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Yang–Mills All-Plus: Two Loops for the Price of One

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We present work on two-loop amplitudes in pure Yang–Mills theory with all gluons of identical helicity. We show how to obtain their rational terms — the hardest parts to compute — via well-understood one-loop unitarity techniques.

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1. Introduction

Increasing integrated luminosity at the Large Hadron Collider (LHC) in the coming decade will drive experimenters' search for physics beyond the Standard Model (SM). The LHC will be sensitive to ever-fainter discrepancies from SM predictions, thanks to increased statistics and to a better understanding of systematic uncertainties. Greater experimental sensitivity does not suffice. We also need higher-precision calculations in perturbative QCD to reduce theoretical uncertainties.

The current frontier for perturbative QCD calculations is at next-to-next-to-leading order (NNLO), where one expects a reduction in these latter uncertainties to below a few percent.

We explore a technique for computing certain contributions to a simple class of two-loop Yang–Mills amplitudes, the so-called "all-plus" amplitudes, with all external gluons of identical helicity. This technique is based on a conjecture by Badger, Mogull and Peraro (BMP) in ref. [1]. It relies solely on well understood one-loop generalized unitarity technology, and turns out to apply to all partial amplitudes in the color decomposition, in particular also to the nonplanar ones. We have given a much more detailed discussion of our calculations in ref. [2].

2. All-Plus Amplitudes

All-plus amplitudes are simpler than general two-loop amplitudes. At tree level, they vanish. At one-loop, the dimensionally regulated amplitude ($D = 4 - 2\epsilon$) can be decomposed in a form exposing its universal singular structure [3–6],

$$A^{(1)} = A^{(0)} I^{(1)} + F^{(1)} + O(\epsilon)\,. \tag{1}$$

Here, $I^{(1)}$ is a universal function of the Lorentz invariants with double and single poles in ϵ. The vanishing of the tree makes one-loop amplitudes free of ultraviolet and infrared divergences. Indeed the latter are purely rational in the external spinors [7–12]. For the leading-color partial amplitude, an all-n conjecture ref. [9] came from demanding correct collinear factorization,

$$A^{(1)}(1^+ \dots n^+) = -\frac{1}{3} \frac{\sum_{1\le i<j<k<l\le n} \langle i|jkl|i]}{\langle 12\rangle \langle 23\rangle \dots \langle (n-1)n\rangle \langle n1\rangle} + O(\epsilon)\,, \tag{2}$$

a form which was later proven in ref. [11], and rederived in ref. [12]. The subleading-color amplitudes at one-loop can always be obtained from the leading-color ones through color relations [13]. Compact forms for them are also known [9, 14]. Their finiteness and absence of branch-cuts makes these expressions more like tree-level amplitudes than one-loop ones.

The four-point all-plus amplitude at two loops was computed long ago in ref. [15]; the five-point one was derived much later [16–20]. The full six-gluon amplitude was derived in refs. [24], and there exist partial seven-gluon [25] and n-gluon expressions [26]. More recently, first results for the four-gluon amplitude at three loops have been derived in refs. [27, 28].

As in the one-loop case, two-loop amplitudes can be decomposed with respect to their singularity structure [29], which leads us to a relation similar to that of eq. (1),

$$A^{(2)} = A^{(0)} I^{(2)} + A^{(1)} I^{(1)} + F^{(2)} + O(\epsilon). \tag{3}$$

Here, $I^{(2)}$ is (like $I^{(1)}$) a universal function of the Lorentz invariants with divergences up to ϵ^{-4}; $I^{(1)}$ is the same function as in eq. (1). The remainder $F^{(2)}$ is finite in dimensional regularization, and can be split into polylogarithmic and rational parts $P^{(2)}$ and $R^{(2)}$,

$$F^{(2)} = P^{(2)} + R^{(2)} . \tag{4}$$

Two-loop all-plus amplitudes have singularities in dimensional regularization of the same degree as general one-loop amplitudes. The polylogarithmic part $P^{(2)}$ has ordinary branch cuts, and may therefore be computed using four-dimensional generalized unitarity. In contrast, the rational part $R^{(2)}$ does not contain such discontinuities and requires separate treatment.

