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Progress in two-loop Master Integrals computation

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Over the last years, master integral families at one, two and three loops, with up to five external particles, including off-shell legs and internal masses have been computed analytically based on the Simplified Differential Equations approach. In this presentation we focus on the latest results for two-loop five-point Feynman Integrals with one off-shell leg. The three planar and one of the non-planar families have been fully expressed in terms of Goncharov polylogarithms. For the other two non-planar families, we introduce a new approach to obtain the boundary terms and establish a one-dimensional integral representation of the master integrals in terms of generalised polylogarithms, when the alphabet contains non-factorizable square roots. The results are relevant to the study of NNLO QCD corrections for W, Z and Higgs-boson production in association with two hadronic jets.

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1. Introduction

The upcoming High Luminosity upgrade of the LHC will provide us with experimental data of unprecedented precision. Making sense of the data and exploiting the machine's full potential will require theoretical predictions of equally high precision. In recent years, the theoretical community has made tremendous effort to meet the challenge of performing notoriously difficult perturbative calculations in Quantum Field Theory. The current precision frontier for the QCD dominated processes studied at the LHC lies at the Next-to-Next-to-Leading-Order (NNLO) for massless $2 \to 3$ scattering with one off-shell external particle [1, 2].

A typical NNLO calculation involves, among other things, the computation of two-loop Feynman diagrams [3]. The established method for performing such calculations is by solving first-order differential equations (DE) satisfied by the relevant Feynman integrals (FI) [4–7]. Working within dimensional regularisation in $d = 4 - 2\epsilon$ dimensions, allow us the derivation of linear relations in the form of Integration-By-Parts (IBP) identities satisfied by these integrals [8], which allows one to obtain a minimal and finite set of FI for a specific scattering process, known as master integrals (MI).

It has been conjectured that FI with constant leading singularities in d dimensions satisfy a simpler class of DE [9], known as canonical DE [10]. A basis of MI satisfying canonical DE is known as a *pure* basis. The study of the special functions which appear in the solutions of such DE has provided a deeper understanding of their mathematical properties. These special functions often admit a representation in the form of Chen iterated integrals [11]. For a large class of FI, their result can be written in terms of a well studied class of special functions, known as Multiple or Goncharov polylogarithms (GPLs) [12–14]. Several computational tools have been developed for their algebraic manipulation [15] and numerical evaluation [16, 17].

For the case of two-loop five-point MI with one massive leg, pure bases of MI have been recently presented in [18] for the planar topologies, which we will call *one-mass pentaboxes*, and more recently in [19] for some of the non-planar topologies, which we will call *one-mass hexaboxes*. All one-mass pentaboxes have been computed both numerically [18], using generalised power-series expansions [20, 21], as well as analytically in terms of GPLs [22, 23], by employing the Simplified Differential Equations (SDE) approach [24]. Recently, analytic results were also obtained in the form of Chen iterated integrals and have been implemented into the so-called *one-mass pentagon functions* [25], similar to the two-loop five-point massless results [26, 27]. These results, along with fully analytic solutions for the relevant one-loop integral family [28], have lead to the production of the first phenomenological studies at the leading-colour approximation for $2 \to 3$ scattering processes involving one massive particle at the LHC [29–31]. For the one-mass hexabox topologies, numerical results were first presented in [32], using a method which emulates the Feynman parameter technique, for one of the non-planar integral families. All three integral families were treated numerically in [19] using the same methods as in [18] and analytically using the SDE approach in [33].

