

## DIS coefficient functions at four loops in QCD and beyond

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We report results for the lowest even- $N$  moments of the flavor-nonsinglet structure functions  $F_2$  and  $F_L$  in QCD at the fourth order in the perturbative expansion in the strong coupling constant  $\alpha_s$ . Our results are presented in numerical form and we compare them with the leading and subleading terms of the threshold expansion for large values of  $N$ , which corresponds to the limit  $x \rightarrow 1$ .

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## 1. Introduction

The deep-inelastic scattering (DIS) of leptons off a nucleon target is one of the basic scattering reactions in Quantum Chromodynamics (QCD), which has been studied in many collider experiments during the past 50 years. In recent times high precision inclusive DIS data has been measured at HERA [1] and in the future the forthcoming Electron-Ion-Collider (EIC) [2, 3] is expected to contribute as well. Unpolarized inclusive lepton-nucleon DIS proceeds through the reaction

$$l(k) + \text{nucl}(p) \rightarrow l'(k') + X, \quad (1)$$

where the scattered leptons (or neutrinos) are denoted by  $l$  and  $l'$  and the nucleon state by ‘nucl’, with the respective momenta  $k$ ,  $k'$  and  $p$ , and  $X$  is the inclusive hadronic final state. The charge of the exchanged gauge boson  $V(q)$  with momentum  $q = k - k'$ , is used to classify neutral- ( $V = \gamma^*$ ,  $Z$ ) or charged-current ( $V = W^\pm$ ) DIS. The theoretical predictions for the reaction (1) are based on the well known structure functions  $F_i$ , with  $i = 1, 2, 3$  and  $F_L = F_2 - 2xF_1$ , which are functions of the Bjorken  $x$  variable  $x = Q^2/(2p \cdot q)$ , where  $0 < x \leq 1$ , and the scale  $Q^2 = -q^2 > 0$  of the exchanged virtual boson.

Perturbative QCD allows for the computation of the scale dependence of the structure functions as a series in the strong coupling  $\alpha_s$ , together with the coefficient functions of the hard scattering process. The current state-of-the art for massless perturbative QCD is the next-to-next-to-next-to-leading order approximation (abbreviated as (next-to)<sup>3</sup>-leading order or N<sup>3</sup>LO) for the structure functions  $F_i$ , with  $i = 1, 2, 3$ . This encompasses the complete three-loop coefficient functions for neutral- [4] and charged-current [5–7] DIS and partial results for the four-loop splitting functions (Mellin moments [8–13], the large- $n_f$  contributions [14, 15], and the planar limit of the nonsinglet case [11]) governing the  $Q^2$  dependence, so that the N<sup>3</sup>LO predictions are robust within the kinematic range of past (HERA) and future (EIC) DIS experiments.

For theory predictions beyond this order, i.e., at the (next-to)<sup>4</sup>-leading order (N<sup>4</sup>LO), it is necessary to consider the coefficient functions at four loops in perturbative QCD, which is ongoing work and will be reported in these proceedings.

## 2. Computation

The focus in these proceedings is on the structure functions  $F_2^{\gamma^*}$  and  $F_L^{\gamma^*}$ , which describe the one-photon exchange in neutral-current DIS. The standard QCD factorization in leading twist approximation, i.e., disregarding terms suppressed by powers of  $1/Q^2$ , allows to express them as convolutions of the coefficient functions with the parton distributions (PDFs),

$$F_a^{\gamma^*}(x, Q^2) = [C_a(Q^2) \otimes q_{+, \text{ns}}(Q^2)](x), \quad a = 2, L, \quad (2)$$

where  $q_{+, \text{ns}}$  is the nonsinglet quark PDF in the nucleon, the dependence on the factorization scale  $\mu^2$  is suppressed, and  $\otimes$  denotes the standard convolution. The coefficient functions in Eq. (2) have an expansion in powers of the strong coupling  $a_s = \alpha_s/(4\pi)$ ,

$$C_{a, \text{ns}} = \delta_{a2} + \sum_{l=1}^{\infty} a_s^l c_{a, \text{ns}}^{(l)}, \quad a = 2, L, \quad (3)$$

where we have suppressed the dependence on  $x$ , on the number of effectively massless flavors  $n_f$  and on the ratio of scales  $Q^2/\mu^2$ . Their Mellin moments are defined as

$$C_{a,\text{ns}}(N) = \int_0^1 dx x^{N-1} C_{a,\text{ns}}(x), \quad a = 2, L, \quad (4)$$

and the even ones are accessible within the framework of the operator product expansion (OPE) by computing suitable projections of the imaginary part of the forward Compton amplitude for the scattering process of a virtual photon off a quark. The computational set-up, building on the OPE, is well established [16], and has been used and described in detail in previous publications, see e.g., Refs. [4–7, 10, 17, 18].