The structure of the all-plus amplitude at two loops has made it possible to compute the polylogarithmic terms for an arbitrary number of external gluons for the leading-color [23] and a special subleading-color partial amplitude [26]. In addition, the authors of refs. [14, 21–25] used recursive techniques to compute the rational terms in the five- and six-point amplitudes at leading and subleading color, as well as the leading-color seven-point amplitude [25]. Dunbar, Perkins, and Strong (DPS) presented an all-n conjecture [26] for the special subleading-color amplitude. BMP [1] have also computed the leading-color five- and six-point rational parts through a reconstruction of the integrand. In addition, they presented a conjecture for the all-n integrand at leading color on which we shall rely in our calculations.

3. Separability

In dimensional regularization, scattering amplitudes generally depend on two types of dimensional parameter: the dimension of loop-momentum integrations, D; and D_s, which controls the number of states, with $D_s \geq D$. Integrands of loop amplitudes depend *polynomially* on D_s, while integrals depend in a general analytic fashion on D. The conjecture of BMP [1] is given in terms of the all-plus amplitude's dependence on the dimension D_s.

To extract this dependence, we follow a modified approach originally introduced in ref. [15], and later exploited at one loop in ref. [30] and at two loops by BMP [1] to help isolate rational contributions. An all-loop discussion can be found in ref. [31].

Any two-loop amplitude $A^{(2)}_{D_s}$ in Yang–Mills theory can be written as a quadratic polynomial in D_s. By computing the amplitude for three different (ideally integer) values of D_s, we can fix the coefficients of this polynomial, allowing us to interpolate to non-integer D_s. We refer to this method as *dimensional reconstruction*.

Choosing $D_s = 6, 7, 8$ as sampling dimensions, we can determine $A^{(2)}_{D_s}$ [31],

$$A^{(2)}_{D_s} = A^{(2)}_0 + (D_s - 6)A^{(2)}_{1s} + (D_s - 6)^2 A^{(2)}_{2s} + (D_s - 6)(D_s - 5)A^{(2)}_{\times} . \tag{5}$$

Here, $A^{(2)}_0$, $A^{(2)}_{1s}$, $A^{(2)}_{2s}$, and $A^{(2)}_{\times}$ are six-dimensional amplitudes with either two gluons ($A^{(2)}_0$), one gluon and one scalar ($A^{(2)}_{1s}$), or two scalars running in the loops ($A^{(2)}_{2s}$, $A^{(2)}_{\times}$). The difference between $A^{(2)}_{2s}$ and $A^{(2)}_{\times}$ lies in the way the two scalar loops are connected. In the former, they interact through the exchange of a gluon, while in the latter they are connected by a four-scalar contact term [31].

In six dimensions, there are four different gluon polarization states. The scalars arise via Kaluza–Klein reduction from the additional polarization states of seven- and eight-dimensional gluons, and are massless in six dimensions.

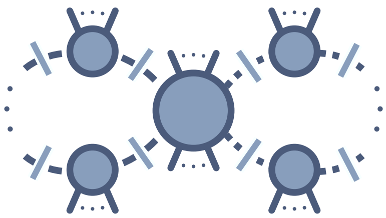


Figure 1: The generic of one-loop squared cuts contributing to $R^{(2)}_{ss}$.

In ref. [1], BMP show how to decompose the leading-color two-loop all-plus amplitudes into polylogarithmic terms $P^{(2)}_{n:1}$ and rational parts $R^{(2)}_{n:1}$ associated to different powers of the state dimension D_s. More precisely, through $O(\epsilon^0)$, BMP conjectured that these terms are always associated to different powers of $D_s - 2$,

$$F^{(2)}_{n:1}(1^+ \dots n^+) = \frac{1}{2}(D_s - 2)P^{(2)}_{n:1}(1^+ \dots n^+) + \frac{1}{4}(D_s - 2)^2 R^{(2)}_{n:1}(1^+ \dots n^+) + O(\epsilon)\,. \qquad (6)$$

BMP verified this decomposition for the five- and six-gluon leading color partial amplitudes.

