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Scattering Amplitudes and Conservative Dynamics at the Fourth Post-Minkowskian Order

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We review recent progress using amplitudes methods to obtain new results for the conservative classical scattering of two massive black holes at order $O(G^4)$. We include contributions from both potential and radiation modes which effect the conservative dynamics through the tail effect. We compute the $2 \to 2$ scattering amplitudes and extract the radial action through the amplitude-action formula and derive the scattering angle. By matching we also obtain a Hamiltonian valid for arbitrary scattering trajectories.

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Introduction. Gravitational-wave observations [1] and their interpretation necessitate high-precision theoretical input through modeling of waveforms. Much higher precision will be required to fully interpret results at next generation detectors [2], requiring further theoretical advances. Waveforms are constructed by combining information from numerical relativity [3], self force [4, 5] and perturbative methods, as implemented, for example, through the effective one-body model [6]. In this presentation, we summarize new recent results for the conservative contributions to two-body dynamics at $O(G^4)$ in the scattering amplitudes approach [7, 8].

Traditionally analytic input, in the form of corrections to Newton's potential, is obtained via the post-Newtonian (PN) expansion [9–15]. Recently, the post-Minkowskian (PM) expansion, which is particularly suited to describe hyperbolic motion, has seen a surge of interest [16, 17]. The appeal of the PM approach follows from the fact that it keeps all orders in velocity and from its close connection to the modern amplitude program. This has resulted in important improvements to analytic computations through the incorporation of on-shell methods relating loop integrands to tree amplitudes [18], double copy relations between gravity and gauge theory [19, 20], advances in multiloop integration [21–24] and effective field theory (EFT) methods [10, 25–27].

The computation at fourth post-Minkowskian order summarized in this presentation is particularly interesting as it unveils qualitatively new features in the way radiation modes, i.e. soft gravitons that are close to their mass shell, contribute to observables. While radiation through $O(G^3)$ is purely dissipative and can therefore be cleanly separated and computed [28–34], at $O(G^4)$ contributions from radiative modes enter the conservative dynamics via the so-called tail effect [35, 36]. Critically, this requires some care in the definition of the conservative dynamics. We isolate conservative part through use of the Feynman-Wheeler time-symmetric propagator advocated in Ref. [37]. At this order, we find that this prescription is equivalent to taking the real part of the Feynman $i\varepsilon$-prescription, as natural in the amplitudes approach. The methods and results reviewed here have been confirmed in multiple studies [14, 15, 38–44]

1. Classical limit

We extract classical two-body dynamics from the classical limit the $2 \to 2$ scattering amplitude. To efficiently compute the latter it is important to make early use of the hierarchy of scales present in the classical limit (see e.g. Refs. [27, 45, 46])

$$m_\star^2 \sim s, |u|, m_1^2, m_2^2 \sim J^2|t| \gg |t| = |q|^2\,, \tag{1}$$

where $m_\star$ is a characteristic hard scale. The dimensionless expansion parameter λ is the ratio of the (reduced) Compton wavelength associated to the hard scale $m_\star$ to the impact parameter b,

$$\lambda = \frac{1}{m_\star b} \sim \frac{|q|}{m_\star} \ll 1\,. \tag{2}$$

In terms of the hierarchy (1), the phase space splits into two regions

$$\text{hard (h):} \quad \ell \sim m_\star\,, \qquad \text{soft (s):} \quad \ell \lesssim \lambda m_\star\,. \tag{3}$$

The hard region correspond to short-range interactions and can be neglected in the present analysis which is aimed at extracting the long-range classical potential. The soft region still has a sub-region that does not correspond to classical physics, which at $O(G^4)$ scales as a single power of

q. To refine the analysis we resort to a threshold expansion [47] around the two-particle threshold $s = (m_1 + m_2)^2$, characterized by the velocity in the center of mass (COM) $v = \sqrt{\sigma^2 - 1}$, with $\sigma = p_1 \cdot p_2/(m_1 m_2)$ in mostly minus signature. We identify the following regions

$$\text{potential (p)} : \ell \sim (v\lambda, \lambda)\,, \quad \text{radiation (r)} : \ell \sim (v\lambda, v\lambda)\,. \tag{4}$$

