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Constructing Compact Ansätze for Scattering Amplitudes

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In these proceedings, we discuss the recent approach of Ref. [1] for the construction of Ansätze for scattering amplitudes. The method builds powerful constraints on the analytic structure of the rational functions in amplitudes from numerical tests of their behavior close to singularity surfaces. We discuss how we systematically understand these surfaces and how the singular behavior of the rational function can be incorporated into an Ansatz using techniques from algebraic geometry. To perform the numerical sampling, we make use of p-adic numbers, a number-theoretical field that can be considered a cousin of finite fields. The p-adic numbers admit a non-trivial absolute value, as well as analytic functions such as the p-adic logarithm. We provide a detailed example of the approach applied to an NMHV tree amplitude and discuss the efficacy when applied to the two-loop leading-color amplitude for three-photon production at hadron colliders.

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1. Introduction

Meeting the calculational requirements of modern hadron collider experiments is theoretically very demanding. Due to the high precision projected at the LHC, it is necessary to push fixed-order cross-section computations to NNLO QCD precision for a variety of multi-scale observables, see e.g. [2]. Many such predictions require the computation of two-loop QCD scattering amplitudes with five external particles. By now, the frontier is represented by a collection of two-loop five-point amplitudes involving a massive particle [3–6].

In the modern approach, scattering amplitudes are computed in a divide-and-conquer fashion, where they are broken down into a linear combination of special functions, whose coefficients depend rationally on the external kinematics. The special functions and rational coefficients are then computed separately. In recent multi-scale loop calculations, this is often considered at the level of the finite remainder. Specifically, after ultra-violet renormalization, one can subtract away the infra-red poles to define a finite remainder $\mathcal{H}_n^{(l)}$ [7–10]

$$\mathcal{A}_{n,R}^{(l)} = \sum_{l'=0}^{l-1} \mathbf{I}_\epsilon^{(l-l')} \mathcal{A}_{n,R}^{(l')} + \mathcal{H}_n^{(l)} + \mathcal{O}(\epsilon)\,. \tag{1}$$

This finite remainder admits a decomposition into a basis of special functions as

$$\mathcal{H}_n^{(l)} = \sum_i C_i(\lambda, \tilde{\lambda}) \times F_i(\lambda, \tilde{\lambda})\,. \tag{2}$$

The special functions F_i can be constructed in a way that depends only on the kinematics of the process under study, see, e.g. the recent development of sets of pentagon functions [11–13]. However, the coefficients C_i are complicated process-dependent rational functions. In recent years, a new paradigm has emerged which has enabled an explosion of computations of multi-scale amplitudes. In short, one writes an Ansatz for the C_i, and fixes the unknown parameters from numerical evaluations over finite fields [14, 15].

However, as we look towards amplitudes which depend on an increased number of scales, we see that the number of unknowns in the frequently used "common denominator" Ansatz is growing significantly, as one can see in Table 1. This has motivated us to search for a strategy for constructing improved Ansätze for the rational coefficients in QCD scattering amplitudes.

	$\mathcal{H}^{(2)}_{pp\to jjj}$	$\mathcal{H}^{(2)}_{pp\to\gamma\gamma\gamma}$	$\mathcal{H}^{(2)}_{pp\to W+2j}$	$\mathcal{A}^{(1)}_{0\to 6g} + \mathcal{O}(\epsilon)$
Ref.	[16]	[17, 18]	[3]	[19]
Ansatz size	$\mathcal{O}(10^5)$	$\mathcal{O}(10^5)$	$\mathcal{O}(10^7)^{(*)}$	$\mathcal{O}(10^9)$

Table 1: Ansatz size for the numerators when written in common denominator form.
$^{(*)}$ Reduced to $\mathcal{O}(10^6)$ after the univariate partial-fraction decomposition of Ref. [4].

In recent years, it has been observed that if the rational prefactors C_i are cast in a multivariate partial fractions decomposition, significant cancellations occur (see e.g. Ref. [20]). However, in practice, these simplifications are achieved after first reconstructing the result in a less compact form. In this talk, we discuss the approach of Ref. [1], where an Ansatz inspired by the structure of a partial fractions decomposition is constructed from numerical evaluations in singular limits. In practice, these Ansätze exhibit a reduced number of free parameters in comparison to the corresponding common denominator form and thereby significantly reduce the computational cost of analytic reconstruction.

