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Feynman Integral Relations from GKZ Hypergeometric Systems

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We study Feynman integrals in the framework of Gel'fand-Kapranov-Zelevinsky (GKZ) hypergeometric systems. The latter defines a class of functions wherein Feynman integrals arise as special cases, for any number of loops and kinematic scales. Utilizing the GKZ system and its relation to D-module theory, we propose a novel method for obtaining differential equations for master integrals. This note is based on the longer manuscript [1].

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1. Introduction

Feynman integrals (FIs) are ubiquitous in perturbation theory calculations for research areas as diverse as quantum field theory, condensed matter theory, the weak field limit of gravity and fluid dynamics. However, despite immense progress spanning more than half a century, many analytic properties of FIs remain to be understood in full generality; for instance their linear [2, 3] and quadratic [4] relations, the differential equations they are subject to [5], their ϵ-expansions under the dimensional regularization scheme [6], and more.

A key step towards uncovering the analytic properties of FIs, in full generality, is to first understand the space of functions to which they belong. Given any number of scales or loops, FIs are now known to evaluate to Gel'fand-Kapranov-Zelevinsky (GKZ) hypergeometric functions [7, 8] when the GKZ variables take on special values [9–19] . The study of GKZ hypergeometric functions is, accordingly, beneficial for uncovering generic, analytic properties of FIs.

In this note, we shall employ the GKZ perspective to study Pfaffian systems (first-order PDEs) for FIs [1]. We especially benefit from the relation of GKZ systems to D-modules [20, 21] and intersection theory [22–33] (see also the proceedings [34]). We have organized the remaining part of this note as follows. In section 2, we define the GKZ system of PDEs, and show how FIs arise as solutions to these equations. In section 3, we introduce Pfaffian systems, and show to obtain them using the Macaulay matrix method. We conclude in section 4.

2. GKZ hypergeometric systems

A GKZ hypergeometric function is defined as the solution to a particular system of PDEs, generally of higher order. These PDEs are fixed by an integer matrix A and a complex vector β. For FIs, the A matrix is determined by the set of propagators, while the β vector depends on the propagator powers and spacetime dimension of the integral.

Let us write the GKZ system of PDEs. Consider N integer vectors $a_1, \ldots, a_N \in \mathbb{Z}^{n+1}$, each of length $n+1$. We collect these vectors into the columns of an $(n+1) \times N$ matrix

$$A = \begin{pmatrix} a_1 & \ldots & a_N \end{pmatrix} . \tag{1}$$

The (left) kernel of A is defined as the set of vectors annihilated by A: $\mathrm{Ker}(A) = \left\{ u = (u_1, \ldots, u_N) \in \mathbb{Z}^N \mid A \cdot u = \mathbf{0} \right\}$. Moreover, fix $n+1$ complex parameters $\beta = (\beta_0, \ldots, \beta_n) \in \mathbb{C}^{n+1}$ and let $z = (z_1, \ldots, z_N) \in \mathbb{C}^N$ be N complex variables. A *GKZ hypergeometric function* $f_\beta(z)$ is defined by satisfying the *GKZ system*

$$E_j \bullet f_\beta(z) = 0 \,, \quad j = 1, \ldots, n+1 \tag{2}$$

$$\Box_u \bullet f_\beta(z) = 0 \,, \quad \forall u \in \mathrm{Ker}(A) \,, \tag{3}$$

where the differential operators E_j and $\Box_u$ are given by

$$E_j = \sum_{i=1}^{N} a_{i,j} \, z_i \, \frac{\partial}{\partial z_i} - \beta_{j-1} \quad , \quad \Box_u = \prod_{u_i > 0} \left(\frac{\partial}{\partial z_i} \right)^{u_i} - \prod_{u_i < 0} \left(\frac{\partial}{\partial z_i} \right)^{-u_i} \,, \tag{4}$$

and $a_{i,j}$ denotes the jth component of the column vector a_i.

[1]This is work in collaboration with Vsevolod Chestnov, Federico Gasparotto, Manoj K. Mandal, Pierpaolo Mastrolia, Saiei J. Matsubara-Heo, and Nobuki Takayama.