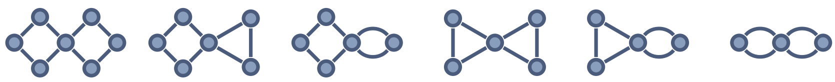
Assuming this conjecture, we can use dimensional reconstruction to express the leading-color rational part in terms of six-dimensional amplitudes' rational parts. Comparing eqs. (5) and (6),

$$R^{(2)}_{n:1}(1^+ \dots n^+) = 4\left[R^{(2)}_{2s} + R^{(2)}_{\times}\right] \equiv 4R^{(2)}_{ss}\,, \qquad (7)$$

where $R^{(2)}_{2s}$, $R^{(2)}_{\times}$ are the rational parts of $A^{(2)}_{2s}$ and $A^{(2)}_{\times}$, and $R^{(2)}_{ss}$ is a shorthand for their sum.

Phrasing the BMP conjecture in the dimensional reconstruction picture means we only need the rational part $R^{(2)}_{ss}$. The scalar Feynman rules forbid diagrams with propagators carrying both loop momenta. All contributing two-loop Feynman integrals then factorize into a product of one-loop integrals. We can then determine the two-loop rational parts $R^{(2)}_{ss}$—and therefore $R^{(2)}(1^+ \dots n^+)$—using only one-loop generalized unitarity techniques, in what we call the *separable approach*.

We wish to compute the rational part $R^{(2)}_{ss}$ using one-loop D-dimensional generalized unitarity. As all integrals have to factorize, we can limit ourselves to a basis of integrals which factorize as well. Box, triangle and bubble integrals form a basis of one-loop Feynman integrals. A basis of factorizing two-loop integrals is therefore given by all integrals of the following six topologies,



. (8)

We use generalized unitarity cuts in each loop to determine the coefficients of these types of integrals, following refs. [32, 33]. A generic "one-loop squared" cut contributing to the leading-color rational part is shown in Fig. 1. The dashed lines represent the internal six-dimensional scalar loop propagators that are cut, while the circles are six-dimensional, color-ordered, on-shell tree amplitudes. To compute the coefficient of each integral, we treat the loops sequentially. The first loop is computed from tree amplitudes using standard one-loop unitarity techniques; the second loop is computed from tree amplitudes and the coefficient computed at the first step, with the latter playing the role of a tree amplitude. In the second step, we again use one-loop unitarity. As the loops are equivalent we are free to choose, which loop is computed first, and which one second. The coefficient of the required two-loop integral is then the result of this two-stage computation.

The BMP conjecture leads to the separable approach for leading-color amplitudes [1]. We extend the conjecture to the rational contributions of subleading-color partial amplitudes as well, specifically the nonplanar ones. That is, the nonplanar rational part $R_{ss}^{(2)}$ is related to the nonplanar two-loop rational parts of all-plus amplitudes as in eq. (7). Furthermore, thanks to the factorization of the loop integrals, we only require one-loop generalized-unitarity technology.

4. Cuts for Color Structures

We compute $R_{ss}^{(2)}$ for nonplanar amplitudes via color-dressed unitarity, similar to the procedure for polylogarithmic contributions in refs. [14, 24, 26]. We can write a complete two-loop SU(N_c) Yang–Mills amplitude as follows [14, 24],

