PROCEEDINGS OF SCIENCE

Epsilon Factorized Differential Equations for Elliptic Feynman Integrals

PoS(LL2022)039

Hjalte Frellesvig[a,*]

[a] *Niels Bohr International Academy, University of Copenhagen, Blegdamsvej 17, 2100 København Ø, Denmark*

E-mail: hjalte.frellesvig@nbi.ku.dk

This is a contribution to the proceedings of the 2022 "Loops and Legs" conference. It is based on a talk discussing recent work on epsilon factorized differential equations for Feynman integrals which are beyond polylogarithmic and have an elliptic highest sector. It introduces a systematic method to find such epsilon factorized bases, that works by requiring the period matrix of the integrands to be diagonal. In this contribution we will go through a new example (the elliptic box-triangle) not discussed in the original publication.

Loops and Legs in Quantum Field Theory - LL2022,
25-30 April, 2022
Ettal, Germany

*Speaker

1. Introduction

The method of differential equations is the main method used to compute Feynman integrals for phenomenological use. It consist of doing a derivative with respect to a kinematic variable of the members of the relevant integral family, and then map the results back to the original integrals using IBP relations. This will give a system of coupled first-order differential equations, which in principle can be solved using traditional methods. Yet a great amount of progress happened with the introductions of *canonical forms* [2] of Feynman integrals, which are bases with the property that differential equations has the form

$$\frac{\partial}{\partial s} J = \epsilon A^{(s)} J \tag{1}$$

where J is the vector of master integrals, and where $A^{(s)}$ is a matrix free of dependence on the space-time dimension. One additional requirement for canonical forms in the traditional sense is the requirement that the entries of A are of d-log form, which means that they can be written as s-derivatives of logarithms of algebraic function of s. If those algebraic functions are rational the equation system can trivially be integrated up order by order in ϵ yielding results in the function class of generalized polylogarithms.

Yet it is not all Feynman integrals that can be integrated to generalized polylogs, nor can they all be brought to a canonical form in the traditional sense. A simple class of integrals for which that is not possible are those known as elliptic, characterized by the presence of an elliptic curve at the maximal cut. Such functions have been the object of intense study in recent years (see ref. [3] for an overview and the references therein). Yet it is a good open question how much of the benefits of canonical forms that can be made to apply beyond the polylogarithmic case. For the purpose of these proceedings that refers specifically to eq. (1), and the goal is to make an algorithm that can bring an arbitrary Feynman integral, polylogarithmic, elliptic, or beyond, to a form where its differential equations are epsilon factorized.

This is inspired by a number of recent publications[1] [4–6] in which various elliptic Feynman integrals successfully have been put to such an epsilon factorized form.

Central to the approach discussed here is the idea of using combinations of maximal cuts and varying integration contours to analyze Feynman integrals. This in itself is not new but has a history [7–12] in the context of IBPs, differential equations, dimension shift relations, and more. The work discussed here can be considered a continuation of the research direction of those works. Likewise such ideas have been used in the past to find integrals with epsilon factorized differential equations, by requiring that the integrals are pure on the maximal (and sub-maximal) cuts [13–15], and the approch discussed in these proceedings reduce to that in the polylogarithmic case. Additionally our approach is similar to methods used to find integrands in $\mathcal{N}$=4 sYM theory under the name of prescriptive unitarity [16, 17]. Finally we should mention the duality between integrands and contours, and the related fact that the set of integrands spans a vector space, is playing a significant role. That connection has been clarified in recent works on the relation between Feynman integrals and the mathematical field of intersection theory[2] [18, 19], from which a lot of the notation here is borrowed.

[1]See also the talk and proceedings by Stefan Weinzierl.

[2]See also the talks and proceedings by Vsevolod Chestnov and Henrik Munch.