All contributing Feynman diagrams for the process

$$\gamma^*(q) + q(p) \rightarrow X, \quad (5)$$

up to four loops are generated with QGRAF [19] and their color coefficients are obtained for a general  $SU(n_c)$  gauge group using the algorithms of [20], which determine the group invariants for (semi-)simple Lie groups. At four loops, the color factors are given by powers of the quadratic Casimirs  $C_F^{4-k} C_A^k$  for  $k = 0, \dots, 3$  as well as by combinations of the cubic ones  $C_F d_F^{abc} d_F^{abc}$  and  $C_A d_F^{abc} d_F^{abc}$  and the quartic ones  $d_F^{abcd} d_A^{abcd}$  and  $d_F^{abcd} d_F^{abcd}$ , the latter being obtained from the symmetrized trace  $d_r^{abc}$  ( $d_r^{abcd}$ ) of three (four)  $SU(n_c)$  generators in representation  $r$  of the  $SU(n_c)$ . In addition,  $n_f$ -dependent terms arise by replacing  $C_F \rightarrow n_f$  for up to three powers of  $C_F$ . Besides their color factors diagrams are grouped according to their nonsinglet flavor topologies. For neutral-current diagrams up to four loops the corresponding charge factors read [4, 21]

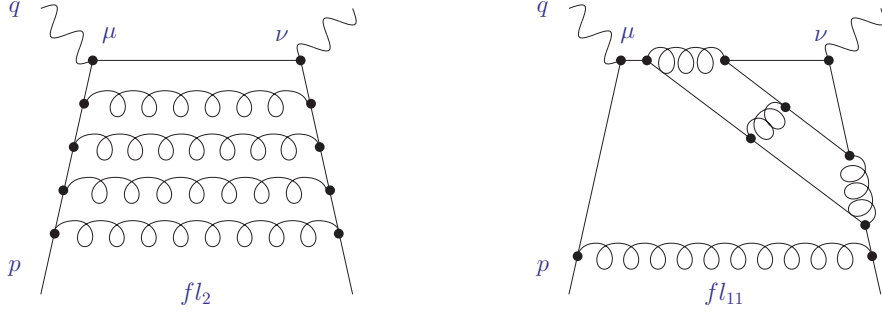
$$fl_2 = 1, \quad fl_{11} = 3\langle e \rangle = \frac{3}{n_f} \sum_{i=1}^{n_f} e_{q_i}, \quad (6)$$

where  $\langle e \rangle$  represents the average of the quark charges  $e_{q_i}$ . Thus, for  $n_f = \{1, \dots, 6\}$  the coefficient  $fl_{11}$  takes the numerical values  $fl_{11} = \{-1, 1/2, 0, 1/2, 1/5, 1/2\}$  and diagrams of the  $fl_{11}$  flavor topology arise only with cubic color factors  $d_F^{abc} d_F^{abc}$ . An illustration of the nonsinglet flavor topologies at four loops is given in Fig. 1.

The matrix elements for the process (5) contain divergencies at higher orders, which are regularized in  $D = 4 - 2\epsilon$  dimensions. Choosing a fixed Mellin moment  $N$  as defined in Eq. (4) then reduces the expressions for the Feynman diagrams to two-point functions up to four-loop order, i.e., massless propagator-type diagrams, which can be computed with the help of standard integration-by-parts algorithms [22, 23], whose solutions are encoded in the program FORCER [24] for the computer algebra system FORM [25–27] and its multi-threaded version TFORM [28], which is used for all symbolic manipulations.

This computation provides an analytic result for the bare forward Compton amplitude at a chosen fixed value of  $N$  which is then subject to the standard ultraviolet renormalization of the strong coupling  $\alpha_s$  and the subsequent removal of the remaining collinear singularities in powers of  $1/\epsilon$  in the minimal subtraction scheme [29]. The single poles in  $1/\epsilon$  provide the anomalous dimensions of the quark nonsinglet operator matrix elements [11].