A third possible region with modes $\ell \sim (\lambda, \lambda)$ corresponds to quantum corrections and will not be included in the following. The classical contribution is obtained by considering all possible assignments of scalings to the three independent loop momenta ℓ_i. We have three possible assignments $(\ell_1 \ell_2 \ell_3) \sim$ (ppp), (ppr), (prr) with the fourth possibility (rrr) leading to scaleless integrals vanishing in dimensional regularization. The contribution from the assigment (ppr) has odd parity under $v \mapsto -v$ and therefore is purely dissipative. Therefore the conservative part can be obtained by adding the contributions from the (ppp) and (prr) region. The integration in the (prr) region is sensitive to the boundary condition chosen for the propagator and the classical physics is isolated using the principal-value (PV) prescription corresponding to a time-symmetric graviton propagator [37, 48],[1]

$$G^{\mathrm{PV}}(k) = \mathcal{P}\frac{1}{k^2}\,. \tag{5}$$

Intuitively, this prescription prevents gravitons from going on shell and escaping as gravitational radiation by removing the pole of the propagator of gravitons in the radiation region.

2. Integrand Construction

The integrand of the amplitude at $O(G^4)$ is constructed using generalized unitarity [18], making use of generalized Ward identities to simplify the construction [50]. The region analysis performed in the previous chapter serves as an important guidance for the construction of the integrand. In particular we discard diagrams that do not contribute in either the (ppp) or (prr) region. Examples of diagrams contributing in different regions are shown in Figure 2. The definition of the conservative dynamics in terms of the PV prescription requires the energy integral to be performed by picking up positive-energy residues of matter propagators [7, 27, 45, 46]. This implies that each integral must have at least one matter line per loop and therefore we only need to consider cuts with this property. This significantly reduces the number of unitary cuts needed in order to determine the relevant parts of the integrand. The unitary cuts needed to fix the integrand are shown in Figure 1.

The integrand is organized by assigning the individual integrals to one of 93 cubic Feynman-type topologies, examples of which are shown in Figure 2.

3. Integration

In order to evaluate the integrals we use the strategy established in Ref. [24]. Following the method of regions [47] we expand the integrals asigning the scaling of the soft region, keeping terms up to $O(|q|)$, resulting in integrals with eikonal-type propagators (see Ref. [24] for details). The integrals are subsequently reduced to a basis of master integrals using IBP identities [21]

[1]For a thorough discussion of propagator prescriptions see Ref. [49].

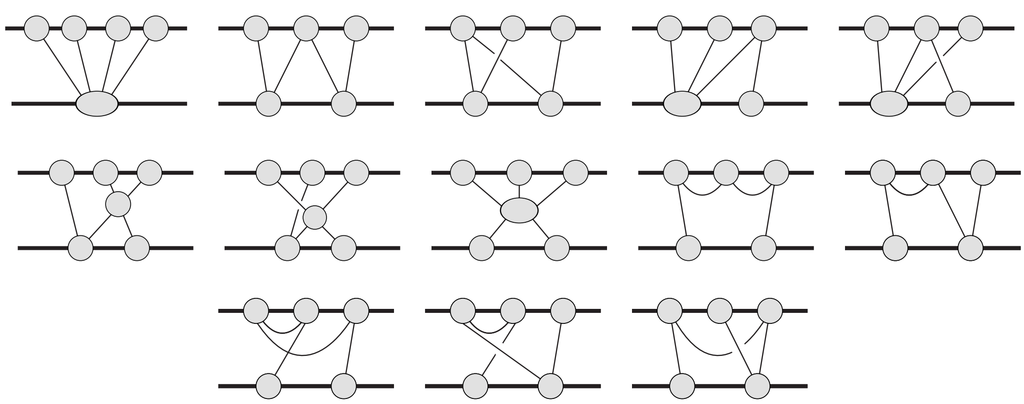


Figure 1: Unitary cuts required to fix the part of the integrand encoding all information fo conservative classical scattering at $O(G^4)$. Thick lines represent on-shell massive scalars, thin lines on-shell gravitons and the gray ovals represent tree-level amplitudes. The first eight cuts are enough to determine all contributions due to potential gravitons.