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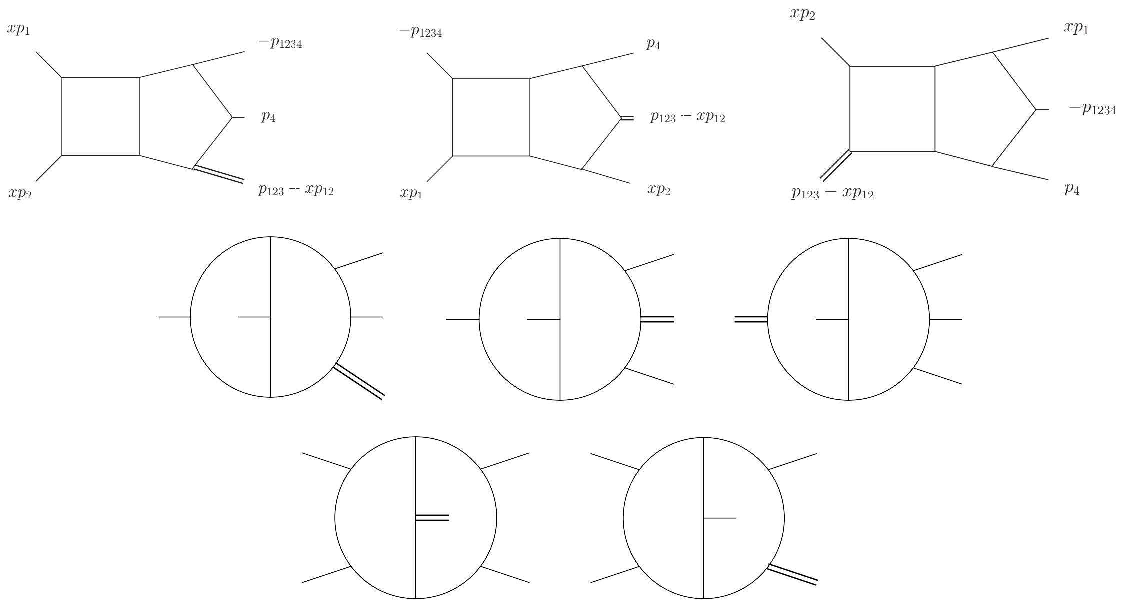

Figure 1: The eight families with one external massive leg. The first row corresponds to the so-called planar pentaboxes (from left to right P_1, P_2, P_3), the second to the hexabox topologies (from left to right N_1, N_2, N_3), whereas the diagrams of the third row are known as double-pentagons (from left to right N_4, N_5). All diagrams have been drawn using `Jaxodraw` [34].

2. The Simplified Differential Equations approach

In figure 1, Feynman Integrals at the top sector corresponding to all eight different families are shown, with the convention that double lines for external legs are associated to off-shell momenta ($p^2 \neq 0$), single lines to light-like momenta, whereas internal single lines correspond to massless particles.

In the SDE approach [24] the external momenta are parametrized by introducing a dimensionless variable x, mapping the external momenta configuration $E = \{q_i; \quad i = 1, \ldots, N\}$ to $\tilde{E} = \{x\} \cup \{p_i; \quad i = 1, \ldots N\}$ with the property that if there are n off-shell momenta in E there are $n - 1$ off-shell momenta in $\tilde{E}$. To be more specific, let us consider the following mapping, as depicted in the first row of figure 1,

$$q_1 \to p_{123} - xp_{12}, \;\; q_2 \to p_4, \;\; q_3 \to -p_{1234}, \;\; q_4 \to xp_1 \tag{1}$$

where the new momenta $p_i, \; i = 1 \ldots 5$ satisfy $\sum_1^5 p_i = 0, \; p_i^2 = 0, \; i = 1 \ldots 5$, whereas $p_{i \ldots j} := p_i + \ldots + p_j$. The explicit mapping between the two sets of invariants, in $E : \{s_{ij}\}$ and $\tilde{E} : \{S_{ij}\}$, is given by

$$\begin{gathered} q_1^2 = (1 - x)(S_{45} - S_{12}x) \neq 0, \;\; s_{12} = (S_{34} - S_{12}(1 - x))\, x, \;\; s_{23} = S_{45}, \;\; s_{34} = S_{51}x, \\ s_{45} = S_{12}x^2, \;\; s_{15} = S_{45} + (S_{23} - S_{45})x \end{gathered} \tag{2}$$

and as usual the $x = 1$ limit corresponds to the on-shell kinematics.