2.1 Euler integrals

GKZ hypergeometric functions enjoy an *Euler integral* representation [2]

$$f_\beta(z) = \int_C g(z;x)^{\beta_0} \, x_1^{-\beta_1} \cdots x_n^{-\beta_n} \frac{\mathrm{d}x}{x} \quad , \quad \frac{\mathrm{d}x}{x} := \frac{\mathrm{d}x_1}{x_1} \wedge \cdots \wedge \frac{\mathrm{d}x_n}{x_n} \, , \tag{5}$$

where C is an integration contour and $g = g(z;x)$ is a Laurent polynomial in integration variables x with monomial coefficients z:

$$g(z;x) = \sum_{i=1}^{N} z_i \, x^{\alpha_i} \quad , \quad x^{\alpha_i} := x_1^{\alpha_{i,1}} \cdots x_n^{\alpha_{i,n}} \quad , \quad \alpha_i \in \mathbb{Z}^n \, . \tag{6}$$

Note that we have n integration variables $x = (x_1, \ldots, x_n)$ and N monomials in g.

The column vectors $a_1, \ldots, a_N \in \mathbb{Z}^{n+1}$ of the A matrix (1) are related to the exponent vectors $\alpha_1, \ldots, \alpha_N \in \mathbb{Z}^n$ of g via $a_i = (1 \; \alpha_i)^T$.

Example. *Euler integral.* Consider the $(2+1) \times 4$ matrix

$$A = \begin{array}{cc} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix} & \begin{array}{c} \\ x_1 \\ x_2 \end{array} \\ \begin{array}{cccc} z_1 & z_2 & z_3 & z_4 \end{array} & \end{array} \; . \tag{7}$$

It defines the Euler integral

$$f_\beta(z) = \int_C g(z;x)^{\beta_0} x_1^{-\beta_1} x_2^{-\beta_2} \frac{\mathrm{d}x}{x} \quad , \quad g(z;x) = z_1 x_1 + z_2 x_2 + z_3 x_1^2 + z_4 x_1 x_2 \tag{8}$$

in terms of $n = 2$ integration variables $x = (x_1, x_2)$ and $N = 4$ monomial coefficients $z = (z_1, z_2, z_3, z_4)$.

Let us also write the GKZ system of PDEs satisfied by $f_\beta(z)$. The A matrix in question leads to the following E_j operators:

$$E_1 = 1 \cdot \theta_1 \; + \; 1 \cdot \theta_2 \; + \; 1 \cdot \theta_3 \; + \; 1 \cdot \theta_4 \; - \; \beta_0 \tag{9a}$$

$$E_2 = 1 \cdot \theta_1 \; + \; 0 \cdot \theta_2 \; + \; 2 \cdot \theta_3 \; + \; 1 \cdot \theta_4 \; - \; \beta_1 \tag{9b}$$

$$E_3 = 0 \cdot \theta_1 \; + \; 1 \cdot \theta_2 \; + \; 0 \cdot \theta_3 \; + \; 1 \cdot \theta_4 \; - \; \beta_2 \, , \tag{9c}$$

where we defined the Euler operators $\theta_i := z_i \frac{\partial}{\partial z_i}$. To write the $\Box_u$ operators, we first calculate that $\mathrm{Ker}(A) = \mathrm{span}\{u\}$, $u = \big(1, -1, -1, 1\big)^T$. We therefore have a single $\Box_u$ operator given by $\Box_u = \partial_1 \partial_4 - \partial_2 \partial_3$, where we defined $\partial_i := \frac{\partial}{\partial z_i}$.

