

## Recent progress in intersection theory for Feynman integrals decomposition

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**Vsevolod Chestnov**<sup>a,b,\*</sup>

<sup>a</sup>*Dipartimento di Fisica e Astronomia, Università degli Studi di Padova, Via Marzolo 8, I-35131 Padova, Italy.*

<sup>b</sup>*INFN, Sezione di Padova, Via Marzolo 8, I-35131 Padova, Italy.*

*E-mail:*

[vsevolod.chestnov@pd.infn.it](mailto:vsevolod.chestnov@pd.infn.it)

High precision calculations in perturbative QFT often require evaluation of big collection of Feynman integrals. Complexity of this task can be greatly reduced via the usage of linear identities among Feynman integrals. Based on mathematical theory of intersection numbers, recently a new method for derivation of such identities and decomposition of Feynman integrals was introduced and applied to many non-trivial examples.

In this note based on [1] we discuss the latest developments in algorithms for the evaluation of intersection numbers, and their application to the reduction of Feynman integrals.

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\*Speaker

## 1. Introduction

Feynman integrals (FIs) are a corner stone of the perturbative Quantum Field Theory (pQFT), at least in its contemporary formulation. Within that branch of research one fruitful discovery were the integration-by-parts identities (IBPs) [2, 3], i.e. linear identities among FIs. For a given pQFT problem (such as, for example, scattering amplitudes) IBPs allow to reduce an infinite set of contributing FIs to a linear combination of finite number of basic objects known as the master integrals (MIs).

Recently a novel framework [4–8] based on the twisted cohomology theory [9–19] was proposed to describe relations among FIs. It was shown that (for fixed topology of the corresponding Feynman graphs) FIs form a finite dimensional vector space endowed with a scalar product called the intersection number. Among other applications, this structure then helped to derive novel algorithms for the direct projection of FIs onto the basis of MIs.

In Section 2 review the basics of twisted cohomology theory and computation of intersection numbers. Then in Section 3 we present another algorithm<sup>1</sup> [1] for reduction of FIs exploiting the connection with the Gel'fand-Kapranov-Zelevinsky (GKZ) hypergeometric systems and the secondary equation [19].

## 2. Twisted cohomology

Here we review some aspects of the twisted cohomology and intersection theory see also [20–23] and [24]. Our central subject of study is going to be generalized hypergeometric integrals of the form:

$$\mathcal{I} = \int_C u(x) \varphi(x) , \quad (1)$$

where  $u(x) = \prod_i \mathcal{B}_i^{\gamma_i}$  is a multivalued function,  $C$  is an  $n$ -dimensional integration contour such that  $\prod_i \mathcal{B}_i(\partial C) = 0$ , and  $\varphi \equiv \hat{\varphi} dx_1 \wedge \dots \wedge dx_n$  is a holomorphic  $n$ -form (meaning that the coefficient  $\hat{\varphi}(x)$  is a rational function).

Integrals such as (1) often appear as parametric representation of FIs. For example, the Baikov representation of a FI with  $\ell$  loops and  $E$  external legs in  $d$  dimensions, the multivalued function  $u(x)$  contains a single factor  $u(x) = \mathcal{B}(x)^\gamma$ , where  $\mathcal{B}$  is the Baikov polynomial [25], and the exponent  $\gamma = (d - \ell - E - 1)/2$ . Hence in the following we will refer to the integrals (1) as generalized Feynman Integrals (GFI).

Linear equivalence relation between FIs:

$$\mathcal{I} = \int_C u \varphi \equiv \int_C u (\varphi + \nabla_\omega \xi) , \quad (2)$$

where we introduced the covariant derivative:  $\nabla_\omega := d + \omega \wedge$  and the 1-form

$$\omega := d \log u , \quad (3)$$

<sup>1</sup>This section is based on the joint work with Federico Gasparotto, Manoj K. Mandal, Pierpaolo Mastrolia, Saiei J. Matsubara-Heo, Henrik J. Munch, Nobuki Takayama.

which will be very useful in the following. The equivalence relation (2) follows from the Stokes theorem:  $0 = \int_{\partial C} u \xi = \int_C u \nabla_{\omega} \xi$ , where  $\nabla_{\omega} \xi := d\xi + \omega \wedge \xi$  is the covariant derivative.