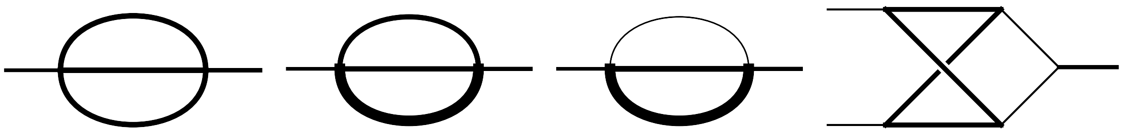

Figure 1: The four elliptic examples considered in ref. [1]. These are three variations on the elliptic sunrise integral, and the non-planar double triangle.

2. The algorithm

The approach to finding epsilon factorized differential equations discussed here is similar to what has been used in the past for the polylogarithmic case, but formulated in a way that makes it generalize to the elliptic case and beyond. We will write our family of Feynman integrals as

$$J_i = \mathcal{K} \int_C u \hat{\varphi}_i \mathrm{d}^n z \tag{2}$$

where u is a multivalued function, and $\hat{\varphi}$ a rational function. Additionally we have $\mathcal{K}$ as a prefactor, C as an integration contour, and z as the n integration variables. That this form of the integrals exist follows from the Baikov representation. In this language the claim is that if

$$\mathcal{K} u \hat{\varphi}_i = \sigma \hat{\Phi}_i \tag{3}$$

(where σ is a *pure function*, and $\hat{\Phi}_i$ an algebraic function), and additionally the *period matrix*

$$P_{ij} := \int_{\gamma_j} \hat{\Phi}_i \mathrm{d}^n z \tag{4}$$

is proportional to the unit matrix $P = (2\pi i)^n I$ then the integrals J_i will have differential equations in ϵ-factorized form.

To apply this algorithm in practice we will write the integrals J as linear combination of a known intermediate basis I as $J_i = f_{ij} I_j$ or correspondingly $\hat{\Phi}_i = f_{ij} \hat{\Phi}_j^{\text{int}}$ at the integrand level. In that case $P = (2\pi i)^n I$ becomes a constraint, that uniquely will fix all the free coefficients f_{ij}, giving a basis with ϵ-factorized differential equations.

3. Example

In ref. [1] we did (in addition to two polylogarithmic warm-up examples) four elliptic examples, all on the maximal cut, of the Feynman integrals depicted in fig. 1.

But for these proceedings we will look at a different example depicted on fig. 2, which we will refer to as the elliptic box-triangle (ebt). Such integrals will be of relevance for for instance NNLO QCD corrections to double-top production at hadron colliders. Its kinematics is such that $p_1^2 = p_2^2 = 0$, $p_3^2 = p_4^2 = m^2$, $(p_1{+}p_2)^2 = s$, $(p_2{+}p_3)^2 = t$, and it is defined by the propagators

$$\begin{aligned} D_1 &= (k_1{-}p_1{-}p_2)^2 - m^2\,, & D_3 &= k_1^2 - m^2\,, & D_5 &= (k_2{+}p_3)^2\,,\\ D_2 &= (k_1{-}p_1)^2 - m^2\,, & D_4 &= k_2^2 - m^2\,, & D_6 &= (k_1{-}k_2)^2\,. \end{aligned} \tag{5}$$

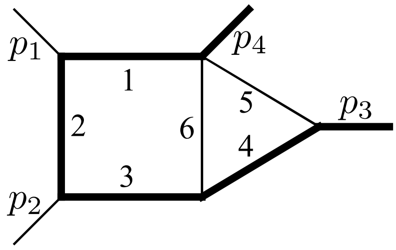

Figure 2: The elliptic box-triangle.