High values of  $N$ , however, give rise to high powers of both, propagators in the denominator and scalar products in the numerator in the propagator-type loop diagrams, to be computed with



**Figure 1:** Typical Feynman diagrams for the different nonsinglet flavor topologies,  $fl_2$  (left) and  $fl_{11}$  (right). The latter admits a  $d_F^{abc} d_F^{abc}$  color factor starting from three loop order through the coupling of three gluons.

**Forcer.** This leads to large-size intermediate expressions, sometimes of the order of several TByte and to long run times. For example, the four-loop result for the Mellin moment  $N = 10$  of the coefficient function  $C_{L,ns}$  in Eq. (3) required the evaluation  $\mathcal{O}(3200)$  Feynman diagrams with intermediate expressions of sizes up to  $\mathcal{O}(20)$  TByte and a total of  $\mathcal{O}(800000)$  hrs CPU time. The multi-threaded version TFORM delivers an average speed-up factor of  $\mathcal{O}(10)$ .

### 3. Coefficient functions at four loops

We can now provide the exact even Mellin moments for  $N = 2, 4, 6, 8, 10$  of the coefficient functions  $C_{2,ns}$  and  $C_{L,ns}$  up to fourth order in  $\alpha_s$  for QCD, i.e., taking the numerical values of the  $SU(n_c)$  color coefficients for  $n_c = 3$ .

The complete perturbative expansion for  $C_{2,ns}(N)$  for  $n_f = 4$  reads

$$\begin{aligned}
C_{2,ns}(2) &= 1 + 0.0354 \alpha_s - 0.0231 \alpha_s^2 - 0.0613 \alpha_s^3 - 0.4746 \alpha_s^4 \\
&\quad + fl_{11} \left( -0.0486 \alpha_s^3 - 0.1424 \alpha_s^4 \right), \\
C_{2,ns}(4) &= 1 + 0.4828 \alpha_s + 0.4711 \alpha_s^2 + 0.4727 \alpha_s^3 - 0.2458 \alpha_s^4 \\
&\quad + fl_{11} \left( -0.0367 \alpha_s^3 - 0.0893 \alpha_s^4 \right), \\
C_{2,ns}(6) &= 1 + 0.8894 \alpha_s + 1.2054 \alpha_s^2 + 1.7572 \alpha_s^3 + 1.7748 \alpha_s^4 \\
&\quad + fl_{11} \left( -0.0325 \alpha_s^3 - 0.0684 \alpha_s^4 \right), \\
C_{2,ns}(8) &= 1 + 1.2358 \alpha_s + 2.0208 \alpha_s^2 + 3.5294 \alpha_s^3 + 5.3921 \alpha_s^4 \\
&\quad + fl_{11} \left( -0.0304 \alpha_s^3 - 0.0591 \alpha_s^4 \right), \\
C_{2,ns}(10) &= 1 + 1.5359 \alpha_s + 2.8608 \alpha_s^2 + 5.6244 \alpha_s^3 + 10.324 \alpha_s^4 \\
&\quad + fl_{11} \left( -0.0291 \alpha_s^3 - 0.0547 \alpha_s^4 \right), \tag{7}
\end{aligned}$$