Introducing pure bases, as presented in [18, 19], turns the system of differential equation satisfied by the elements of the basis $\vec{g}$, into the so-called d log or canonical form.

$$d\vec{g} = \epsilon \sum_i d\log(W_i)\,\tilde{\mathbf{M}}_a\vec{g} \tag{3}$$

where W_i are algebraic functions of the kinematics and $\tilde{\mathbf{M}}_i$ are matrices independent of the kinematical invariants, whose matrix elements are pure rational numbers. In the SDE approach the functions, in most cases, as for instance in the planar families and the first non-planar one, W_a assume a very simple form in terms of the variable x and the corresponding DE takes the form

$$\partial_x\mathbf{g} = \epsilon\left(\sum_{a=1}^{l_{max}}\frac{\mathbf{M}_a}{x-l_a}\right)\mathbf{g} \tag{4}$$

where l_a are functions of the kinematics *but independent* of x and $\mathbf{M}_a$ are matrices independent of the kinematical invariants, whose matrix elements are pure rational numbers. SDE provides a mechanism to fully factorise the $x-$dependence, and the DE becomes a Fuchsian ODE whose solution, up to the desired order in ϵ, can be directly cast in the form

$$\mathbf{g} = \sum_{i=0}^{4}\epsilon^i\sum_{k=0}^{i}\mathcal{G}\left(l_{a_1}\dots l_{a_{i-k}};x\right)\mathbf{M}_{a_1}\dots\mathbf{M}_{a_{i-k}}\mathbf{b}_0^{(k)} \tag{5}$$

where $\mathbf{g}$ and $\mathbf{M}$ are taken from Eq. (4), $\mathbf{b}_0^{(i)}$ are the boundary values of the basis elements in the limit $x\to 0$ (see Eq.(3.6) of reference [23]) at order ϵ^i, $i = 0\dots 4$ and $\mathcal{G}(l_{a_1}l_{a_2}\dots;x)$ stands for Goncharov polylogarithms.

In the case of the two last non-planar hexabox topologies not all algebraic functions W_i factorise in x and the DE assumes the more general form

$$\partial_x\mathbf{g} = \epsilon\left(\sum_{a=1}^{l_{max}}\frac{dL_a}{dx}\mathbf{M}_a\right)\mathbf{g} \tag{6}$$

where most of the L_a are simple rational functions of x, as in (4), whereas the rest are algebraic functions of x involving the non-rationalisable square roots of the alphabet [33].

A detailed analysis of (6) reveals that these non-factorizable in x functions start appearing at weight two. In practise this means that we can use the mapping (2) and solve the respective canonical simplified DE by integrating with respect to x up to weight one in terms of ordinary logarithms. For weight two, analytic expressions in terms of GPLs can be achieved [1]. In fact, most of the basis elements are straightforwardly expressed in terms of GPLs by integrating the corresponding simplified DE. For the rest, an educated ansatz can be constructed involving only specific weight-two GPLs, which are identified by inspecting the DE, modulo the boundary terms that one needs to compute as we will discuss in the next section. Thus analytic expressions in terms of GPLs up to weight two are obtained for all elements belonging in these families. In fact only one element per family requires an ansatz, whereas there are three more elements in lower sectors that cannot be obtained by direct integration but are known in terms of GPLs up to weight 4 [22, 35], based though on different variants of the parametrisation (1).