Within the loop-by-loop Baikov parametrization we may write this integral as a seven fold integral, introducing the seventh propagator

$$z = D_7 = (k_1{+}p_3)^2 \tag{6}$$

On the maximal cut (i.e. the cut of D_1-D_6) this becomes univariate and the integral family may be written as

$$I^{\text{ebt}} = \mathcal{K} \int_C u\varphi \tag{7}$$

with

$$u = s^{\frac{d-6}{2}} z^{\frac{d-5}{2}} (4m^2 - z)^{(3-d)/2} \big(z^2 - 2(m^2{+}t)z - (4m^2{-}s)(m^2{-}t)^2/s\big)^{\frac{d-5}{2}} \tag{8}$$

and

$$\mathcal{K} = \frac{2^{2-d}(m^2)^{\frac{d-4}{2}} \big((m^2 - t)^2 + st\big)^{\frac{4-d}{2}}}{\pi^{\frac{5}{2}} \Gamma\big(\frac{d-3}{2}\big)\Gamma\big(\frac{d-2}{2}\big)} \tag{9}$$

The expression for φ will depend on the powers of the propagators of the given integral. We can indeed write this integrand as

$$\mathcal{K}u\varphi = \sigma \frac{\phi}{Y} \qquad \text{where} \qquad Y = \sqrt{z\,(4m^2{-}z)\,\big(z^2 - 2(m^2{+}t)z - (4m^2{-}s)(m^2{-}t)^2/s\big)} \tag{10}$$

and where σ is pure and ϕ is rational.

There are three master integrals on the cut, and we pick as an intermediate basis

$$I_1 = I^{\text{ebt}}_{1111110} \qquad\qquad I_2 = I^{\text{ebt}}_{1111210} \qquad\qquad I_3 = I^{\text{ebt}}_{111111\text{-}1} \tag{11}$$

corresponding to

$$\hat{\phi}_1 = \frac{1}{s} \qquad\qquad \hat{\phi}_2 = \frac{2\epsilon}{sz} \qquad\qquad \hat{\phi}_3 = \frac{z}{s} \tag{12}$$

Likewise we need a set of three independent integration cycles. Defining

$$R := \sqrt{\frac{(m^2{-}t)^2 + st}{s\,m^2}} \tag{13}$$

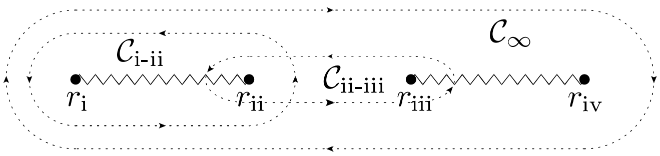

Figure 3: The three independent contours for the elliptic box-triangle.

we can name the four roots of Y

$$r_{\text{i}} = t{+}m^2(1{-}2R)\,, \qquad r_{\text{ii}} = 0\,, \qquad r_{\text{iii}} = t{+}m^2(1{+}2R)\,, \qquad r_{\text{iv}} = 4m^2\,. \tag{14}$$

The we may pick as our three independent contours

$$\gamma_1 = \mathcal{C}_{\text{ii-iii}}\,, \qquad \gamma_2 = i\mathcal{C}_{\text{i-ii}}\,, \qquad \gamma_3 = \mathcal{C}_{\infty}\,. \tag{15}$$

which are depicted on fig. 3. This choice is identical to the case of the two-mass elliptic sunrise integral discussed in ref. [1].

We may then compute the integrals of the intermediate basis over the three cycles. We get