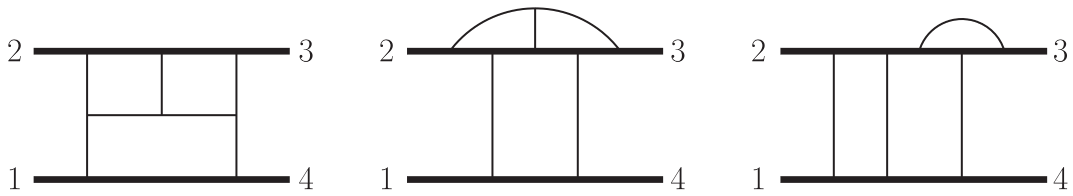


Figure 2: Sample diagrams at $O(G^4)$. Thick lines are massive scalars, thin lines gravitons. From left to right: a diagram that contributes in both the (ppp) and (prr) region, a diagram only contributing to the (prr) region and a diagram containing iteration terms that cancel in the total amplitude.

implemented in `FIRE6` [23]. The separation of the classical regions (4) is done at the level of the master integrals, which are evaluated via the method of differential equations adapted to classical physics [22, 24]. The boundary conditions for the differential equations are determined using earlier methods for evaluating the integrals by series expanding in velocity [27, 46].

Given that each individual integral is representable as a sum of different regions and that the differential equations are linear, we find it advantageous to evaluate the integrals in the (prr) and (ppp) regions separately. This has the effect of setting to zero integrals that do not have support in a given region.

3.1 Potential region

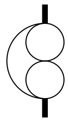
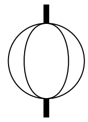
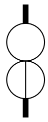
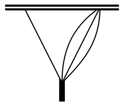
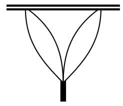
The differential equation for the potential region cannot be brought to canonical form [7] which is connected to the presence of elliptic contact integrals depicted in Figure 3. To isolate the elliptic integrals we note that upon expanding in the static limit $v \to 0$ all integrals are expressible in terms of nine three-dimensional Euclidean integrals

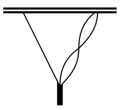
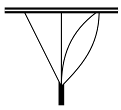
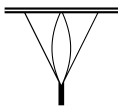
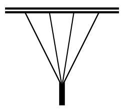
$$I = \sum_{j=1}^{9} I^{(j)}(\sigma, \epsilon) g_{3D}^{(j)} \,. \tag{6}$$

(a) **(b)**

Figure 3: Integral topologies that contain elliptic integrals. Not shown is the u-channel flip of Figure a which also contains elliptic integrals.

The basis integrals are given by the following scalar expressions

$g_{3\mathrm{D}}^{(1)} =$ , $g_{3\mathrm{D}}^{(2)} =$ , $g_{3\mathrm{D}}^{(3)} =$ , $g_{3\mathrm{D}}^{(4)} =$ , $g_{3\mathrm{D}}^{(5)} =$ ,

$g_{3\mathrm{D}}^{(6)} =$ , $g_{3\mathrm{D}}^{(7)} =$ , $g_{3\mathrm{D}}^{(8)} =$ , $g_{3\mathrm{D}}^{(9)} =$ , (7)

with all integrals, in $D = 3 - 2\epsilon$ spatial dimensions. The double lines denote eikonal propagators, thin line massless propagators and thick lines represent an injection of an off-shell spatial momentum $\boldsymbol{q}$. Since the three-dimensional integrals do not depend on σ, we can straightforwardly trade the differential equation for 9 independent differential equations for the coefficients $I^{(j)}$. The integrals $g_{3\mathrm{D}}^{(4)}$–$g_{3\mathrm{D}}^{(9)}$ have poles that will be subtracted by removing lower-loop iterations. Therefore, the new contribution at $O(G^4)$ is proportional to the contributions from $g_{3\mathrm{D}}^{(1)}$–$g_{3\mathrm{D}}^{(3)}$. We find that the elliptic integrals depicted in Figure 3 are proportional to $g_{3\mathrm{D}}^{(1)}$, leaving the contributions proportional to $g_{3\mathrm{D}}^{(2)}$ and $g_{3\mathrm{D}}^{(3)}$ to be purely polylogarithmic and easily obtainable to any order in the dimensional regulator ϵ leading to classical polylogs with complex arguments. For the contribution proportional to $g_{3\mathrm{D}}^{(1)}$, we first expand all integrals up to order v^{60} and assemble the amplitude. We then fit the resulting expression to an ansatz based on the functions that are obtained by exactly solving the integrals in Figure 3 (details are given in Ref. [51]). The resulting expression is then confirmed by expanding to up to order v^{400}.