2.2 Generalized Feynman integrals

Having defined the GKZ hypergeometric system and its associated Euler integral, we are now well equipped to make the connection to FIs. We begin by defining a *generalized FI* (GFI) as

$$\mathcal{I}(d_0; \nu) := c(d_0; \nu) \, f_\beta(z) \tag{10}$$

[2] It is possible to generalize to the case of several polynomial factors in the integrand, though we shall not require it here.

involving a special choice for the β vector:

$$\beta = (\epsilon, -\epsilon\delta, \dots, -\epsilon\delta) \; - \; (d_0/2, \nu_1, \dots, \nu_n) \tag{11}$$
$$d_0 \in \mathbb{Z}_{>0}\,, \;\; \nu = (\nu_1, \dots, \nu_n) \in \mathbb{Z}^n\,, \;\; 0 < \epsilon, \delta \ll 1\,.$$

The constant $c(d_0; \nu)$ is given by a ratio of Γ-functions (see section 5 of [1]). Now, consider the following identifications:

- Integration contour of $f_\beta(z)$: $C = (0, \infty)^n$
- δ appearing in (11): $\delta \to 0$
- z variables of $f_\beta(z)$: $z_i \in \mathbb{Z}_{>0} \cup \{m_i^2, p_i^2, p_i \cdot p_j\}$.

The m_i denote masses and p_i are external momenta. Under these identifications, the GFI reduces to the *Lee-Pomeransky representation* (LPr) [35] of an L-loop FI in $d = d_0 - 2\epsilon$ spacetime dimensions, with propagator powers $\nu = (\nu_1, \dots, \nu_n)$. In the LPr, the polynomial g in the integrand takes the form $g = \mathcal{U} + \mathcal{F}$, where $\mathcal{U}, \mathcal{F}$ are the first and second Symanzik polynomials.

The key difference between the LPr and its corresponding GFI has to do with the status of the monomial coefficients in g. For the GFI, each monomial coefficient z_i is regarded as an independent variable. For the LPr, the z_i may not be independent. At the end of a calculation involving GFIs, we may use the identifications itemized above to match with the LPr.

Example. *Bubble integral.* To illustrate the difference between the LPr and its associated GFI, let us give the example of a one-loop bubble diagram with one internal mass. The propagators are $D_1 = -\ell^2 + m^2$ and $D_2 = -(\ell + p)^2$, where ℓ is the integration momentum, m is the mass, and p is the external momentum.

The LPr is proportional to

$$\int_{(0,\infty)^2} \left(x_1 + x_2 + m^2 x_1^2 + \left(m^2 - p^2\right) x_1 x_2\right)^{\epsilon - d_0/2} x_1^{\nu_1} x_2^{\nu_2} \, \frac{\mathrm{d}x}{x}\,, \tag{12}$$

with ν_i denoting the exponent of propagator D_i in momentum space representation.

The corresponding GFI is proportional to

$$\int_C \left(z_1 x_1 + z_2 x_2 + z_3 x_1^2 + z_4 x_1 x_2\right)^{\epsilon - d_0/2} x_1^{\nu_1 + \epsilon\,\delta} \, x_2^{\nu_2 + \epsilon\,\delta} \, \frac{\mathrm{d}x}{x}\,. \tag{13}$$

Note that the GKZ system for (13) was presented in the example of section 2.1.

3. Pfaffian systems

Using the GKZ framework, let us outline how to obtain the system of first-order PDEs obeyed by a basis of master integrals. In other words, given a basis $\vec{\mathcal{I}}$ and kinematic variables $z = (z_1, \dots, z_N)$, we seek the PDEs

$$\partial_i \vec{\mathcal{I}} = P_i \cdot \vec{\mathcal{I}}\,, \quad i = 1, \dots, N\,, \tag{14}$$

where the matrices $P_i = P_i(z)$ contain rational functions in z and satisfy the integrability conditions $\partial_i P_j - \partial_j P_i = [P_i, P_j]$. We dub (14) the *Pfaffian system* for $\vec{\mathcal{I}}$ and the P_i are called *Pfaffian matrices*.