[1]For details, please see reference [33]

3. Boundary terms

Our main approach is the one introduced in [22] and elaborated in detail in [36]. In general we need to calculate the $x \to 0$ limit of each pure basis element. At first we exploit the canonical SDE at the limit $x \to 0$ and define through it the resummation matrix

$$\mathbf{R} = \mathbf{S}\mathrm{e}^{\epsilon \mathbf{D} \log(x)} \mathbf{S}^{-1} \tag{7}$$

where the matrices $\mathbf{S}$, $\mathbf{D}$ are obtained through the Jordan decomposition of the residue matrix for the letter $l_1 = 0$, $\mathbf{M}_1$,

$$\mathbf{M}_1 = \mathbf{S}\mathbf{D}\mathbf{S}^{-1}. \tag{8}$$

Secondly, we can relate the elements of the pure basis to a set of MI $\mathbf{G}$ through IBP reduction,

$$\mathbf{g} = \mathbf{T}\mathbf{G}. \tag{9}$$

Using the expansion by regions method [37] as implemented in the `asy` code which is shipped along with `FIESTA4` [38], we can obtain the $x \to 0$ limit of the MI in terms of which we express the pure basis (9),

$$G_i \underset{x\to 0}{=} \sum_j x^{b_j + a_j \epsilon} G_i^{(b_j + a_j \epsilon)} \tag{10}$$

where a_j and b_j are integers and G_i are the individual members of the basis $\mathbf{G}$ of MI in (9). This analysis allows us to construct the following relation

$$\mathbf{R}\mathbf{b} = \lim_{x\to 0} \mathbf{T}\mathbf{G}\Big|_{\mathcal{O}\left(x^{0 + a_j \epsilon}\right)} \tag{11}$$

where the right-hand side implies that, apart from the terms $x^{a_i \epsilon}$ coming from (10), we expand around $x = 0$, keeping only terms of order x^0. Equation (11) allows us in principle to determine all boundary constants $\mathbf{b} = \sum_{i=0}^{4} \epsilon^i\, \mathbf{b}_0^{(i)}$.

To reduce the number of region-integrals involved in (11) we have investigated a different approach. The idea is rather simple and straightforward. The pure basis elements can be written in general as follows:

$$g = C e^{2\gamma_E \epsilon} \int \frac{d^d k_1}{i\pi^{d/2}} \frac{d^d k_2}{i\pi^{d/2}} \frac{P\left(\{D_i\}, \{S_{ij}, x\}\right)}{\prod\limits_{i \in \tilde{S}} D_i^{a_i}} \tag{12}$$

where D_i, $i = 1 \ldots 11$, represent the inverse scalar propagators, $\tilde{S}$ the set of indices corresponding to a given sector, S_{ij}, x the kinematic invariants, P is a polynomial, a_i are positive integers and C a factor depending on S_{ij}, x. This form is usually decomposed in terms of FI, F_i,

$$g = C \sum c_i \left(\{S_{ij}, x\}\right) F_i$$

with c_i being polynomials in S_{ij}, x. The limit $x = 0$, is then obtained, after IBP reduction, through Feynman parameter representation of the individual MI, as described in the previous paragraphs.

An alternative approach, would be to build-up the Feynman parameter representation for the whole basis element, by considering the integral in (12) as a tensor integral and making use of the formulae from the references [39, 40], to bring it in its Feynman parameter representation. Then, by using the expansion by regions approach [37, 38], we determine the regions[2] in the limit $x = 0$.

[2] Only the corresponding scalar integral of (12) determines the regions.

Rescaling the Feynman parameters by appropriate powers of x, keeping the leading power in x, we then obtain the final result that can be written as follows:

$$b = \sum_I N_I \int \prod_{i\in S_I} dx_i\, U_I^{a_I} F_I^{b_i} \Pi_I$$

where I runs over the set of contributing regions, U_I and F_I are the limits of the usual Symanzik polynomials, Π_I is a polynomial in the Feynman parameters, x_i, and the kinematic invariants S_{ij}, and S_I the subset of surviving Feynman parameters in the limit. In this way a significant reduction of the number of regions to be calculated is achieved, namely from 208 to 9 for the N_3 family [33]. Notice that in contrast to the approach described in the previous paragraphs, only the regions $x^{-2\epsilon}$ and $x^{-4\epsilon}$ contribute to the final result, making thus the evaluation of the region-integrals simpler. Moreover, this approach overpasses the need for an IBP reduction of the basis elements in terms of MI.