Fixing the contour of integration  $C$  allows us to interpret relation (2) as an equivalence of integrands. Namely, we collect  $n$ -forms  $\varphi$  into equivalence classes  $\langle \varphi \rangle : \varphi \sim \varphi + \nabla_{\omega} \xi$  generated by adding covariant derivatives of  $(n-1)$ -forms. Their totality forms the twisted cohomology group:

$$\langle \varphi \rangle \in \mathbb{H}_{\omega}^n := \{n\text{-forms } \varphi \mid \nabla_{\omega} \varphi = 0\} / \{\nabla_{\omega} \xi\}, \quad (4)$$

which can be thought of as the space of linearly independent FIs (of a given topology).

Analogously we can introduce the dual integrals  $\mathcal{I}^{\vee}$ , whose definition mimics (1) up to  $u \mapsto u^{-1}$  and  $\nabla_{\omega} \mapsto \nabla_{-\omega}$ . Elements of the dual twisted cohomology group will be denoted by kets  $|\psi\rangle$ .

## 2.1 Counting the number of Master integrals

The framework of twisted cohomology unites several seemingly independent methods for computation of the number of MIs  $r$ :

1. Number of unreduced integrals produced by the Laporta algorithm [26].
2. Number of critical points, i.e. solutions of  $d \log u(x) = 0$  [27, 28].
3. Number of independent integration contours  $C_{\lambda}$  [29, 30].
4. Number of independent  $n$ -forms, i.e.  $\dim(\mathbb{H}_{\pm\omega}^n)$  [4, 8].
5. Holonomic rank of GKZ system (volumes of certain polytopes) [1, 31].

## 2.2 Scalar product between Feynman integrals

The twisted cohomology theory allows us to view the set of FIs (of a given topology) as a finite dimensional vector space. A set of MIs  $\langle e_{\lambda} \rangle$  for  $\lambda \in \{1, \dots, r\}$  then forms a basis in that space.

The dual FIs really form a dual vector space to FIs due to the existence of a scalar product:

$$\langle \varphi | \psi \rangle = \frac{1}{(2\pi i)^n} \int \iota(\varphi) \wedge \psi, \quad (5)$$

called the intersection number. This scalar product allows to directly decompose a given integral  $\mathcal{I}$  in a basis of MIs  $\mathcal{J}_{\lambda} := \int_C u e_{\lambda}$ , namely  $\mathcal{I} = \sum_{\lambda=1}^r c_{\lambda} \mathcal{J}_{\lambda}$ . Linear algebra leads us to the master decomposition formula [4, 8]:

$$\langle \varphi \rangle = \sum_{\lambda=1}^r c_{\lambda} \langle e_{\lambda} \rangle, \quad c_{\lambda} = \sum_{\mu=1}^r \langle \varphi | h_{\mu} \rangle (C^{-1})_{\mu\lambda}, \quad (6)$$

$$C_{\lambda\mu} := \langle e_{\lambda} | h_{\mu} \rangle, \quad (7)$$

for any choice of the dual basis  $|h_{\mu}\rangle$ . Therefore the intersection numbers (5) completely determine the decomposition coefficients. Let's see now how they can be computed.

## 2.3 Univariate intersection numbers

In the  $n=1$  case, intersection numbers (5) turn into a sum of residues [10, 11, 22]:

$$\langle \varphi | \psi \rangle \equiv \frac{1}{2\pi i} \int_X \left( \varphi - \sum_{p \in \mathbb{P}_{\omega}} \nabla_{\omega} (\theta_p(x, \bar{x}) f_p) \right) \wedge \psi = \sum_{p \in \mathbb{P}_{\omega}} \operatorname{Res}_{x=p} [f_p \psi], \quad (8)$$

where

- Integration goes over  $X = \mathbb{CP}^1$ .
- $\mathbb{P}_\omega := \{p \mid \text{poles of } \omega\}$ , including the  $\infty$ .
- Terms with Heaviside  $\theta$ -functions regulate the integral with the help of a local potential  $f_p$ , which satisfies  $\nabla_\omega f_p \equiv (d + \omega \wedge) f_p = \varphi$  around the pole  $p$ . This differential equation can be solved via an Ansatz:  $f_p = f_{p, \min}(x-p)^{\min} + f_{p, \min+1}(x-p)^{\min+1} + \dots + f_{p, \max}(x-p)^{\max}$ .