3.2 Radiation region

The differential equation for in the (prr) region can be brought to canonical form [52], and effectively solved up to the required order in the dimensional regulator. While for the potential region the gravitons are always off-shell and therefore the choice of boundary conditions is unambiguous, for the (prr) region there are possible different choices and we use the PV prescription (5). After imposing scaling in the limit $v \to 0$ we find that there is only one non-trivial boundary condition and it is related to the integral depicted in Figure 4. The matter propagators are cut after adding up integrals that are related by permutations of the matter-graviton vertices. This integral has only two propagators that are in the radiation region. In order to evaluate this integrals we make use of the

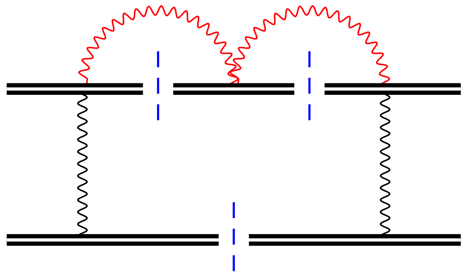
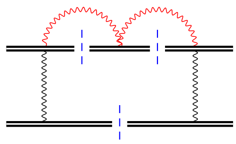
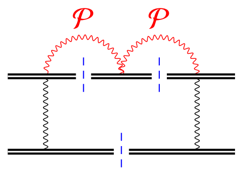


Figure 4: The boundary integral for the (prr) region. Doubled lines represent eikonal matter propagators, wiggly lines are massless propagators. Momenta flowing though the red lines are in the radiation region.

Poincaré-Bertrand theorem in the form [53]

$$\frac{1}{x-a+\mathrm{i}\varepsilon}\frac{1}{x-b+\mathrm{i}\varepsilon} = \left[\mathcal{P}\left(\frac{1}{x-a}\right) - \mathrm{i}\pi\delta(x-a)\right]\left[\mathcal{P}\left(\frac{1}{x-b}\right) - \mathrm{i}\pi\delta(x-b)\right] + \pi^2\delta(x-a)\delta(x-b)\,. \tag{8}$$

Applying this relation to the boundary integral in Figure 4, we find

Re [] = 

, (9)

where we used that the cut and potential propagators are real after separating appropriate factors of the imaginary unit. Since the differential equations are linear, this property extends to all integrals and eventually to the full amplitude

$$\mathcal{M}_4^{\text{cons}} = \mathrm{Re}(\,\mathcal{M}_4|_{v\text{-even}})\,. \tag{10}$$

Remarkably, at this order the PV prescription is equivalent to taking the real part of the Feynman-iε prescription. This is not expected to hold at higher orders in perturbation theory. Adding up all pieces, we find the following expression for the conservative part of the scattering amplitude at order $O(G^4)$

$$\begin{aligned}
\mathcal{M}_4^{\text{cons}} &= G^4 M^7 \nu^2 |\boldsymbol{q}|\pi^2 \left[\mathcal{M}_4^{\text{p}} + \nu\left(4\mathcal{M}_4^{\text{t}}\log\left(\tfrac{v}{2}\right) + \mathcal{M}_4^{\pi^2} + \mathcal{M}_4^{\text{rem}}\right)\right] && (11)\\
&\qquad + \text{terms proportional to } g_{3\text{D}}^{(4)},\ldots g_{3\text{D}}^{(9)}\,, \\
\mathcal{M}_4^{\text{p}} &= -\frac{35\left(1-18\sigma^2+33\sigma^4\right)}{8\left(\sigma^2-1\right)}\,, && (12)\\
\mathcal{M}_4^{\text{t}} &= r_1 + r_2\log\left(\tfrac{\sigma+1}{2}\right) + r_3\frac{\operatorname{arccosh}(\sigma)}{\sqrt{\sigma^2-1}}\,, && (13)\\
\mathcal{M}_4^{\pi^2} &= r_4\pi^2 + r_5\,K\left(\tfrac{\sigma-1}{\sigma+1}\right)E\left(\tfrac{\sigma-1}{\sigma+1}\right) + r_6\,K^2\left(\tfrac{\sigma-1}{\sigma+1}\right) + r_7\,E^2\left(\tfrac{\sigma-1}{\sigma+1}\right), && (14)
\end{aligned}$$