3.1 From integrals to operators

We propose to obtain the Pfaffian system using tools from D-module theory [20, 21]. In this setting, we begin by representing GFIs as partial differential operators w.r.t. the z variables. This means that our computations will be performed inside a rational Weyl algebra [3]

$$\mathcal{R}_N = \left\{ \sum_{k\in K} h_k(z)\partial^k \;\middle|\; K \subset \mathbb{Z}_{>0} \text{ is finite }, \; h_k(z) \text{ is rational in } z \right\} \tag{15}$$

$$[z_i, z_j] = [\partial_i, \partial_j] = 0 \,, \;\; [\partial_i, z_j] = \delta_{ij} \,. \tag{16}$$

It can be shown [32] that there exists $\mathcal{D} = \mathcal{D}(d_0; \nu; z) \in \mathcal{R}_N$ such that

$$\mathcal{D} \bullet \mathcal{I}(0;0) = \mathcal{I}(d_0;\nu) \quad , \quad \mathcal{I}(0;0) = c(0;0) \int_C g(z;x)^{\epsilon} x_1^{\epsilon\,\delta} \cdots x_n^{\epsilon\,\delta} \frac{\mathrm{d}x}{x} \,. \tag{17}$$

Using this fact, we can let $\mathcal{D}$ *represent* the GFI $\mathcal{I}(d_0;\nu)$, thereby opening the door to perform manipulations on operators instead of integrals.

The existence of $\mathcal{D}$ follows from the isomorphism between GKZ hypergeometric systems and twisted cohomology groups [31]. The $\mathcal{D}$ operators can be obtained using the package `mt_gkz` [32] implemented in the computer algebra system `Risa/Asir` [36].

Example. *Bubble integral as an operator.* Let use find the Weyl algebra element corresponding to the bubble GFI (13) with $\nu_1 = \nu_2 = 1$ and $d_0 = 4$. We hence seek $\mathcal{D}$ such that $\mathcal{D} \bullet \mathcal{I}(0;0,0) = \mathcal{I}(4;1,1)$, i.e.

$$\mathcal{D} \bullet \int_C g(z;x)^{\epsilon} x_1^{\epsilon\,\delta}\, x_2^{\epsilon\,\delta} \frac{\mathrm{d}x}{x} = \int_C g(z;x)^{\epsilon-2} x_1^{1+\epsilon\,\delta}\, x_2^{1+\epsilon\,\delta} \frac{\mathrm{d}x}{x} \,, \tag{18}$$

with $g(z;x) = z_1x_1 + z_2x_2 + z_3x_1^2 + z_4x_1x_2$. By inspection, $\mathcal{D} = \frac{\partial_1\partial_2}{\epsilon(\epsilon-1)}$.

3.2 Macaulay matrices

Our algorithm for computing Pfaffian systems begins by choosing a special operator basis, namely that of *standard monomials*: $\mathcal{D}_i = \mathrm{Std}_i$ [4]. It is a monomial basis, i.e. $\mathrm{Std}_i = \partial^{k_i}$ for some $k_i \in \mathbb{Z}^N$ - see appendix B of [1] for more details.

Consider the Pfaffian system in the standard monomial basis: $\partial_i\,\mathrm{Std} = P_i \cdot \mathrm{Std}$. We may split the LHS into two terms,

$$\partial_i\,\mathrm{Std} = C_{\mathrm{Ext},i} \cdot \mathrm{Ext} \;+\; C_{\mathrm{Std},i} \cdot \mathrm{Std} \,, \tag{19}$$

with *external monomials* Ext defined as those monomials in the vector $\partial_i\,\mathrm{Std}$ not already contained in Std. $C_{\mathrm{Ext},i}$ and $C_{\mathrm{Std},i}$ are sparse matrices consisting of 1s and 0s.

Choose an integer $d > 0$ and set $\mathrm{Der}_d := \{\partial^k \mid k_1 + \ldots + k_N \le d\}$. Moreover, let Mons_d be the set of all monomials in ∂_i appearing in the set $\left\{\partial^k\, E_j \,,\, \partial^k\, \Box_u\right\}$, for all $\partial^k \in \mathrm{Der}_d$, $j = 1, \ldots, n+1$ and $u \in \mathrm{Ker}(A)$. The *Macaulay matrix* of degree d, $M_d = M_d(\beta; z)$, is defined by the relation

$$\left\{\partial^k\, E_j \,,\, \partial^k\, \Box_u\right\}_{\forall j,k,u} = M_d \cdot \mathrm{Mons}_d \,. \tag{20}$$

[3] We use the notation $\partial^k := \partial_1^{k_1} \cdots \partial_N^{k_N}$ given a non-negative integer vector $k \in \mathbb{Z}^N_{>0}$.