2. Ansatz Construction Approach

To introduce our strategy, let us consider the 6-point NMHV tree amplitude $A_{0\to q^+g^+g^+\bar{q}^-g^-g^-}$. We work with the assumption that we do not know its analytical form, as if we were numerically computing a loop-integral coefficient. The least common denominator is easily obtained from numerical evaluations, thus we can write

$$A_{0\to q^+g^+g^+\bar{q}^-g^-g^-} = \frac{\mathcal{N}}{\langle 12\rangle\langle 23\rangle\langle 34\rangle[45][56][61]s_{345}}\,, \tag{3}$$

where $\mathcal{N}$ is an unknown polynomial in spinor brackets. That is, $\mathcal{N}$ belongs to a vector space $\mathcal{M}_{d,\vec{\phi}}$ spanned by spinor-bracket monomials of appropriate mass dimension d and little-group weights $\vec{\phi}$. The mass dimension (6) as well as the little-group weights (-1, 0, 0, 1, 0, 0) of $\mathcal{N}$ are easily obtained, thus we have

$$\mathcal{N} \in \mathcal{M}_{6,[-1,0,0,1,0,0]}\,, \quad \text{where} \quad \dim(\mathcal{M}_{6,[-1,0,0,1,0,0]}) = 143\,. \tag{4}$$

We aim to construct an Ansatz for $\mathcal{N}$, based on the expectation that the result can be cast in some partial-fractioned form, where the numerator has been partially cancelled against the denominator. For example, if we are able to show that $\mathcal{N}$ can be written in the form

$$\mathcal{N} = n_{12}\langle 12\rangle + n_{23}\langle 23\rangle\,, \tag{5}$$

then we can cancel this against factors of $\langle 12\rangle$ and $\langle 23\rangle$ in the denominator. In Ref. [19], it was suggested that the validity of such a decomposition can be deduced from evaluations of the amplitude on phase-space points where $\langle 12\rangle$ and $\langle 23\rangle$ are $\mathcal{O}(\epsilon)$, for some small quantity ϵ. Note that there are actually two separate possibilities to achieve this, either the spinor λ_2 is nearly soft, or the spinors λ_1, λ_2 and λ_3 are simultaneously almost collinear. In practice, for $A_{0\to q^+g^+g^+\bar{q}^-g^-g^-}$ one finds that

$$\begin{gathered}\lambda_2 \sim \epsilon \Rightarrow \mathcal{N} \sim \epsilon^0, \\ \langle 12\rangle \sim \langle 23\rangle \sim \langle 13\rangle \sim \epsilon \Rightarrow \mathcal{N} \sim \epsilon^1.\end{gathered} \tag{6}$$

Therefore, the tentative decomposition of Eq. (5) does not hold. Nevertheless, we still have found a constraint from the vanishing of the numerator on the simultaneously collinear surface. It can be shown that the subspace of $\mathcal{M}_{d,\vec{\phi}}$ that satisfies this constraint has dimension 84, i.e. 1 evaluation fixed 59 parameters. We naturally wish to incorporate this style of constraint into an Ansatz. To systematize this strategy, we therefore find ourselves needing to answer a number of questions:

- How do we understand the different ways in which we can set the denominators small?
- How do we numerically obtain the degree of vanishing for the C_i in two-loop amplitudes?
- How do we incorporate into an Ansatz the constraints provided by the vanishing of $\mathcal{N}$?

Let us consider these problems one by one.

2.1 Branching of singular varieties

First, we interpret the task of numerically setting a pair of denominators $\mathcal{D}_i$ and $\mathcal{D}_j$ to be small as finding phase-space points which are close to the surfaces where the denominators are zero. That is, we are interested in understanding the solutions of