4. Integral representation

After obtaining all boundary terms in section 3 and constructing analytic expressions for families N_2 and N_3 up to $O(\epsilon^2)$ in terms of GPLs up to weight two, we will now introduce an one-fold integral representation for $O(\epsilon^3)$ and $O(\epsilon^4)$. This representation will allow us to obtain numerical results through direct numerical integration [26, 41].

Weight 3: The differential equation (6) can be written in the form:

$$\partial_x g_I^{(3)} = \sum_a \left(\partial_x \log L_a\right) \sum_J c_{IJ}^a g_J^{(2)} \tag{13}$$

where a runs over the set of contributing letters, I, J run over the set of basis elements, c_{IJ}^a are $\mathbb{Q}$−number coefficients read off from the matrices $\mathbf{M}_a$ and $g_J^{(2)}$ are the basis elements at weight 2, known in terms of GPLs. Since the lower limit of integration corresponds to $x = 0$, we need to subtract the appropriate term so that the integral is explicitly finite. This is achieved as follows:

$$\partial_x g_I^{(3)} = \sum_a \frac{l_a}{x} \sum_J c_{IJ}^a g_{J,0}^{(2)} + \left(\sum_a \left(\partial_x \log L_a\right) \sum_J c_{IJ}^a g_J^{(2)} - \sum_a \frac{l_a}{x} \sum_J c_{IJ}^a g_{J,0}^{(2)} \right) \tag{14}$$

where $g_{I,0}^{(2)}$ are obtained by expanding $g_I^{(2)}$ around $x = 0$ and keeping terms up to order $O\big(\log(x)^2\big)$, and $l_a \in \mathbb{Q}$ are defined through

$$\partial_x \log L_a = \frac{l_a}{x} + O\big(x^0\big). \tag{15}$$

The DE (14) can now be integrated from $x = 0$ to $x = \bar{x}$, and the result is given by

$$g_I^{(3)} = g_{I,\mathcal{G}}^{(3)} + b_I^{(3)} + \int_0^{\bar{x}} \mathrm{d}x \left(\sum_a \left(\partial_x \log L_a\right) \sum_J c_{IJ}^a g_J^{(2)} - \sum_a \frac{l_a}{x} \sum_J c_{IJ}^a g_{J,0}^{(2)} \right) \tag{16}$$

with $b_I^{(3)}$ being the boundary terms at $O(\epsilon^3)$ and

$$g_{I,\mathcal{G}}^{(3)} = \int_0^{\bar{x}} \mathrm{d}x \sum_a \frac{l_a}{x} \sum_J c_{LJ}^a g_{J,0}^{(2)} \bigg|_{\mathcal{G}} \tag{17}$$

with the subscript $\mathcal{G}$, indicating that the integral is represented in terms of GPLs (see ancillary files of reference [33]), following the convention

$$\int_0^{\bar{x}} dx \frac{1}{x} \mathcal{G}\left(\underbrace{0, \ldots 0}_{n}; x\right) = \mathcal{G}\left(\underbrace{0, \ldots 0}_{n+1}; \bar{x}\right). \tag{18}$$