[4] There is a fast algorithm for finding the standard monomials of GKZ systems [37].

On the LHS, we employ the Weyl algebra (16) to commute all the derivatives to the right, thereby exposing the vector Mons_d and its coefficient matrix M_d. The identity (19) turns out to induce a natural block structure of the Macaulay matrix: $M_d = (M_{\mathrm{Ext}} \,|\, M_{\mathrm{Std}})$, where the columns of M_{Ext} are labeled by the monomials in Ext, and similarly for M_{Std}.

The heart of our algorithm for computing Pfaffian matrices is then the following. We first solve for an unknown matrix C in [5]

$$C_{\mathrm{Ext},i} - C \cdot M_{\mathrm{Ext}} = 0\,, \tag{21}$$

whereafter C is inserted into

$$C_{\mathrm{Std},i} - C \cdot M_{\mathrm{Std}} = P_i\,, \tag{22}$$

thereby yielding the Pfaffian matrix. Note that the matrices $C_{\mathrm{Ext},i}$, $C_{\mathrm{Std},i}$, M_{Ext}, M_{Std} are known. In solving (21), we benefit from codes employing rational reconstruction over finite fields [38, 39].

Once the Pfaffian system is found in the Std basis, it is swift to gauge transform the system to any other basis of choice.

Example. *Pfaffian system.* Let us use the Macaulay matrix method to derive the Pfaffian system for the Euler integral (8) (associated to the bubble topology (13)).

To simplify matters, we may rescale $n + 1 = 3$ of the z-variables to 1 (see appendix A of [1]): $(z_1, z_2, z_3, z_4) \longrightarrow (1, 1, 1, z)$ with $z := \frac{z_1 z_4}{z_2 z_3}$. This leaves us with a single derivative $\partial := \frac{\partial}{\partial z}$. One then finds that

$$\mathrm{Std} = \begin{pmatrix} \partial \\ 1 \end{pmatrix} \quad , \quad \mathrm{Ext} \;=\; \partial^2 \quad , \quad C_{\mathrm{Ext}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad , \quad C_{\mathrm{Std}} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad , \quad M_{\mathrm{Ext}} = z(1-z) \tag{23}$$

$$M_{\mathrm{Std}} = \Big(z(2b_2 + b_1 - b_0 - 1) - 2b_2 - b_1 + 2b_0 + 1\,,\; b_2(b_0 - b_1 - b_2) \Big)\,.$$

The solution to $C_{\mathrm{Ext}} - C \cdot M_{\mathrm{Ext}} = 0$ is then $C = \begin{pmatrix} \frac{1}{z(z-1)} \\ 0 \end{pmatrix}$. Finally, we get the Pfaffian matrix P in $\partial\mathrm{Std} = P \cdot \mathrm{Std}$ from

$$P = C_{\mathrm{Std}} - C \cdot M_{\mathrm{Std}} = \begin{pmatrix} \frac{(z-2)b_0 - (z-1)(2b_2 + b_1 - 1)}{z(z-1)} & \frac{b_2(b_0 - b_1 - b_2)}{z(z-1)} \\ 1 & 0 \end{pmatrix}\,. \tag{24}$$

From here, one may relate z to the kinematic ratio p^2/m^2, include the $c(d_0; \nu)$ prefactors, and send β_i to the relevant propagator powers, in order to obtain the DEQ matrix for the bubble FI.

4. Conclusion

We have considered FIs in the framework of GKZ hypergeometric systems. Utilizing the Macaulay matrix, built from the GKZ system, we presented a method for the calculation of Pfaffian systems, i.e. the set of first-order PDEs satisfied by master integrals. In the future, it would be interesting to apply these tools to study stringy canonical forms [40], which are also integrals of the GKZ type.

[5] The integer d is chosen such that this equation has a solution. Typically, $d \leq 2$.

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