and for  $C_{L,ns}(N)$  for  $n_f = 4$

$$\begin{aligned}
C_{L,ns}(2) &= 0.14147 \alpha_s \left( 1 + 1.7270 \alpha_s + 3.7336 \alpha_s^2 + 9.5619 \alpha_s^3 \right. \\
&\quad \left. + fl_{11} \left( -0.1102 \alpha_s^2 - 0.7865 \alpha_s^3 \right) \right), \\
C_{L,ns}(4) &= 0.08488 \alpha_s \left( 1 + 2.5619 \alpha_s + 6.9208 \alpha_s^2 + 20.251 \alpha_s^3 \right. \\
&\quad \left. + fl_{11} \left( -0.1201 \alpha_s^2 - 0.9983 \alpha_s^3 \right) \right), \\
C_{L,ns}(6) &= 0.06063 \alpha_s \left( 1 + 3.1557 \alpha_s + 9.6370 \alpha_s^2 + 30.573 \alpha_s^3 \right. \\
&\quad \left. + fl_{11} \left( -0.1232 \alpha_s^2 - 1.1174 \alpha_s^3 \right) \right), \\
C_{L,ns}(8) &= 0.04716 \alpha_s \left( 1 + 3.6191 \alpha_s + 12.040 \alpha_s^2 + 40.535 \alpha_s^3 \right. \\
&\quad \left. + fl_{11} \left( -0.1245 \alpha_s^2 - 1.1997 \alpha_s^3 \right) \right), \\
C_{L,ns}(10) &= 0.03858 \alpha_s \left( 1 + 4.0020 \alpha_s + 14.215 \alpha_s^2 + 50.164 \alpha_s^3 \right. \\
&\quad \left. + fl_{11} \left( -0.1253 \alpha_s^2 - 1.2629 \alpha_s^3 \right) \right), \tag{8}
\end{aligned}$$

where the numerical values for  $C_{2,ns}$  and  $C_{L,ns}$  (flavor class  $fl_2$  only) at  $N = 2, 4, 6$  have already been quoted in [10]. Further results for  $C_{2,ns}$  and  $C_{L,ns}$  at  $N = 12$  and  $N = 14$  have also been obtained, but are limited to the large- $n_c$  approximation. All Mellin moments agree with the published all-order large- $n_f$  limits of the nonsinglet structure functions  $F_2$  [30] and  $F_L$  [31].

In addition to those moments more information about the four-loop coefficient functions is available in the kinematic limits of large  $N$  ( $x \rightarrow 1$ ) and small  $N \simeq 0$  ( $x \rightarrow 0$ ). Near threshold the coefficient function  $c_{2,ns}^{(4)}$  is dominated by singular plus-distributions  $[\ln^k(1-x)/(1-x)]_+$  with  $7 \geq k \geq 0$  for  $x \rightarrow 1$  ( $\alpha_s^n \ln^{k+1}(N)$  in Mellin- $N$  space) and the coefficients of all terms with  $k \geq 1$  are known, see, e.g. [32]. For  $n_c = 3$  in QCD  $c_{2,ns}^{(4)}$  reads

$$\begin{aligned}
c_{2,ns}^{(4)}(x) &= 16.85596708 \left[ \frac{\ln^7(1-x)}{1-x} \right]_+ + \dots \\
&+ \{ (3.88405 \pm 0.1) \cdot 10^4 + (-3.49648951 \pm 0.00000004) \cdot 10^4 n_f \\
&+ 2062.715183 n_f^2 - 12.08488248 n_f^3 + 47.55183089 n_f fl_{11} \} \left[ \frac{1}{1-x} \right]_+ \\
&+ \delta(1-x) c_{4,\delta,DIS} \\
&- 16.85596708 \ln^7(1-x) + \{ 504.6255144 - 14.74897119 n_f \} \ln^6(1-x) \\
&+ \{ -5135.705824 + 416.4828532 n_f - 4.213991769 n_f^2 \} \ln^5(1-x) \\
&+ \{ 20935.61036 - 4034.293546 n_f + 108.5761316 n_f^2 - 0.3950617283 n_f^3 \} \ln^4(1-x) \\
&+ \mathcal{O}(\ln^3(1-x)), \tag{9}
\end{aligned}$$

where all exact values have been rounded to ten digits and the flavor factor  $fl_{11}$  is given in Eq. (6). The new result for the term proportional to  $[1/(1-x)]_+$  is based on the Mellin moments up to  $N \leq 15$  for  $C_{2,ns}$  and  $C_{3,ns}$ . The latter denotes the coefficient function of the charged-current