Weight 4: At weight 4, the differential equation (6) can be written in the form:

$$\partial_x g_I^{(4)} = \sum_a \left(\partial_x \log L_a\right) \sum_J c^a_{IJ} g_J^{(3)} \tag{19}$$

which after doubly-subtracting, in order to obtain integrals that are explicitly finite as in (14), is written as

$$\partial_x g_I^{(4)} = \sum_a \partial_x(\log L_a - LL_a) \sum_J c^a_{IJ} g_J^{(3)} + \sum_a \partial_x(LL_a) \sum_J c^a_{IJ}(g_J^{(3)} - g_{J,0}^{(3)}) + \sum_a \frac{l_a}{x} \sum_J c^a_{IJ} g_{J,0}^{(3)} \tag{20}$$

where LL_a are obtained by expanding $\log(L_a)$ around $x = 0$ and keeping terms up to order $O\big(\log(x)\big)$, and

$$g_{I,0}^{(3)} = g_{I,\mathcal{G}}^{(3)} + b_I^{(3)}. \tag{21}$$

Now, by integrating by parts and using (14) we can write the final result as follows:

$$\begin{aligned} g_I^{(4)} = & g_{I,\mathcal{G}}^{(4)} + b_I^{(4)} + \left(\sum_a \log L_a \sum_J c^a_{IJ} g_J^{(3)}\right) - \left(\sum_a LL_a \sum_J c^a_{IJ} g_{J,0}^{(3)}\right) \\ & - \int_0^{\bar{x}} \mathrm{d}x \sum_a (\log L_a - LL_a) \sum_J c^a_{IJ} \sum_b \frac{l_b}{x} \sum_K c^b_{JK} g_{K,0}^{(2)} \\ & - \int_0^{\bar{x}} \mathrm{d}x \sum_a \log L_a \sum_J c^a_{IJ} \left(\sum_b (\partial_x \log L_b) \sum_K c^b_{JK} g_K^{(2)} - \sum_b \frac{l_b}{x} \sum_K c^b_{JK} g_{K,0}^{(2)}\right) \end{aligned} \tag{22}$$

with a, b running over the set of contributing letters, I, J, K running over the set of basis elements, $b_I^{(4)}$ being the boundary terms at $O(\epsilon^4)$ and

$$g_{I,\mathcal{G}}^{(4)} = \int_0^{\bar{x}} \mathrm{d}x \left(\sum_a \frac{l_a}{x} \sum_J c^a_{IJ} g_{J,0}^{(3)}\right)\Bigg|_{\mathcal{G}} \tag{23}$$

where the subscript $\mathcal{G}$ indicates that the integral is represented in terms of GPLs (see ancillary files of reference [33]), following (18).

Numerical results concerning all planar and hexabox families based on the SDE can be found in references [22, 33].

5. Conclusions

The frontier of precision calculations at NNLO currently targets $2 \to 3$ scattering process involving massless propagators and one off-shell external particle. At the level of FI, all planar two-loop MI have been recently computed through the solution of canonical DE both numerically [18], via generalised power series expansions, and analytically in terms of GPLs up to weight 4 [22], using the SDE approach [24]. More recently, results in terms of Chen iterated integrals were presented and implemented in the so-called pentagon functions [25].

Concerning the two-loop non-planar topologies, these can be classified into the three so-called hexabox topologies and two so-called double-pentagons, see figure 1. One of the hexabox topologies, denoted as N_1 in figure 1, was calculated numerically a few years ago using an approach which introduces a Feynman parameter and uses analytic results for the sub-topologies that are involved [32]. More recently, pure bases for the three hexabox topologies satisfying DE in $d\log$ form were presented in reference [19] and solved numerically using the same methods as in [18].

In this presentation we addressed the calculation of the three two-loop hexabox topologies, N_1, N_2, N_3 in figure 1, using the SDE approach [33].. For the N_1 family results up to weight 4 in terms of GPLs are obtained. For the N_2 and N_3 families we have established an one-dimensional integral representation involving up to weight-2 GPLs. This allows to extend the scope of the SDE approach when non-factorizable square roots appear in the alphabet [33]. We have also introduced a new approach to compute the boundary terms directly for the basis elements, that significantly reduces the complexity of the problem. With these new developments, we hope to complete the full set of five-point one-mass two-loop MI families in the near future and provide a solid implementation for their numerical evaluation.