$$
\begin{aligned}
\mathcal{M}_4^{\text{rem}} &= r_8 + r_9 \log\left(\tfrac{\sigma+1}{2}\right) + r_{10}\tfrac{\operatorname{arccosh}(\sigma)}{\sqrt{\sigma^2-1}} + r_{11}\log(\sigma) + r_{12}\log^2\left(\tfrac{\sigma+1}{2}\right) \\
&+ r_{13}\tfrac{\operatorname{arccosh}(\sigma)}{\sqrt{\sigma^2-1}}\log\left(\tfrac{\sigma+1}{2}\right) + r_{14}\tfrac{\operatorname{arccosh}^2(\sigma)}{\sigma^2-1} \\
&+ r_{15}\,\mathrm{Li}_2\left(\tfrac{1-\sigma}{2}\right) + r_{16}\,\mathrm{Li}_2\left(\tfrac{1-\sigma}{1+\sigma}\right) + r_{17}\tfrac{1}{\sqrt{\sigma^2-1}}\left[\mathrm{Li}_2\left(-\sqrt{\tfrac{\sigma-1}{\sigma+1}}\right) - \mathrm{Li}_2\left(\sqrt{\tfrac{\sigma-1}{\sigma+1}}\right)\right].
\end{aligned} \tag{15}
$$

In this expression $\boldsymbol{q}$ is the three-momentum transfer in the COM frame, $M = m_1 + m_2$ is the total mass, $\nu = m_1 m_2/M^2$ is the symmetric mass ratio, while the r_i are rational functions listed in Table 1.

Importantly, the result has no contribution of order $O(\nu^2)$ with respect to the probe contribution. The absence of such terms is in agreement with the "mass polynomiality rule" [40, 54]. For the amplitude this fact is simple to understand: such a term would require rational factors of $1/(m_1+m_2)$ which can never appear.

The conservative contribution to the amplitude has been computed in a PN expansion in Ref. [34], where the authors explicitly give the value of the contribution $\mathcal{M}_4^{\text{radgrav,f}} := 14\mathcal{M}_4^{\text{t}} + \mathcal{M}_4^{\text{rem,(prr)}}$, where the superscript (prr) denotes the contribution of the (prr) region. Expanding our all-order in velocity result to ninth PN order, we find

$$
\begin{aligned}
\mathcal{M}_4^{\text{radgrav,f}} &= \frac{12044}{75}v^2 + \frac{212077}{3675}v^4 + \frac{115917979}{793800}v^6 - \frac{9823091209}{76839840}v^8 + \frac{115240251793703}{1038874636800}v^{10} \\
&- \frac{411188753665637}{4155498547200}v^{12} + \cdots ,
\end{aligned} \tag{16}
$$

where the first three terms match those given in Ref. [34], providing a strong check on our results.

4. The Amplitude-Action Relation

To extract the classical physics we have to subtract the IR divergences related to iterations of lower-loop interactions. We do this by making use of unitarity in partial wave space. This relates the amplitude to the phase-shift $\delta_J(E)$. The final connection to classical physics is the realization that to leading order in the classical limit the phase-shift is related to the radial action $2\delta_J(E) = I_r(J,E)$ [7]. (See also Refs. [55].) Explicitly the connection between the radial action and the conservative amplitude is encoded in the following *amplitude-action relation* [7]

$$
\mathrm{i}\mathcal{M}^{\text{cons}}(\boldsymbol{q}) = 4E|\boldsymbol{p}|\mu^{-2\epsilon}\int \mathrm{d}^{D-2}\boldsymbol{b}\, e^{\mathrm{i}\boldsymbol{b}\cdot\boldsymbol{q}}\left(e^{\mathrm{i}I_r(J)} - 1\right), \qquad J = |\boldsymbol{p}||\boldsymbol{b}|. \tag{17}
$$