$$\begin{aligned}
&\int_{\gamma_1} \frac{\phi_1}{Y} = \frac{1}{m^2\sqrt{R}s} K(k^2) \qquad\qquad \int_{\gamma_2} \frac{\phi_1}{Y} = \frac{1}{m^2\sqrt{R}s} K(\bar{k}^2) \\
&\int_{\gamma_1} \frac{\phi_2}{Y} = \frac{-2\epsilon}{(4m^2{-}s)(m^2{-}t)^2} \left(4\sqrt{R}E(k^2) + \frac{m^2(1-2R)+t}{m^2\sqrt{R}} K(k^2) \right) \\
&\int_{\gamma_2} \frac{\phi_2}{Y} = \frac{2\epsilon}{(4m^2{-}s)(m^2{-}t)^2} \left(4\sqrt{R}E(\bar{k}^2) - \frac{m^2(1+2R)+t}{m^2\sqrt{R}} K(\bar{k}^2) \right) \\
&\int_{\gamma_1} \frac{\phi_3}{Y} = \frac{(4m^2{-}s)(m^2{-}t)}{2m^2\sqrt{R}s^2} \left(K(k^2) - \frac{(1{+}R)((3-2R)m^2+t)}{(1{-}R)((1{+}2R)m^2+t)} \Pi(n^2,k^2) \right) \\
&\int_{\gamma_2} \frac{\phi_3}{Y} = \frac{(4m^2{-}s)(m^2{-}t)}{2m^2\sqrt{R}s^2} \left(K(\bar{k}^2) + \frac{(1{-}R)((3+2R)m^2-t)}{(1{+}R)((1{-}2R)m^2+t)} \Pi(\bar{n}^2,\bar{k}^2) \right) \\
&\int_{\gamma_3} \frac{\phi_1}{Y} = 0 \qquad\qquad \int_{\gamma_3} \frac{\phi_2}{Y} = 0 \qquad\qquad \int_{\gamma_3} \frac{\phi_3}{Y} = \frac{-\pi}{s}
\end{aligned} \tag{16}$$

where the arguments of the elliptic integrals are

$$k^2 := \frac{((3+2R)m^2-t)((1{+}2R)m^2+t)}{16m^4R} \qquad \bar{k}^2 := \frac{((3-2R)m^2-t)((2R{-}1)m^2-t)}{16m^4R} \tag{17}$$

$$n^2 := \frac{(m^2{-}t)^2}{4m^4(1{-}R)^2} \qquad\qquad \bar{n}^2 := \frac{(m^2{-}t)^2}{4m^4(1{+}R)^2} \tag{18}$$

which obey the relations $\bar{k}^2 = 1 - k^2$ and $\bar{n}^2/\bar{k}^2 = 1 - n^2/k^2$.

If we construct the integrands of our candidate integrals as

$$\varphi_i = f_{ij}\phi_j \tag{19}$$

we may now construct our period matrix. If we name the above entries as $g_{kj} := \int_{\gamma_j} \frac{\phi_k}{Y}$ the entries of the period matrix are given as

$$P_{ij} = f_{ik}g_{kj} \tag{20}$$

Imposing that $P = (2\pi i)I$ corresponds to fixing the f_{ij} coefficients as $f_{ij} = 2\pi i(g^{-1})_{ij}$ and with this we have our candidate integrals. The expressions for the f_{ij} will not be written here.

Those candidate integrals do indeed have epsilon factorized differential equations

$$\frac{\partial}{\partial x} J = \epsilon A^{(x)} J \tag{21}$$

We will not write all the entries of the A-matrices here, one example is

$$\begin{aligned} A_{13}^{(m^2)} &= \frac{2}{m^2(m^2-t)\sqrt{R}(4m^4-4m^2t-9m^2s+st)} \\ &\times \Big(\big(t^3-(m^2-s)t^2+3m^2(m^2+s)t-3m^6\big)\big(K(k^2)-2E(k^2)\big) + \big(m^2t(5s+2t)-st^2-2m^6\big)\sqrt{R}K(k^2)\Big) \end{aligned} \tag{22}$$

and the other entries will have a similar form containing linear or quadratic combinations of the complete elliptic integrals.

4. Discussion

The differential equation matrix, of which one example entry is given by eq. (22) looks in many cases rather cumbersome. In the cases in the literature in which other examples of elliptic Feynman integrals have been put to epsilon-factorized form [4–6] the results have properties that allow them to be integrated up in a systematic fashion. The outcomes of my algorithm (as discussed here and in ref. [1]) do not straightforwardly have such properties (even though there are hints, such as the fact that the matrix entries never have poles of degree higher than one), so to find a modification of my approach that produces such a form directly is an interesting research direction for the future.