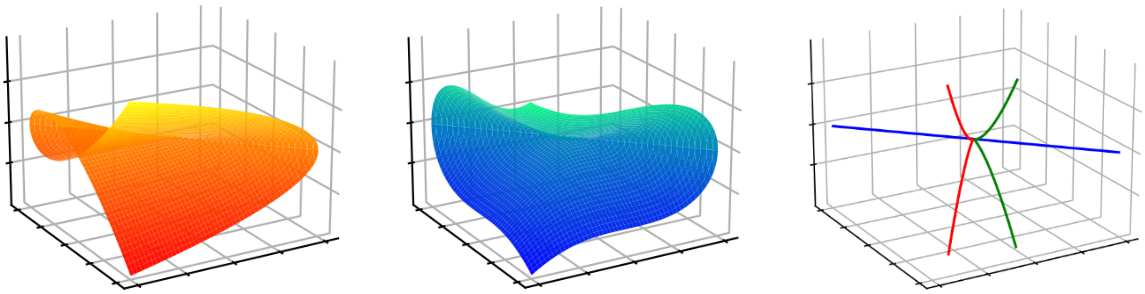
$$\mathcal{D}_i = \mathcal{D}_j = 0. \tag{7}$$

These are surfaces defined by the zero sets of polynomials and are known as algebraic varieties. The fact that there are different ways of setting the denominators small corresponds to the fact that these varieties may have multiple branches. More mathematically, varieties associated to pairs of denominator factors may be reducible.

A variety is said to be irreducible if it cannot be written as the union of simpler varieties, i.e. U is irreducible if $U = U_1 \cup U_2$ implies $U_1 = U$ or $U_2 = U$. A variety U admits a minimal decomposition in terms of irreducible varieties U_k,

$$U = \bigcup_{k=1}^{n_B(U)} U_k \quad \text{s.t.} \quad U_i \not\subseteq U_j \; \forall \, i \neq j. \tag{8}$$

In Fig. 1 we provide simple, graphical examples of irreducible and reducible varieties in $\mathbb{R}^3$.



(a) $V(\langle xy^2 + y^3 - z^2 \rangle)$ **(b)** $V(\langle x^3 + y^3 - z^2 \rangle)$ **(c)** $V(\langle xy^2 + y^3 - z^2, x^3 + y^3 - z^2 \rangle)$

Figure 1: Two irreducible codimension-one varieties in Fig. 1a and Fig. 1b, and the three branches of their reducible codimension-two intersection in Fig. 1c.

In higher dimensions, the reducibility of a given variety may be non-trivial to detect from its defining polynomials. In order to systematically approach the problem of determining the branches of a variety, we formulate the problem algebraically. Specifically, to the varieties U and U_k in the decomposition in Eq. (8), we associate the algebraic ideals, $I(U)$ and $I(U_k)$ respectively. In this language, the decomposition is translated to

$$I(U) = \bigcap_{k=1}^{n_B(U)} I(U_k). \tag{9}$$

The task of describing the branching of U now becomes the task of finding generators for each of the $I(U_k)$. As described in Ref. [1], this can be achieved by applying primary decomposition techniques to the ideal $\langle \mathcal{D}_i, \mathcal{D}_j \rangle_{R_n}$, making use of the computer algebra system `Singular` [21].

Returning to the motivating example, we see that the fact that there are two ways to make $\langle 12\rangle$ and $\langle 23\rangle$ simultaneously small algebraically corresponds to the primary decomposition

$$\big\langle\langle 12\rangle,\langle 23\rangle\big\rangle_{R_6} = \big\langle\langle 12\rangle,\langle 23\rangle,\langle 13\rangle\big\rangle_{R_6} \cap \langle\lambda_2^\alpha\rangle_{R_6}. \tag{10}$$

To be able to numerically check if numerator decompositions of the form of Eq. (5) apply for any pair of denominators, it is necessary to compute the primary decompositions of all ideals generated by such pairs. In Ref. [1] we perform this for all singularities arising from symbol letters for two-loop five-point massless processes.

2.2 p-adic numbers

Next, we discuss our approach for the sampling of the rational functions C_i close to varieties where a pair of denominators are zero. To this end we make use of the p-adic numbers $\mathbb{Q}_p$, which can be thought of as a close cousin of a finite field $\mathbb{F}_p$ and inherit some of their stability properties. A p-adic number x can be defined in terms of a formal series in powers of a prime number p,

$$x = \sum_{i=\nu_p(x)}^{\infty} a_i p^i = a_{\nu_p(x)} p^{\nu_p(x)} + \cdots + a_{-1}p^{-1} + a_0 + a_1 p + a_2 p^2 + \cdots\,, \tag{11}$$

where $\nu_p(x)$ is called the *valuation* of x, and $a_{\nu_p(x)} \neq 0$. The subset of $\mathbb{Q}_p$ with $\nu_p(x) = 0$ is known as the set of p-adic integers $\mathbb{Z}_p$.