Similiar to other exponentiation schemes like the eikonal approximation [56], the relation (17) encodes the divergences of the amplitude corresponding to iterations. When perturbatively expanding the relation (17) one encounters terms that can be brought to the form [51]

$$
\int \mathrm{d}^{D-2}\boldsymbol{b}\, e^{\mathrm{i}\boldsymbol{b}\cdot\boldsymbol{q}}\frac{[\mathrm{i}I_r(J)]^n}{n!} = \mathrm{i}\int_{\boldsymbol{\ell}}\frac{\tilde{I}_r(\boldsymbol{\ell}_1)\ldots\tilde{I}_r(\boldsymbol{\ell}_n)}{Z_1\ldots Z_{n-1}}, \tag{18}
$$

where $\int_{\boldsymbol{\ell}} := \int\prod_{i=1}^n \frac{d^{D-1}\boldsymbol{\ell}_i}{(2\pi)^{D-1}}(2\pi)^{D-1}\delta(\sum_{j=1}^n \boldsymbol{\ell}_j - \boldsymbol{q})$ and the Z poles, in terms of the spatial unit vector $\hat{z}$, are

$$
Z_j = -4E|\boldsymbol{p}|\left[(\boldsymbol{\ell}_1 + \boldsymbol{\ell}_2 + \cdots + \boldsymbol{\ell}_j)\cdot\hat{z} + \mathrm{i}\varepsilon\right]. \tag{19}
$$

$$
\begin{aligned}
r_1 &= \frac{1151 - 3336\sigma + 3148\sigma^2 - 912\sigma^3 + 339\sigma^4 - 552\sigma^5 + 210\sigma^6}{12(\sigma^2-1)} \\
r_2 &= 4g_1 - 2g_2 - \frac{1}{2}g_3 \\
r_3 &= \frac{\sigma\left(-3+2\sigma^2\right)}{4(\sigma^2-1)}g_3 \\
r_4 &= -\frac{16}{3}g_1 + \frac{4}{3}g_2 - \frac{1}{3}g_4 \\
r_5 &= -\frac{1183 + 2929\sigma + 2660\sigma^2 + 1200\sigma^3}{2(\sigma^2-1)} \\
r_6 &= \frac{834 + 2095\sigma + 1200\sigma^2}{2(\sigma^2-1)} \\
r_7 &= \frac{7\left(169 + 380\sigma^2\right)}{4(\sigma-1)} \\
r_8 &= \frac{1}{72(\sigma^2-1)^2\sigma^7}\Big(3600\sigma^{16} + 4320\sigma^{15} - 35360\sigma^{14} + 33249\sigma^{13} \\
&\quad + 27952\sigma^{12} - 25145\sigma^{11} - 15056\sigma^{10} - 32177\sigma^9 + 64424\sigma^8 \\
&\quad - 38135\sigma^7 + 13349\sigma^6 - 1471\sigma^4 + 207\sigma^2 - 45\Big) \\
r_9 &= -\frac{2\left(75\sigma^7 - 416\sigma^5 - 612\sigma^4 - 739\sigma^3 - 136\sigma^2 - 2520\sigma - 152\right)}{3(\sigma^2-1)} \\
r_{10} &= -\frac{\sigma(-3+2\sigma^2)}{(\sigma^2-1)}2r_1 + 28\sigma^2(-3+2\sigma^2) \\
r_{11} &= \frac{4\sigma}{3(\sigma^2-1)}\left(-852 - 283\sigma^2 - 140\sigma^4 + 75\sigma^6\right) \\
r_{12} &= 6g_2 + g_3 - \frac{1}{2}g_4 \\
r_{13} &= -8\frac{\sigma(-3+2\sigma^2)}{\sigma^2-1}g_1 \\
r_{14} &= -\frac{\sigma^2(-3+2\sigma^2)^2}{4(\sigma^2-1)^2}g_3 \\
r_{15} &= -16g_1 - 4g_2 - g_4 \\
r_{16} &= -2g_4 \\
r_{17} &= -\frac{\sigma(-3+2\sigma^2)}{\sigma^2-1}(8g_1 - 4g_2) \\
g_1 &= 2 + 15\sigma^2 \\
g_2 &= 19\sigma + 15\sigma^3 \\
g_3 &= 11 - 30\sigma^2 + 35\sigma^4 \\
g_4 &= -20 - 111\sigma^2 - 30\sigma^4 + 25\sigma^6
\end{aligned}
$$

Table 1: Functions specifying the amplitude in Eq. (11).