References

[1] H. Frellesvig, *On epsilon factorized differential equations for elliptic Feynman integrals*, *JHEP* **03** (2022) 079, [arXiv:2110.07968].

[2] J. M. Henn, *Multiloop integrals in dimensional regularization made simple*, *Phys. Rev. Lett.* **110** (2013) 251601, [arXiv:1304.1806].

[3] J. L. Bourjaily et al., *Functions Beyond Multiple Polylogarithms for Precision Collider Physics*, in *2022 Snowmass Summer Study*, 3, 2022. arXiv:2203.07088.

[4] L. Adams and S. Weinzierl, *The ε-form of the differential equations for Feynman integrals in the elliptic case*, *Phys. Lett.* **B781** (2018) 270–278, [arXiv:1802.05020].

[5] C. Bogner, S. Müller-Stach, and S. Weinzierl, *The unequal mass sunrise integral expressed through iterated integrals on* $\overline{\mathcal{M}}_{1,3}$, *Nucl. Phys. B* **954** (2020) 114991, [arXiv:1907.01251].

[6] H. Müller and S. Weinzierl, *A Feynman integral depending on two elliptic curves*, arXiv:2205.04818.

[7] R. N. Lee and V. A. Smirnov, *The Dimensional Recurrence and Analyticity Method for Multicomponent Master Integrals: Using Unitarity Cuts to Construct Homogeneous Solutions*, *JHEP* **12** (2012) 104, [arXiv:1209.0339].

[8] A. Primo and L. Tancredi, *On the maximal cut of Feynman integrals and the solution of their differential equations*, *Nucl. Phys.* **B916** (2017) 94–116, [arXiv:1610.08397].

[9] J. Bosma, M. Sogaard, and Y. Zhang, *Maximal Cuts in Arbitrary Dimension*, *JHEP* **08** (2017) 051, [arXiv:1704.04255].

[10] H. Frellesvig and C. G. Papadopoulos, *Cuts of Feynman Integrals in Baikov representation*, *JHEP* **04** (2017) 083, [arXiv:1701.07356].

[11] A. Primo and L. Tancredi, *Maximal cuts and differential equations for Feynman integrals. An application to the three-loop massive banana graph*, *Nucl. Phys.* **B921** (2017) 316–356, [arXiv:1704.05465].

[12] M. Harley, F. Moriello, and R. M. Schabinger, *Baikov-Lee Representations Of Cut Feynman Integrals*, *JHEP* **06** (2017) 049, [arXiv:1705.03478].

[13] J. M. Henn, *Lectures on differential equations for Feynman integrals*, *J. Phys.* **A48** (2015) 153001, [arXiv:1412.2296].

[14] P. Wasser, *Analytic properties of Feynman integrals for scattering amplitudes*, *M.Sc.* (2016) [https://publications.ub.uni-mainz.de/theses/frontdoor.php?source opus=100001967].

[15] S. Weinzierl, *Feynman Integrals*, arXiv:2201.03593.

[16] J. L. Bourjaily, E. Herrmann, and J. Trnka, *Prescriptive Unitarity*, *JHEP* **06** (2017) 059, [arXiv:1704.05460].

[17] J. L. Bourjaily, N. Kalyanapuram, C. Langer, and K. Patatoukos, *Prescriptive unitarity with elliptic leading singularities*, *Phys. Rev. D* **104** (2021), no. 12 125009, [arXiv:2102.02210].

[18] P. Mastrolia and S. Mizera, *Feynman Integrals and Intersection Theory*, *JHEP* **02** (2019) 139, [arXiv:1810.03818].

[19] H. Frellesvig, F. Gasparotto, S. Laporta, M. K. Mandal, P. Mastrolia, L. Mattiazzi, and S. Mizera, *Decomposition of Feynman Integrals on the Maximal Cut by Intersection Numbers*, *JHEP* **05** (2019) 153, [arXiv:1901.11510].