An important feature of $\mathbb{Q}_p$ is that it admits a non-trivial absolute value,

$$|x|_p = p^{-\nu_p(x)} \quad\implies\quad |p|_p < |1|_p < \left|\frac{1}{p}\right|_p. \tag{12}$$

In contrast to $\mathbb{F}_p$, this allows one to find numerical configurations such that the a pair of denominators are small by building a p-adic phase-space point near varieties. In practice, we first find a finite-finite-valued phase-space point on the variety exactly. Then, we *lift* the $\mathbb{F}_p$ solution to an approximate p-adic solution. Further details are described in Ref. [1].

With a non-trivial absolute value on the field $\mathbb{Q}_p$, one can also introduce functions defined through power series. As an example, let us discuss the p-adic logarithm. This can be defined from the Taylor expansion around $x = 1$, that is

$$\log_p(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}x^k}{k}\,, \tag{13}$$

which converges for $|x|_p < 1$, that is for p-adic numbers x with $\nu_p(x) > 1$. To continue the logarithm outside of the radius of convergence of Eq. (13), we can make use of the following trick. We first recall that the logarithm relates products to sums. That is, for any a, b in $\mathbb{Q}_p$

$$\log_p(ab) = \log_p(a) + \log_p(b)\,. \tag{14}$$

Next, we remind ourselves of Fermat's little theorem: for an integer w, one can show that

$$w^{p-1} = 1 \mod p\,. \tag{15}$$

In $\mathbb{Z}_p$ we see that this implies that $|w^{p-1} - 1|_p < 1$. We can therefore apply the identity of Eq. (14) to rewrite

$$\log_p(w) = \frac{\log_p(w^{p-1})}{p-1}. \tag{16}$$

Now the argument of the logarithm on the right hand side of Eq. (16) is within the radius of convergence. We observe that computing the p-adic logarithm to low precision can be implemented efficiently by computing the exponentiation in Eq. (16) by the approach of repeated squaring. Given that, for large p, linear algebra modulo p^k is very stable, we expect applications to efficiently find linear relations between logarithms of products of functions, e.g. as often required in applications of the symbol calculus.

2.3 Constructing Compact Ansätze

Given the ability to classify all irreducible singular varieties associated to the amplitude, and to p-adically evaluate the rational functions close to these varieties, it remains to mathematically understand the constraint that this places for us on the numerator. We will then use these constraints to construct compact Ansätze for the rational functions.

First, we note that we make the assumption that if the numerator vanishes to a given degree on a single such configuration, that this is a property of the numerator on the entire variety, not just this point. This allows us to make use of a tool from algebraic geometry: the so-called "Zariski-Nagata" theorem [22–24]. Schematically, the theorem states that the set of polynomials which vanish to k^{th} order on a variety U is given by the k^{th} "symbolic power" of the ideal associated to the variety, $I(U)^{\langle k \rangle}$. That is, for a point $(\eta^{(\epsilon)}, \tilde{\eta}^{(\epsilon)})$ near a variety U one has that

$$\mathcal{N}(\eta^{(\epsilon)}, \tilde{\eta}^{(\epsilon)}) = \mathcal{O}(\epsilon^k) \quad \Rightarrow \quad \mathcal{N} \in I(U)^{\langle k \rangle}. \tag{17}$$

The symbolic power of an ideal is strongly related to the standard power of an ideal, and in Ref. [1] we give a number of strategies for computing a generating set of symbolic powers.

Given a collection of varieties $\mathcal{V} = \{U_1, \ldots, U_n\}$, we determine $\kappa(\mathcal{N}, U_i)$, the degree of vanishing of $\mathcal{N}$ on each of these varieties. This tells us that the numerator is a member of the ideal

$$\mathcal{N} \in \mathfrak{J}, \quad \text{where} \quad \mathfrak{J} = \bigcap_{U \in \mathcal{V}} I(U)^{\langle \kappa(\mathcal{N}, U) \rangle}. \tag{18}$$

From knowledge of the mass dimension and little group weights of the numerator, we now know that $\mathcal{N} \in \mathcal{M}_{d,\vec{\phi}} \cap \mathfrak{J}$. This is a finite-dimensional vector space, in which every element has the same vanishing properties as $\mathcal{N}$. Therefore, we take a basis of this space to be our Ansatz for the numerator $\mathcal{N}$. In Ref. [1] we constructed a basis of this vector space using Gröbner basis and linear algebra techniques.