structure function  $F_3^{W^+W^-}$ , with its odd Mellin moments being accessible in the OPE. The limits  $x \rightarrow 1$  for  $C_{2,\text{ns}}$  and  $C_{3,\text{ns}}$  coincide, thus increasing the available information for the estimates in Eq. (6). Also, unpublished results for much higher Mellin moments (up to  $N \simeq 40$ ) for the  $n_f$ -dependent contributions have been used. The coefficient of  $[1/(1-x)]_+$  in Eq. (9) thus supersedes the previous estimate [32], where also numerical estimates for the term  $c_{4,\delta,\text{DIS}}$  proportional to  $\delta(1-x)$  have been presented. The logarithmically enhanced powers  $\ln^k(1-x)$  for  $7 \geq k \geq 4$  of the first subleading corrections at power  $(1-x)^0$  ( $\alpha_s^n \ln^k(N)/N$  in Mellin- $N$  space) for  $c_{2,\text{ns}}^{(4)}$  in Eq. (9) have been derived in [33], with unknown terms starting at  $\mathcal{O}(\ln^3(1-x))$ , as indicated (see also [34] for recent work). For  $c_{L,\text{ns}}^{(4)}$  they are actually leading and the expansion around the limit  $x \rightarrow 1$  for  $n_c = 3$  in QCD reads

$$\begin{aligned} c_{L,\text{ns}}^{(4)}(x) &= 16.85596708 \ln^6(1-x) + \{-306.0638892 + 21.06995884 n_f\} \ln^5(1-x) \\ &\quad + \{2421.032535 - 356.9371659 n_f + 9.481481481 n_f^2\} \ln^4(1-x) \\ &\quad + \mathcal{O}(\ln^3(1-x)). \end{aligned} \quad (10)$$

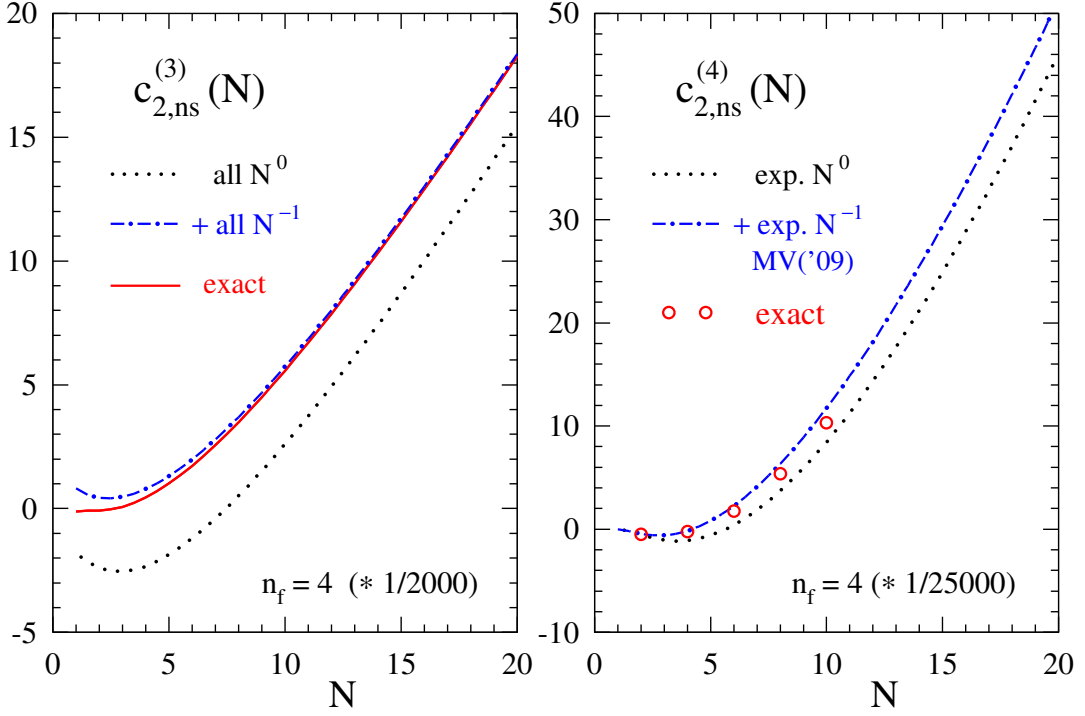
The small- $x$  behavior of the nonsinglet coefficient functions features double logarithms in the limit  $x \rightarrow 0$ , which appear as  $\alpha_s^n \ln^k(x)$ , where  $k = 2n - 1, \dots, 1$ , at all orders ( $\alpha_s^n/N^{k+1}$  in Mellin- $N$  space for  $N \rightarrow 0$ ), which have been resummed to the third logarithmic ( $N^2\text{LL}$ ) order [35, 36]. The fixed-order expansions of the latter resummations gives for  $c_{2,\text{ns}}^{(4)}$  and  $c_{L,\text{ns}}^{(4)}$  the following results [36],