## 2.4 Multivariate intersection numbers

One strategy for dealing with the intersection numbers of multivariate FIs is to apply the univariate procedure recursively one variable at a time [4, 7, 8, 12, 18]. Consider a 2 variable problem: given two 2-forms  $\varphi(x_1, x_2)$  and  $\psi(x_1, x_2)$  we would like to compute  $\langle \varphi | \psi \rangle$  by first integrating out  $x_1$  and then  $x_2$ . To do that we pick a basis  $\langle e_\lambda |$  and its dual  $|h_\mu\rangle$  for the internal  $x_1$ -integration and project  $\varphi, \psi$  onto them (omitting the summation signs):

$$\langle \varphi | = \langle e_\lambda | \wedge \langle \varphi_\lambda |, \quad \langle \varphi_\lambda | = \langle \varphi | h_\mu \rangle (C^{-1})_{\mu\lambda}, \quad (9)$$

$$| \psi \rangle = | h_\mu \rangle \wedge | \psi_\mu \rangle, \quad | \psi_\mu \rangle = (C^{-1})_{\mu\lambda} \langle e_\lambda | \psi \rangle, \quad (10)$$

The internal  $x_1$ -integration can be seen as the insertion of the identity operator  $\mathbb{I}_c = |h_\mu\rangle (C^{-1})_{\mu\lambda} \langle e_\lambda|$ , which consequently allows us to write the remaining integral in  $x_2$  as a sum over residues:

$$\langle \varphi | \psi \rangle = \underbrace{\langle \varphi | h_\mu \rangle (C^{-1})_{\mu\lambda} \langle e_\lambda | \psi \rangle}_{\mathbb{I}_c} = \sum_{p \in \mathbb{P}_P} \text{Res}_{x_2=p} [f_{p,\lambda} C_{\lambda\mu} \psi_\mu]. \quad (11)$$

Similar to Section 2.3, this formula requires the knowledge of a local vector potential  $f_{p,\lambda}$  near each pole  $x_2 = p$ . The potential is fixed by the following system of first order differential equations (omitting the  $p$  subscript):

$$\partial_{x_2} f_\lambda + f_\mu P_{\mu\lambda} = \varphi_\lambda \quad \text{near } x_2 = p. \quad (12)$$

The differential equation matrix  $P$  and it's dual version  $P^\vee$  are made out of  $x_1$ -intersection numbers:

$$P_{\lambda\nu} := \langle (\partial_{x_2} + \omega_2) e_\lambda | h_\mu \rangle (C^{-1})_{\mu\nu}, \quad P_{\mu\xi}^\vee := (C^{-1})_{\mu\lambda} \langle e_\lambda | (\partial_{x_2} - \omega_2) h_\xi \rangle, \quad (13)$$

so they can be computed using the univariate method of Section 2.3. The set  $\mathbb{P}_P$  in eq. (11) is defined as  $\mathbb{P}_P := \{p \mid \text{poles of } P\}$ .