Explicitly, up to fourth order in perturbation theory the amplitude-action relation implies the following relations

$$\mathcal{M}_1^{\text{cons}}(\boldsymbol{q}) = \tilde{I}_{r,1}(\boldsymbol{q}) \,, \tag{20}$$

$$\mathcal{M}_2^{\text{cons}}(\boldsymbol{q}) = \tilde{I}_{r,2}(\boldsymbol{q}) + \int_{\boldsymbol{\ell}} \frac{\tilde{I}_{r,1}^2}{Z_1} \,, \tag{21}$$

$$\mathcal{M}_3^{\text{cons}}(\boldsymbol{q}) = \tilde{I}_{r,3}(\boldsymbol{q}) + \int_{\boldsymbol{\ell}} \frac{\tilde{I}_{r,1}^3}{Z_1 Z_2} + \int_{\boldsymbol{\ell}} \frac{\tilde{I}_{r,1}\tilde{I}_{r,2}}{Z_1} \,, \tag{22}$$

$$\mathcal{M}_4^{\text{cons}}(\boldsymbol{q}) = \tilde{I}_{r,4}(\boldsymbol{q}) + \int_{\boldsymbol{\ell}} \frac{\tilde{I}_{r,1}^4}{Z_1 Z_2 Z_3} + \int_{\boldsymbol{\ell}} \frac{\tilde{I}_{r,1}^2\tilde{I}_{r,2}}{Z_1 Z_2} + \int_{\boldsymbol{\ell}} \frac{\tilde{I}_{r,1}\tilde{I}_{r,3}}{Z_1} + \int_{\boldsymbol{\ell}} \frac{\tilde{I}_{r,2}^2}{Z_1} \,, \tag{23}$$

where a sum over permutations is implicit, for instance, $\tilde{I}_{r,1}\tilde{I}_{r,2} \coloneqq \tilde{I}_{r,1}(\boldsymbol{\ell}_1)\tilde{I}_{r,2}(\boldsymbol{\ell}_2) + \tilde{I}_{r,2}(\boldsymbol{\ell}_1)\tilde{I}_{r,1}(\boldsymbol{\ell}_2)$ while $\tilde{I}_{r,1}^3 \coloneqq \tilde{I}_{r,1}(\boldsymbol{\ell}_1)\tilde{I}_{r,1}(\boldsymbol{\ell}_2)\tilde{I}_{r,1}(\boldsymbol{\ell}_3)$. We checked explicitly that the iterations in Eq. (11) are exactly as predicted from Eq. (23). Extracting the remainder and performing an inverse Fourier-transform we we obtain the radial action for hyperbolic orbits at $O(G^4)$

$$I_{r,4}^{\text{hyp}} = -\frac{G^4 M^7 \nu^2 \pi \boldsymbol{p}^2}{8EJ^3}\left[\mathcal{M}_4^{\text{p}} + \nu\left(4\mathcal{M}_4^{\text{t}}\log\left(\tfrac{\nu}{2}\right) + \mathcal{M}_4^{\pi^2} + \mathcal{M}_4^{\text{rem}}\right)\right] \,, \tag{24}$$

where the individual terms are given in Eqs. (12)–(15).

5. Scattering Observables and Hamiltonian

The radial action constructed in the previous section can be directly used to compute scattering observable. The standard observable is the scattering angle, for which the new contribution at order $O(G^4)$ is

$$\chi_4 = -\frac{\partial I_{r,4}^{\text{hyp}}}{\partial J} = \frac{3G^4 M^7 \nu^2 \pi \boldsymbol{p}^2}{8EJ^4}\left[\mathcal{M}_4^{\text{p}} + \nu\left(4\mathcal{M}_4^{\text{t}}\log\left(\tfrac{\nu}{2}\right) + \mathcal{M}_4^{\pi^2} + \mathcal{M}_4^{\text{rem}}\right)\right] \,. \tag{25}$$

We have checked that, when expanded for small velocities, this expression agrees with the sixth-order in the PN expansion given in Ref. [34]. Furthermore, we agree with the fifth PN order result of Ref. [57] up to a term which is of order $O(\nu^2)$ with respect to the probe computation. As noted before such a term cannot be produced in our approach; this discrepancy warrants further study. Since this result has been presented the odd-in-velocity dissipative contribution has been given in Ref. [58] using the linear response formula [30]. The radial action also yields other observables such as the time delay $\partial I_{r,4}^{\text{hyp}}/\partial E$.