3. Examples

In order to demonstrate the details of our approach, as well as its impact when constructing compact Ansätze for two-loop amplitudes, we will now discuss two examples. First we describe in detail an application to the color-ordered tree amplitude $A_{0\to q^+g^+g^+\bar{q}^-g^-g^-}$ introduced in Eq. (3). Second, we will summarize the result of an analogous procedure applied to two-loop finite reminders for the process $q\bar{q} \to \gamma\gamma\gamma$.

3.1 A six-point NMHV tree from its poles and zeros

Returning to $A_{0\to q^+g^+g^+\bar{q}^-g^-g^-}$ of Eq. (3), we examine the behavior of $\mathcal{N}$ on irreducible codimension-two varieties where pairs of invariants vanish. Specifically, we consider those corresponding to physical singularities, that is $\langle ij\rangle$, $[ij]$ and s_{ijk}. Up to permutations and parity, there are 8 inequivalent pairs of these invariants. In other words, there are 8 inequivalent, possibly reducible, codimension-two varieties. The corresponding ideals are given by

$$\begin{aligned} J_1 &= \big\langle\langle 12\rangle,\langle 13\rangle\big\rangle_{R_6} = P_1\cap P_2\,, & J_5 &= \big\langle\langle 12\rangle,[34]\big\rangle_{R_6} = P_6\,,\\ J_2 &= \big\langle\langle 12\rangle,\langle 34\rangle\big\rangle_{R_6} = P_3\,, & J_6 &= \big\langle\langle 12\rangle,s_{123}\big\rangle_{R_6} = P_1\cap P_7\,,\\ J_3 &= \big\langle\langle 12\rangle,[12]\big\rangle_{R_6} = P_4\,, & J_7 &= \big\langle\langle 12\rangle,s_{134}\big\rangle_{R_6} = P_8\,,\\ J_4 &= \big\langle\langle 12\rangle,[13]\big\rangle_{R_6} = P_5\,, & J_8 &= \big\langle s_{123},s_{124}\big\rangle_{R_6} = P_9\,. \end{aligned} \tag{19}$$

The right-hand sides provide primary decompositions and all ideals denoted as P_i are prime. The ones which do not directly correspond to the J_i ideals are

$$P_1 = \big\langle\langle 12\rangle,\langle 13\rangle,\langle 23\rangle\big\rangle_{R_6}\,,\quad P_2 = \big\langle\lambda_1^\alpha\big\rangle_{R_6}\,,\quad P_7 = \big\langle\langle 12\rangle,\lambda_1^\alpha[13]+\lambda_2^\alpha[23]\big\rangle_{R_6}\,. \tag{20}$$

We now sample the numerator of the NMHV amplitude on phase space points near to the set of codimension two varieties. In practice, we observe that on 28 varieties of codimension-two the numerator vanishes. On 2 of these 28 it vanishes quadratically, while on the remaining ones it vanishes linearly. By application of Eq. (18), we have $\mathcal{N}\in\mathfrak{J}$, where $\mathfrak{J}$ is

$$\begin{aligned} \mathfrak{J} = \Big[&P_1\cap\overline{P}_1^{\langle 2\rangle}\cap P_1(216543)\cap\overline{P}_1(216543)\cap P_1(432165)\cap\overline{P}_2\cap P_3\cap P_5(126345)\cap P_6(432165)\\ &\cap P_7\cap\overline{P}_7\cap P_7(321654)\cap P_7(543216)\cap P_8(231654)\Big]\cap\Big[123456\to\overline{456123}\Big]\,, \end{aligned} \tag{21}$$

where we have made use of the anti-symmetry of the amplitude ($\overline{456123}$) to write the intersection of 28 ideals in terms of two related sets of 14. The arguments of the ideals denote permutations of the external legs in the one-line notation $(\sigma(1)\ldots\sigma(6))$, and an overline denotes the swap $\lambda_{i,\alpha}\leftrightarrow\bar{\lambda}_{i,\dot{\alpha}}$. The only second symbolic powers that is needed is of the prime ideal P_1 (or permutations thereof). In this case, it can be shown that $P_1^{\langle 2\rangle} = P_1^2$, i.e. the second symbolic power is easily computed as it coincides with the second (standard) power.