In practice, to compute the residue at, say,  $x_2 = 0$  we solve for  $\rho$  the following system:

$$\begin{cases} [x_2 \partial_{x_2} + P(x_2)] \vec{f} = \vec{\varphi} \\ \rho = \text{Res}_{x_2=0} [\vec{f} \cdot \vec{\psi}] \end{cases}, \quad (14)$$

where we rescaled  $\vec{\varphi}(x_2) \mapsto 1/x_2 \vec{\varphi}(x_2)$  and  $P(x_2) \mapsto 1/x_2 P(x_2)$ , and canceled the  $C$  matrix in the residue (11) against the  $C^{-1}$  coming from eq. (10). The series expansion of the system (14) is build from:

$$P(x_2) = \sum_{i \geq 0} x_2^i P_i, \quad \vec{\varphi} = \sum_{i \geq k} x_2^i \vec{\varphi}_i, \quad \vec{\psi} = \sum_{i \geq m} x_2^i \vec{\psi}_i, \quad (15)$$

for integer  $k, m \in \mathbb{Z}$ . Inserting an Ansatz  $\vec{f} = \sum_i x_2^i \vec{f}_i$ , and matching the powers of  $x_2$  order by order, we obtain the linear system (here dots  $\cdot$  denote zeros):

$$\begin{bmatrix} -1 & \vec{\psi}_1 & \vec{\psi}_0 & \vec{\psi}_{-1} & \vec{\psi}_{-2} & \vec{\psi}_{-3} & \cdot \\ \cdot & P_0 - 2 & \cdot & \cdot & \cdot & \cdot & \vec{\varphi}_{-2} \\ \cdot & P_1 & P_0 - 1 & \cdot & \cdot & \cdot & \vec{\varphi}_{-1} \\ \cdot & P_2 & P_1 & P_0 & \cdot & \cdot & \vec{\varphi}_0 \\ \cdot & P_3 & P_2 & P_1 & P_0 + 1 & \cdot & \vec{\varphi}_1 \\ \cdot & P_4 & P_3 & P_2 & P_1 & P_0 + 2 & \vec{\varphi}_2 \end{bmatrix} \cdot \begin{bmatrix} \rho \\ \vec{f}_{-2} \\ \vdots \\ \vec{f}_2 \\ -1 \end{bmatrix} = 0. \quad (16)$$

This equation has to be solved only for  $\rho$ . Row reduction of this matrix can be carried out only until the first row is filled with zeros except for the element in the last column (highlighted with grey), which will contain the needed residue. Other poles of eq. (11) are treated in the same manner and the sum of their residues produces the intersection number  $\langle \varphi | \psi \rangle$ .

### 3. Decomposition via the secondary equation

As was observed in [1] (see also [31]), the twisted cohomology framework provides another method for computation of the decomposition coefficients (7). The first key idea is the so-called secondary equation [8, 19, 32], which is a matrix differential equation satisfied by the intersection matrix  $C$ :

$$\begin{cases} \partial_{z_i} \langle e_\lambda | = (P_i)_{\lambda\nu} \langle e_\nu | \\ \partial_{z_i} |h_\mu\rangle = |h_\xi\rangle (P_i^\vee)_{\xi\mu} \end{cases} \implies \partial_{z_i} C = P_i \cdot C + C \cdot (P_i^\vee)^\top, \quad (17)$$

where  $z_i$  are some external kinematical variables. The other key step is computation of the differential equation matrices  $P$  and  $P^{\text{aux}}$  made available thanks to the connection of the twisted cohomology theory, the GKZ formalism, and  $D$ -module theory. We assume that this step is completed and refer the interested reader to [1, 31] for the full story. Once the secondary equation (17) is written down, we employ the known algorithms for finding rational solutions of such systems, e.g. the MAPLE package INTEGRABLECONNECTIONS [33].

Finally, to determine the decomposition coefficients (7) we repeat the above procedure for an auxiliary basis  $e^{\text{aux}} := \{e_1, \dots, e_{r-1}, \varphi\}$ , i.e. we compute an auxiliary  $P^{\text{aux}}$  and then  $C^{\text{aux}}$ . The FI decomposition is then encoded in the following matrix product:

$$\begin{bmatrix} e_1 \\ \vdots \\ e_{r-1} \\ \varphi \end{bmatrix} = C^{\text{aux}} \cdot C^{-1} \begin{bmatrix} e_1 \\ \vdots \\ e_{r-1} \\ e_r \end{bmatrix} \implies C^{\text{aux}} \cdot C^{-1} = \begin{bmatrix} & & & 0 \\ & & & \vdots \\ & & \text{id}_{r-1} & 0 \\ c_1 & \cdots & c_{r-1} & c_r \end{bmatrix}, \quad (18)$$

where  $\text{id}_{r-1}$  denotes an identity matrix of size  $(r-1)$ , and the decomposition coefficients  $c_\lambda$  are collected in the last row highlighted with grey.