$$\begin{aligned} c_{2,\text{ns}}^{(4)}(x) &= -\frac{13}{168} C_F^4 \ln^7(x) + \left\{ \frac{263}{180} C_F^4 - \frac{911}{1080} C_F^3 \beta_0 \right\} \ln^6(x) - \left\{ \left( \frac{14}{5} - \frac{734}{15} \zeta_2 \right) C_F^4 - \frac{56}{15} C_F^3 \beta_0 \right. \\ &\quad \left. + \left( \frac{109}{18} + \frac{208}{5} \zeta_2 \right) C_F^3 C_A - 13 \zeta_2 C_F^2 C_A^2 + \frac{1951}{720} C_F^2 \beta_0^2 \right\} \ln^5(x) + \mathcal{O}(\ln^4(x)), \\ c_{L,\text{ns}}^{(4)}(x) &= -2 C_F^4 \ln^5(x) + \left\{ \frac{59}{3} C_F^4 - \frac{124}{9} C_F^3 \beta_0 \right\} \ln^4(x) + \left\{ \left( \frac{322}{3} + \frac{2008}{3} \zeta_2 \right) C_F^4 - \frac{28}{3} C_F^3 \beta_0 \right. \\ &\quad \left. - (80 + 640 \zeta_2) C_F^3 C_A + 200 \zeta_2 C_F^2 C_A^2 - \frac{230}{9} C_F^2 \beta_0^2 \right\} \ln^3(x) + \mathcal{O}(\ln^2(x)), \end{aligned} \quad (11)$$

which have been quoted for a general  $SU(n_c)$  gauge theory. While the partial results are of limited direct use for phenomenology, i.e., still lacking knowledge on the logarithms  $\ln^k(x)$  with  $4 \geq k \geq 1$  for  $c_{2,\text{ns}}^{(4)}$  and  $k = 1, 2$  for  $c_{L,\text{ns}}^{(4)}$ , these expressions provide valuable information for checks of analytic computations or for reconstructions of the exact result from a number of fixed Mellin moments.

In Fig. 2 we plot the results for the coefficient function  $C_{2,\text{ns}}$  for  $n_f = 4$  at three and four loops as a function of the Mellin moment  $N$ . The exact result for  $c_{2,\text{ns}}^{(3)}$  in Fig. 2 (left) is known from [4] and shown to lie in between the curves determined by the threshold logarithms  $\alpha_s^3 \ln^k(N)$  and the first power suppressed terms  $\alpha_s^3 \ln^k(N)/N$ , with  $6 \geq k \geq 1$  in both cases. The new four-loop moments for  $c_{2,\text{ns}}^{(4)}$  in Fig. 2 (right) are also shown in comparison to the threshold logarithms, i.e., the  $N$ -space equivalent of Eq. (9) and including the additional leading  $1/N$  enhanced terms. Again, the two approximations are expected to bracket the, yet unknown, full result.

In Fig. 3 the same information at three and four loops is shown for the coefficient function  $C_{L,\text{ns}}$  for  $n_f = 4$ . Here, the power suppressed terms proportional to  $1/N$  are leading as  $N \rightarrow \infty$ . From the