It is useful to write $\mathfrak{J}$ in terms of ideals generated by two polynomials to see the connection to partial fractions. Specifically, we can write $\mathfrak{J}$ as[1]

$$\begin{aligned} \mathfrak{J} = \Big[&\big\langle[12],[13],[23]\big\rangle^2_{R_6}\cap\big\langle\langle 12\rangle,\langle 34\rangle\big\rangle_{R_6}\cap\big\langle\langle 12\rangle,[16]\big\rangle_{R_6}\cap\big\langle\langle 34\rangle,[12]\big\rangle_{R_6}\cap\big\langle[12],[13]\big\rangle_{R_6}\\ &\cap\big\langle[15],[16]\big\rangle_{R_6}\cap\big\langle\langle 23\rangle,s_{345}\big\rangle_{R_6}\cap\big\langle\langle 12\rangle,s_{123}\big\rangle_{R_6}\cap\big\langle[12],s_{123}\big\rangle_{R_6}\cap\big\langle[12],s_{345}\big\rangle_{R_6}\\ &\cap\big\langle\langle 4|2+3|1],s_{123}\big\rangle_{R_6}\cap\big\langle\langle 6|1+2|3],s_{345}\big\rangle_{R_6}\Big]\cap\Big[123456\to\overline{456123}\Big]\,, \end{aligned} \tag{22}$$

where we have also used the primary decomposition

$$\big\langle\langle 1|3+4|2],s_{134}\big\rangle_{R_6} = P_1(134256)\cap P_1(\overline{256134})\cap P_7(\overline{341562})\cap P_7(562341)\,. \tag{23}$$

[1]We note that the argument of each square bracket is not in one-to-one correspondence with that of Eq. (21).

Interestingly, this involves a spinor chain despite it not being part of the initial set of singularities. Some of these intersections can be further carried out compactly,

$$\begin{aligned}\mathfrak{J} = \Big[&\big\langle[12],[13],[23]\big\rangle^2_{R_6} \cap \big\langle s_{12}, \langle 34\rangle[16][12], \langle 34\rangle\langle 12\rangle[13], \langle 34\rangle[13][16], [16][23]\langle 34\rangle\big\rangle_{R_6} \\ &\cap \big\langle\langle 23\rangle[15], \langle 23\rangle[16], [16]s_{345}, [15]s_{345}\big\rangle_{R_6} \cap \big\langle\langle 12\rangle[12], [12]s_{123}, s_{123}s_{345}\big\rangle_{R_6} \\ &\cap \big\langle\langle 4|2+3|1], s_{123}\big\rangle_{R_6} \cap \big\langle\langle 6|1+2|3], s_{345}\big\rangle_{R_6}\Big] \cap \Big[123456 \to \overline{456123}\Big]\,. \end{aligned} \tag{24}$$

Importantly, such a representation would allow the approach of Ref. [1] to compute $\mathcal{M}_{d,\vec{\phi}} \cap \mathfrak{J}$ more efficiently.

Nevertheless, in this simple example, we can compute the intersection $\mathcal{M}_{d,\vec{\phi}} \cap \mathfrak{J}$ directly by explicitly computing $\mathfrak{J}$ from one of the above Eqs. (21), (22) or (24) up to the required mass dimension $d = 6$. This can be done with `Singular` by setting an appropriate degree bound (`degBound`). The resulting Ansatz is completely fixed, up to an overall numerical prefactor c_0,

$$\mathcal{M}_{d,\vec{\phi}} \cap \mathfrak{J} = c_0\big(\langle 12\rangle[12]\langle 45\rangle[45]\langle 4|2+3|1] + [16]\langle 34\rangle\langle 6|1+2|3]s_{123}\big)\,. \tag{25}$$

Interestingly, we have shown that $A^{(0)}_{0\to q^+g^+g^+\bar{q}^-g^-g^-}$ is in fact the unique function with the given zeros and poles in complex phase space. In general, when considering rational coefficients of master integrals or of special transcendental functions, the ansatz obtained from $\mathcal{M} \cap \mathfrak{J}$ may still contain several terms, rather than a single one. Nevertheless, the imposed constraints are significant.