$$\begin{aligned}
\mathcal{A}_n^{(2)} = {} & N_c^2 \sum_{\sigma\in S_n/Z_n} \mathrm{ITr}(\sigma(1\ldots n))\, A_{n:1}^{(2)}(\sigma(1\ldots n)) \\
&+ N_c \sum_{r=3}^{\lfloor n/2\rfloor+1} \sum_{\sigma\in S_n/P_{n:r}} \mathrm{ITr}(\sigma(1\ldots(r-1)))\,\mathrm{ITr}(\sigma(r\ldots n)) A_{n:r}^{(2)}(\sigma(1\ldots(r-1); r\ldots n)) \\
&+ \sum_{r=2}^{\lfloor n/2\rfloor} \sum_{k=r}^{\lfloor (n-r)/2\rfloor} \sum_{\sigma\in S_n/P_{n:r,k}} \mathrm{ITr}(\sigma(1\ldots r))\,\mathrm{ITr}(\sigma((r+1)\ldots(r+k)))\,\mathrm{ITr}(\sigma((r+k+1)\ldots n)) \\
&\qquad\qquad \times A_{n:r,k}^{(2)}(\sigma(1\ldots r; (r+1)\ldots(r+k); (r+k+1)\ldots n)) \\
&+ \sum_{\sigma\in S_n/Z_n} \mathrm{ITr}(\sigma(1\ldots n)) A_{n:1\mathrm{B}}^{(2)}(\sigma(1\ldots n))\,,
\end{aligned} \tag{9}$$

where the $P_{n:r}$ and $P_{n:r,k}$ account for exchanges of the traces, as well as cyclic permutations of their arguments. The $\mathrm{ITr}(\ldots)$ represent traces over the color generators. Interpreting factors of N_c as empty traces, we can identify two distinct classes of color structures: those with three traces, associated to $A_{n:1}^{(2)}$, $A_{n:r}^{(2)}$,$A_{n:r,k}^{(2)}$, and those with just a single color trace, associated to $A_{n:1\mathrm{B}}^{(2)}$.

We can give a stringy heuristic argument for these two classes. Two-loop gauge theory amplitudes can be obtained from the infinite-tension limit of genus-two open-string amplitudes. External particles are realized through operator insertions on the boundary of the world-sheet. The gauge group is introduced by dressing these insertions with Chan–Paton factors (in this case color generators), which are contracted along the boundary.

Two types of genus-two surfaces contribute: the kind shown Fig. 2a has three boundaries, while the type shown in Fig. 2b has only a single boundary. The former generates three color traces, and

(a) **(b)**

Figure 2: The two types of genus two surfaces contributing to open-string amplitudes. The one on the left has three boundaries, and generates three color traces. The one on the right has only a single boundary, therefore generating only one such trace.

can be interpreted as the origin of partial amplitudes $A_{n:1}^{(2)}$, $A_{n:r}^{(2)}$, $A_{n:r,k}^{(2)}$. The latter can only generate a single trace, and is therefore associated to the $A^{(2)}_{n:1\mathrm{B}}$ partial amplitudes. This string theory picture leads to one-loop squared cuts for the different color structures.

To determine $R^{(2)}_{ss}$ for three-trace amplitudes, we use cuts of form shown in Fig. 1. We identify the outer and two inner edges of the cut with the three color traces, attaching the external gluons to them accordingly. This principle alongside the string theory motivation is illustrated in Figs. 3a and 3b. We can check independently that dressing the color-ordered tree-amplitudes with their color traces and some algebra leads to the correct trace structure. The full rational part $R^{(2)}_{ss}$ is then the sum over unique cuts, taking into account all associations of traces to the three edges.

To obtain $R^{(2)}_{ss}$ for subleading single-trace amplitudes, we have to find cuts generated by the single-edge surface of Fig. 2b. We can smoothly deform this surface to an equivalent form, shown in Fig. 3c. From this it is easy to identify the associated unitarity cuts, see Fig. 3d. The key difference from the cuts for three traces is the attachment of the scalar lines: while before the two lines are separated at the connecting tree amplitude, they now cross. The subleading single-trace rational part $R^{(2)}_{ss}$ is then given by the sum over all unique cuts of this form.