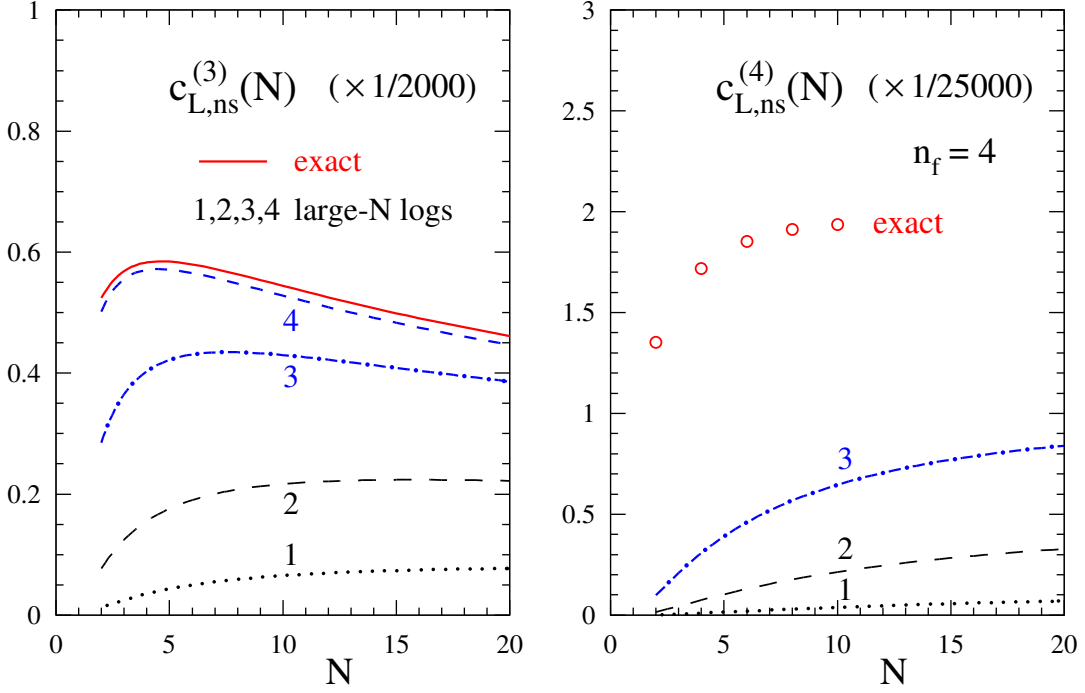


**Figure 2:** The exact  $N$ -space results for the third-order coefficient function  $c_{2,ns}^{(3)}$  at  $n_f = 4$  for neutral-current (one-photon exchange) DIS from [4] (left) and the moments in Eq. (7) of the fourth-order term  $c_{2,ns}^{(4)}$  calculated so far (right). In both cases also the contributions provided by large- $N$  resummations are shown.

three-loop result in Fig. 3 (left) it is obvious, that the complete tower of logarithms  $\alpha_s^3 \ln^k(N)/N$  with  $4 \geq k \geq 1$  is required approximate the exact result. Therefore, it is evident, that at four loops the known first three logarithms  $\alpha_s^4 \ln^k(N)/N$  with  $6 \geq k \geq 4$  for  $c_{L,ns}^{(4)}$  (given in Eq. (10) in  $x$ -space) do not approximate the exact Mellin moments well and the further subleading logarithms are needed as well.

#### 4. Summary and Outlook

We have discussed the computation of the even Mellin moments  $N = 2, \dots, 14$  of the four-loop coefficient functions  $C_{2,ns}$  and  $C_{L,ns}$  in neutral-current (photon-exchange) DIS together with known results on the endpoint behavior for large and small  $x$  from all-order considerations of soft and collinear dynamics or the high energy limit. The results are available for a general  $SU(n_c)$  gauge group and numerical values for QCD ( $n_c = 3$ ) have been presented above. Part of the ongoing studies are also the computation the odd moments  $N = 1, \dots, 15$  for those of the structure function  $F_3$  in charged-current ( $W^+ + W^-$ -boson exchange) DIS, with the results for the highest moments  $N = 11$  to 15 being restricted to the large- $n_c$  limit. This agrees with and extends previous results on the first Mellin moment of  $F_3$ , i.e., the  $\alpha_s^4$  contribution to the Gross-Llewellyn-Smith (GLS) sum rule [37, 38].



**Figure 3:** Same as Fig. 2 for the exact results for  $c_{L,ns}^{(3)}$  from [4] (left) and the moments in Eq. (8) of  $c_{L,ns}^{(4)}$  (right).

The new results for the four-loop coefficient functions can be combined with the low- $N$  moments of the five-loop nonsinglet anomalous dimensions [39] to allow for inclusive DIS predictions at  $N^4$ LO in massless QCD. This situation is somewhat reminiscent to one more than 20 years ago, when the then available three-loop moments of the DIS structure functions [21, 40, 41] were used to derive reliable approximations [42, 43] in the kinematic range relevant for DIS experiments, before the computation of the exact three-loop splitting and coefficient functions [4, 17, 18].

The combination of the new four-loop moments for  $F_2$ ,  $F_L$  and  $F_3$  together with information on the limits  $x \rightarrow 0$  and  $x \rightarrow 1$  will again provide robust approximations for very precise QCD predictions. They will be presented in a future publication [44].

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