi(z) + c_2, \tag{34}$$

$$c_1 = -41.03195676430271058392106810154550945370489789818\ldots \tag{35}$$

$$c_2/c_1^2 = 1.0002016472131992595822805380838324188011572304276\ldots \tag{36}$$

We believe that 6 constants, $S_1, d_1, S_3, f_1, c_1, c_2$, suffice for the complete description of resurgence.

Conjecture [4]: The trans-series solution to (11) and its resurgence take the forms

$$g(x) = \sum_{m=0}^{\infty} \left(x^{-\frac{35}{12}} e^{-\frac{1}{x}} \right)^m \sum_{i=0}^{\lfloor m/2 \rfloor} \sum_{j=0}^{\lfloor (m-2i)/3 \rfloor} \sigma_1^{m-2i-3j}\widehat{\sigma}_2^i\widehat{\sigma}_3^j x^{5(i+j)} \sum_{n \geq 0} a_{i,j}^{(m)}(n)x^n, \tag{37}$$

$$\widehat{\sigma}_2 = \sigma_2 + \tfrac{21265}{2304}\sigma_1^2 \log(x), \quad \widehat{\sigma}_3 = \sigma_3 + \tfrac{21265}{2304}\sigma_1^3 \log(x), \tag{38}$$

$$a_{i,j}^{(m)}(n) \sim -(s+1)S_1 \sum_{k \geq 0} a_{i,j}^{(m+1)}(k)\Gamma(n + \tfrac{35}{12} - k)$$

$$+ S_1 \sum_{k \geq 0} \left(4(i+1)a_{i+1,j}^{(m+1)}(k) + 6(j+1)a_{i,j+1}^{(m+1)}(k) \right) \Gamma(n - \tfrac{25}{12} - k) \left(\tfrac{21265}{4608}\psi(n - \tfrac{25}{12} - k) + d_1 \right)$$

$$+ \tfrac{1}{4}S_3 \sum_{k \geq 0} \left(4(s+1)a_{i-1,j}^{(m-1)}(k) + 6(j+1)a_{i-2,j+1}^{(m-1)}(k) \right) (-1)^{n-k}\Gamma(n + \tfrac{25}{12} - k)$$

$$- 2(s - 2i - 1)S_3 \sum_{k \geq 0} a_{i,j}^{(m-1)}(k)(-1)^{n-k}\Gamma(n - \tfrac{35}{12} - k) \left(\tfrac{21265}{4608}\psi(n - \tfrac{35}{12} - k) + f_1 \right)$$

$$- S_3 \sum_{k \geq 0} \left(8(i+1)a_{i+1,j}^{(m-1)}(k) + 6(j+1)a_{i,j+1}^{(m-1)}(k) \right) (-1)^{n-k}\Gamma(n - \tfrac{95}{12} - k)Q(n - \tfrac{95}{12} - k)$$

$$- (f_1 - c_1)S_3 \sum_{k \geq 0} \left(2(i+1)a_{i+1,j-1}^{(m-1)}(k) + 6(i+j)a_{i,j}^{(m-1)}(k) \right) (-1)^{n-k}\Gamma(n - \tfrac{35}{12} - k), \tag{39}$$

with $s = m - 2i - 3j$ and $Q(z) = \left(\tfrac{21265}{4608} \right)^2 \left(\psi^2(z) + \psi'(z) \right) + 2c_1 \left(\tfrac{21265}{4608} \right)\psi(z) + c_2$.

4. Comments and conclusions

1. The conjecture exhibits 17 resurgent terms, all of which have been intensively tested at high precision, for all actions $m \leq 8$.

2. The 6 Stokes constants have been determined to better than 1000 digits.

3. Excellent freeware, from `Pari-GP` in Bordeaux, was vital to this enterprise.

4. First and second derivatives of Γ and suppressions by $1/n^{5/6}$ make Richardson acceleration infeasible. We used systematic matrix inversion.

5. The presence of logarithms in trans-series has been ascribed to resonant actions. We find this misleading. We have shown that a closely analogous second-order problem is both resonant and log-free.

Acknowledgments

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