References

[1] S. Amoroso et al., *Les Houches 2019: Physics at TeV Colliders: Standard Model Working Group Report*, in *11th Les Houches Workshop on Physics at TeV Colliders: PhysTeV Les Houches*, 3, 2020 [2003.01700].

[2] T. Gehrmann and B. Malaescu, *Precision QCD Physics at the LHC*, 2111.02319.

[3] L. Tancredi, *Calculational Techniques in Particle Theory*, in *European Physical Society Conference on High Energy Physics 2021*, 10, 2021 [2111.00205].

[4] A.V. Kotikov, *Differential equations method: New technique for massive Feynman diagrams calculation*, *Phys. Lett. B* **254** (1991) 158.

[5] A.V. Kotikov, *Differential equations method: The Calculation of vertex type Feynman diagrams*, *Phys. Lett. B* **259** (1991) 314.

[6] A.V. Kotikov, *Differential equation method: The Calculation of N point Feynman diagrams*, *Phys. Lett. B* **267** (1991) 123.

[7] T. Gehrmann and E. Remiddi, *Differential equations for two loop four point functions*, *Nucl. Phys. B* **580** (2000) 485 [hep-ph/9912329].

[8] K.G. Chetyrkin and F.V. Tkachov, *Integration by Parts: The Algorithm to Calculate beta Functions in 4 Loops*, *Nucl. Phys. B* **192** (1981) 159.

[9] J.M. Henn, *Lectures on differential equations for Feynman integrals*, *J. Phys. A* **48** (2015) 153001 [1412.2296].

[10] J.M. Henn, *Multiloop integrals in dimensional regularization made simple*, Phys. Rev. Lett. **110** (2013) 251601 [1304.1806].

[11] K.-T. Chen, *Iterated path integrals*, Bull. Am. Math. Soc. **83** (1977) 831.

[12] A.B. Goncharov, *Multiple polylogarithms, cyclotomy and modular complexes*, Math. Res. Lett. **5** (1998) 497 [1105.2076].

[13] C. Duhr, H. Gangl and J.R. Rhodes, *From polygons and symbols to polylogarithmic functions*, JHEP **10** (2012) 075 [1110.0458].

[14] C. Duhr, *Hopf algebras, coproducts and symbols: an application to Higgs boson amplitudes*, JHEP **08** (2012) 043 [1203.0454].

[15] C. Duhr and F. Dulat, *PolyLogTools — polylogs for the masses*, JHEP **08** (2019) 135 [1904.07279].

[16] J. Vollinga and S. Weinzierl, *Numerical evaluation of multiple polylogarithms*, Comput. Phys. Commun. **167** (2005) 177 [hep-ph/0410259].

[17] L. Naterop, A. Signer and Y. Ulrich, *handyG —Rapid numerical evaluation of generalised polylogarithms in Fortran*, Comput. Phys. Commun. **253** (2020) 107165 [1909.01656].

[18] S. Abreu, H. Ita, F. Moriello, B. Page, W. Tschernow and M. Zeng, *Two-Loop Integrals for Planar Five-Point One-Mass Processes*, JHEP **11** (2020) 117 [2005.04195].

[19] S. Abreu, H. Ita, B. Page and W. Tschernow, *Two-Loop Hexa-Box Integrals for Non-Planar Five-Point One-Mass Processes*, 2107.14180.

[20] F. Moriello, *Generalised power series expansions for the elliptic planar families of Higgs + jet production at two loops*, JHEP **01** (2020) 150 [1907.13234].

[21] M. Hidding, *DiffExp, a Mathematica package for computing Feynman integrals in terms of one-dimensional series expansions*, Comput. Phys. Commun. **269** (2021) 108125 [2006.05510].

[22] D.D. Canko, C.G. Papadopoulos and N. Syrrakos, *Analytic representation of all planar two-loop five-point Master Integrals with one off-shell leg*, JHEP **01** (2021) 199 [2009.13917].