3.2 Application to Leading-Color Two-Loop Tri-Photon Production Amplitudes

The strategy of Ref. [1] was constructed in order to decrease the number of samples required when computing two-loop multi-scale scattering amplitudes. In order to show the benefits of our approach in a two-loop five-point context, we applied it to the finite remainders of the two-loop three-photon production amplitudes of Ref. [18]. Working with the analytic form of these results, we sampled them close to all codimension-two varieties where the amplitude exhibits a (potentially spurious) singularity. We therefore used 317 p-adic evaluations to construct the Ansätze. In Table 2, we characterize the improvements that our strategy provides in terms of the number of free parameters of the Ansatz. Comparing the size of the Ansatz used in Ref. [18], $\dim(\mathcal{M}_{\tilde{d},\vec{0}})$, with the size of Ansatz in spinor variables taking into account numerator vanishing constraints, $\dim(\mathcal{M}_{d,\vec{\phi}}) \cap \mathfrak{J}$, we see an improvement of a factor of 73 in the worst case.

4. Conclusion and Summary

Modern methods for multi-scale two-loop scattering-amplitude computation employ Ansätze and numerical sampling to determine rational coefficients of special functions. The major bottleneck in this approach is the number of numerical samples required, which is dictated by the number of free parameters in the Ansätze. In this talk we discussed a strategy for building compact Ansätze. Specifically, we build Ansätze that match the behavior of the rational coefficients near varieties where amplitudes exhibit (potentially spurious) singularities. It turns out that these varieties may branch, and so we discussed a set of computational tools that one can use to describe this branching. We determine the behavior of the coefficients near irreducible varieties from numerical evaluations.

Remainder	vector space: Mandelstam Ansatz	dim. of $\mathcal{M}_{\vec{d},\vec{0}}$	vector space: spinor Ansatz	dim. of $\mathcal{M}_{d,\vec{\phi}}$	dim. of $\mathcal{M}_{d,\vec{\phi}} \cap \mathfrak{J}$	ratio
$R^{\{2,0\}}_{\gamma^-\gamma^+\gamma^+}$	$\mathcal{M}_{\{50,\vec{0}\}}$	41301	$\mathcal{M}_{35,\{3,0,6,-3,-2\}}$	7358	566	73
$R^{\{2,N_f\}}_{\gamma^-\gamma^+\gamma^+}$	$\mathcal{M}_{\{24,\vec{0}\}}$	2821	$\mathcal{M}_{15,\{-2,-2,0,-3,-3\}}$	378	20	141
$R^{\{2,0\}}_{\gamma^+\gamma^+\gamma^+}$	$\mathcal{M}_{\{32,\vec{0}\}}$	7905	$\mathcal{M}_{20,\{-2,-4,-2,-2,-2\}}$	1140	18	439
$R^{\{2,N_f\}}_{\gamma^+\gamma^+\gamma^+}$	$\mathcal{M}_{\{18,\vec{0}\}}$	1045	$\mathcal{M}_{8,\{1,3,1,1,2\}}$	44	6	174

Table 2: Characterizing data for Mandelstam, spinor and refined spinor Ansätze. Spinor helicity variables alone can be seen to reduce the size of the Ansatz by almost an order of magnitude. The size is reduced by a further order of magnitude by imposing the numerator vanishing constraints on codimension-two varieties.

To perform these evaluations we introduced the p-adic numbers, a cousin of finite fields which allow one to construct phase-space points close to a given variety. Interestingly, the p-adic numbers admit a theory of calculus, allowing one to discuss functions defined through power series such as the p-adic logarithm. With the behavior of the coefficients in singular regions in hand, we then discussed how to interpret the degree of vanishing on the numerator on such varieties algebraically. This then allows us to construct a complete Ansatz matching this behavior. To discuss our approach in practice, we looked at the example of a six-point NMHV tree amplitude and built an Ansatz that required only one sample to determine the full analytic result. Thereafter, we demonstrated the impact of our strategy on a proof-of-concept two-loop example, the leading-color amplitudes for three-photon production at hadron colliders. We look forward to applying this approach to scattering amplitudes involving an increasingly higher number of scales in the future.

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