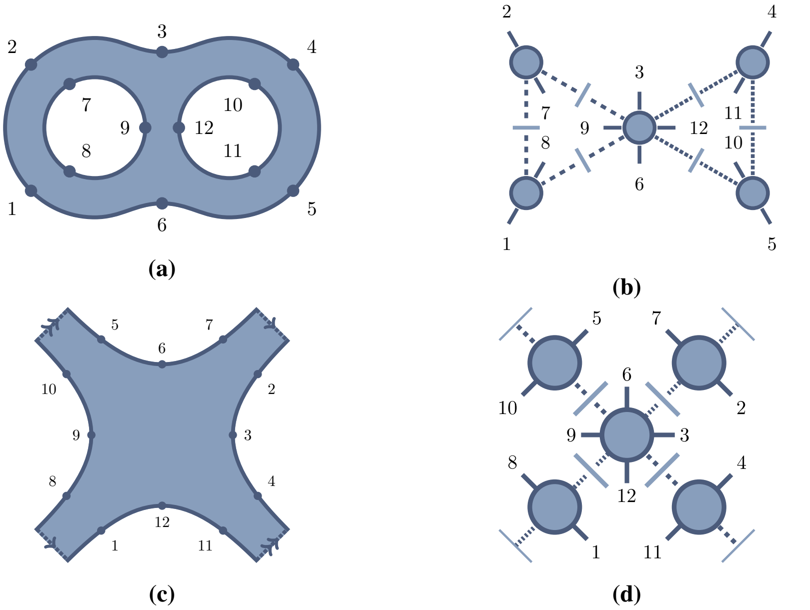


Figure 3: A graphical representation of the heuristic relation between the string world-sheet and unitarity cuts of subleading amplitudes. The configuration shown in (a) and (b) corresponds to the color structure $\mathrm{ITr}(1,2,3,4,5,6)\,\mathrm{ITr}(7,8,9)\,\mathrm{ITr}(10,11,12)$. Figures (c) and (d) correspond to the single-trace structure $\mathrm{ITr}(1,2,\ldots,11,12)$. In the latter two, the dotted lines need to be sewn together according to the arrows shown.

5. Verification

We verify the separability conjecture using an automated generation of nonplanar cuts in *Mathematica*. Their evaluation is carried out using the one-loop D-dimensional unitarity techniques presented in refs. [32, 33]. This procedure is also automated in *Mathematica* using a custom series expansion code specialized for the type of rational functions that appear. The code is capable of handling numerical (rational), as well as analytically parametrized, kinematics. All numeric comparisons were done exactly on rational kinematic points[1].

Using our code we find exact numerical agreement of the separable construction with all known analytic results in the literature. We compared the rational parts of all four, five, and six-gluon partial amplitudes of refs. [14, 21–25], as well as the leading color seven-gluon amplitude of ref. [25]. Using kinematics based on parametrized momentum twistors, we also rederived closed analytic forms for all five-gluon rational terms. We further verified numerically up to nine gluons the agreement with the conjecture for the rational parts of the subleading single-trace amplitudes [26].

6. Conclusions

In this proceeding, we have explored rational terms in two-loop amplitudes. These terms exhibit the least symmetry or structure of all contributions to Yang–Mills amplitudes. We studied the simplest such terms, in the all-plus gluon amplitudes. We relied on the BMP separability conjecture and used a straightforward extension of generalized unitarity techniques for computing one-loop rational terms in order to compute them. We computed the rational terms in the four- and five-point two-loop amplitudes analytically, and those in the six- and seven-point amplitudes numerically. Our results agree with those obtained by Dunbar, Dalgleish, Jehu, Perkins and Strong through a recursive approach. They also agree numerically with the DPS all-n conjecture [26] for the subleading-color single-trace amplitude at eight and nine points. In addition to evidence for the correctness of the results in refs. [14, 24, 25], our calculations also provide evidence for the correctness of the separability conjecture [1] both for leading- and subleading-color amplitudes. The ideas developed here may also help simplify the calculation of other rational terms at two loops, in particular in the other simple helicity configuration, with a lone negative-helicity gluon.

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