[23] C.G. Papadopoulos, D. Tommasini and C. Wever, *The Pentabox Master Integrals with the Simplified Differential Equations approach*, JHEP **04** (2016) 078 [1511.09404].

[24] C.G. Papadopoulos, *Simplified differential equations approach for Master Integrals*, JHEP **07** (2014) 088 [1401.6057].

[25] D. Chicherin, V. Sotnikov and S. Zoia, *Pentagon Functions for One-Mass Planar Scattering Amplitudes*, 2110.10111.

[26] T. Gehrmann, J.M. Henn and N.A. Lo Presti, *Pentagon functions for massless planar scattering amplitudes*, *JHEP* **10** (2018) 103 [1807.09812].

[27] D. Chicherin and V. Sotnikov, *Pentagon Functions for Scattering of Five Massless Particles*, *JHEP* **20** (2020) 167 [2009.07803].

[28] N. Syrrakos, *Pentagon integrals to arbitrary order in the dimensional regulator*, *JHEP* **06** (2021) 037 [2012.10635].

[29] S. Badger, H.B. Hartanto and S. Zoia, *Two-Loop QCD Corrections to Wbb¯Production at Hadron Colliders*, *Phys. Rev. Lett.* **127** (2021) 012001 [2102.02516].

[30] S. Badger, H.B. Hartanto, J. Kryś and S. Zoia, *Two-loop leading-colour QCD helicity amplitudes for Higgs boson production in association with a bottom-quark pair at the LHC*, *JHEP* **11** (2021) 012 [2107.14733].

[31] S. Abreu, F.F. Cordero, H. Ita, M. Klinkert, B. Page and V. Sotnikov, *Leading-Color Two-Loop Amplitudes for Four Partons and a W Boson in QCD*, 2110.07541.

[32] C.G. Papadopoulos and C. Wever, *Internal Reduction method for computing Feynman Integrals*, *JHEP* **02** (2020) 112 [1910.06275].

[33] A. Kardos, C.G. Papadopoulos, A.V. Smirnov, N. Syrrakos and C. Wever, *Two-loop non-planar hexa-box integrals with one massive leg*, *JHEP* **05** (2022) 033 [2201.07509].

[34] D. Binosi, J. Collins, C. Kaufhold and L. Theussl, *JaxoDraw: A Graphical user interface for drawing Feynman diagrams. Version 2.0 release notes*, *Comput. Phys. Commun.* **180** (2009) 1709 [0811.4113].

[35] C.G. Papadopoulos, D. Tommasini and C. Wever, *Two-loop Master Integrals with the Simplified Differential Equations approach*, *JHEP* **01** (2015) 072 [1409.6114].

[36] D.D. Canko and N. Syrrakos, *Resummation methods for Master Integrals*, *JHEP* **02** (2021) 080 [2010.06947].

[37] B. Jantzen, A.V. Smirnov and V.A. Smirnov, *Expansion by regions: revealing potential and Glauber regions automatically*, *Eur. Phys. J. C* **72** (2012) 2139 [1206.0546].

[38] A.V. Smirnov, *FIESTA4: Optimized Feynman integral calculations with GPU support*, *Comput. Phys. Commun.* **204** (2016) 189 [1511.03614].

[39] J. Gluza, K. Kajda, T. Riemann and V. Yundin, *Numerical Evaluation of Tensor Feynman Integrals in Euclidean Kinematics*, *Eur. Phys. J. C* **71** (2011) 1516 [1010.1667].

[40] S.C. Borowka, *Evaluation of multi-loop multi-scale integrals and phenomenological two-loop applications*, Ph.D. thesis, Munich, Tech. U., 2014. 1410.7939.

[41] S. Caron-Huot and J.M. Henn, *Iterative structure of finite loop integrals*, *JHEP* **06** (2014) 114 [1404.2922].

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