### 3.1 A simple example

Let us briefly showcase how the secondary equation can produce the reduction coefficients of a box diagram with a single dot  $\varphi = \text{box with a dot}$  in terms of the basis  $(e_1, e_2, e_3) = \left( \text{circle with a dot}, \text{circle with two dots}, \text{box} \right)$ . This topology has  $u = (x_1 + x_2 + x_3 + x_4 + x_1 x_3 + t/s x_2 x_4)^\gamma$ . Using the algorithm of [1] and the ASIR computer algebra system [34] we generate the differential equation matrices:

$$P = \begin{bmatrix} -\frac{\epsilon(\delta^2(12z+11)+7\delta(z+1)+z+1)}{(3\delta+1)z(z+1)} & -\frac{\delta^2\epsilon}{(3\delta+1)(z+1)} & \frac{\delta^2\epsilon(\delta(z+2)+1)}{2(3\delta+1)z(z+1)(\delta\epsilon+1)} \\ \frac{\delta^2\epsilon}{(3\delta+1)z(z+1)} & -\frac{\delta^2\epsilon}{(3\delta+1)(z+1)} & -\frac{\delta^2\epsilon(\delta+2\delta z+z)}{2(3\delta+1)z(z+1)(\delta\epsilon+1)} \\ -\frac{2(2\delta+1)\epsilon(\delta\epsilon+1)}{(3\delta+1)z(z+1)} & \frac{2(2\delta+1)\epsilon(\delta\epsilon+1)}{(3\delta+1)(z+1)} & -\frac{\epsilon(\delta^2(5z+7)+\delta(2z+5)+1)}{(3\delta+1)z(z+1)} \end{bmatrix}, \quad (19)$$

where  $z = t/s$  is the ratio of the Mandelstam invariants,  $\delta$  is an additional regularization parameter which should be set  $\delta \rightarrow 0$  at the end of the computation, and  $P^\vee = P|_{\epsilon \rightarrow -\epsilon}$  (see [1, 31] for further details). The rational solution to the secondary equation (17) looks like this:

$$C = \begin{bmatrix} -\frac{(2\delta+1)(4\delta+1)}{\delta} & \delta & -2(\delta\epsilon-1) \\ \delta & -\frac{(2\delta+1)(4\delta+1)}{\delta} & -2(\delta\epsilon-1) \\ 2(\delta\epsilon+1) & 2(\delta\epsilon+1) & -\frac{4(10\delta^2+6\delta+1)(\delta\epsilon-1)(\delta\epsilon+1)}{\delta^3} \end{bmatrix}. \quad (20)$$

We repeat the same procedure for the auxiliary  $C^{\text{aux}}$  and mount everything into eq. (18) to produce:

$$\text{box with a dot} = -\frac{2\epsilon(2\epsilon+1)}{z(\epsilon+1)} \cdot \text{circle with a dot} + 0 \cdot \text{circle with two dots} + (2\epsilon+1) \cdot \text{box}. \quad (21)$$

Therefore the secondary equation method allows us to decompose FIs in terms of MIs via solving a first order matrix differential equation [1]!

## 4. Conclusion

We reviewed the connection between FIs and the twisted cohomology theory, focusing on the algorithms for computation of the uni- and multivariate intersection numbers, that is the scalar products between FIs.

Furthermore, following [1], we showed how the twisted cohomology together with the theory of GKZ hypergeometric system provide a way to compute the IBP reduction coefficients via essentially finding rational solutions to a system of PDEs (17) called the secondary equation. In the future it would be interesting to further develop this connection and apply it to other problems and processes within pQFT.

Figures were made with AxoDRAW2 [35].

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