Following Refs. [7, 27, 46] we can construct an effective local Hamiltonian valid for arbitrary scattering trajectories [8]. This Hamiltonian is naturally in the isotropic gauge and has the form

$$H^{\text{hyp}} = E_1 + E_2 + \sum_{n=1}^{\infty} \frac{G^n}{r^n} c_n(\boldsymbol{p}^2) \,, \tag{26}$$

where E_1, E_2 are the energies of the bodies and r is their spatial separation in the COM frame. Performing a matching computation based on the scattering angle, the new coefficient is determined to be

$$
\begin{aligned}
c_4^{\mathrm{hyp}} = {} & \frac{M^7\nu^2}{4\xi E^2}\left[\mathcal{M}_4^{\mathrm{p}} + \nu\left(4\mathcal{M}_4^{\mathrm{t}}\log\left(\frac{\nu}{2}\right) + \mathcal{M}_4^{\pi^2} + \mathcal{M}_4^{\mathrm{rem}}\right)\right] + \mathcal{D}^3\left[\frac{E^3\xi^3}{3}c_1^4\right] \\
& + \mathcal{D}^2\left[\left(\frac{E^3\xi^3}{\boldsymbol{p}^2} + \frac{E\xi(3\xi-1)}{2}\right)c_1^4 - 2E^2\xi^2 c_1^2 c_2\right] \\
& + \left(\mathcal{D} + \frac{1}{\boldsymbol{p}^2}\right)\left[E\xi(2c_1c_3 + c_2^2) + \left(\frac{4\xi-1}{4E} + \frac{2E^3\xi^3}{\boldsymbol{p}^4} + \frac{E\xi(3\xi-1)}{\boldsymbol{p}^2}\right)c_1^4\right. \\
& \left. + \left((1-3\xi) - \frac{4E^2\xi^2}{\boldsymbol{p}^2}\right)c_1^2 c_2\right], \qquad (27)
\end{aligned}
$$

where $\mathcal{D} = \frac{\mathrm{d}}{\mathrm{d}\boldsymbol{p}^2}$ denotes differentiation with respect to $\boldsymbol{p}^2$. The explicit expressions for c_1, c_2, and c_3 are listed in Eq. (10) of Ref. [45]. The superscript 'hyp' emphasizes that the Hamiltonian is only valid for hyperbolic orbits. We remark that part of the expression, in particular the potential contributions, the contribution $\mathcal{M}_4^{\pi^2}$ and the coefficient of the logarithm can be analytically continued between bound and unbound orbits [13, 46, 59].

6. Conclusions

In this presentation we summarized the first computation of conservative scattering observables at order G^4 and to all orders in the velocity of the two heavy bodies. This extends previous results at order G^3 obtained in Ref. [45, 46], and directly confirmed in Refs. [38, 39, 42]. Our approach based on generalized unitarity [18], the double copy [19, 20] effective field theory [25–27] and modern integration methods [21–24], is systematic and scalable, and we see no obstruction to applying it to more complicated problems, for example at the next order in perturbation theory. A particular challenge at higher orders will be the evaluation of integrals.

It will also be important to compute dissipative quantities such as the fluxes of energy and angular momentum. We confirmed the "mass polynomiality rule" [40, 54] for the scattering angle, which has a simple explanation in the amplitude-based framework. Furthermore we have explicitly shown that at least through $O(G^4)$ the PV prescription [37], is consistent with the prescription of Ref. [27].

The quantities presented in this paper are only valid for hyperbolic trajectories. An important task for the future will be to find their relation to corresponding bound state quantities including a Hamiltonian and physical observables [13, 59]. An initial study has recently been carried out comparing scattering angles and energetics based on the results described here and with EOB upgrades to corresponding results from numerical relativity, displaying rather promising agreement [60].

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