

Four-quark states from functional methods

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The discovery of four-quark states attracted a lot of attention from the theoretical as well as the experimental side. To study their properties from QCD we use a functional framework which combines (truncated) Dyson-Schwinger and Bethe-Salpeter equations in Landau gauge. This approach allows us to extract qualitative results for mass spectra, decay widths and wavefunctions of candidates for bound as well as resonant four-quark states. Furthermore, we can investigate the possible internal structure of such states. We report on recent developments and results using this functional framework and give an overview about the current status as well as future developments.

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1. Functional Framework

To study the properties of hadrons we employ a non-perturbative functional framework in which we combine Dyson-Schwinger equations (DSEs), i.e., the quantum equations of motion, with hadronic bound state equations, e.g., Bethe-Salpeter equations (BSEs) (see [1] for a detailed review and references therein). The functional approach has been successfully applied to the meson and baryon spectrum (see, e.g., [1]), glueballs [2, 3] and also the spectrum of light and heavy-light four-quark states [4–9].

A BSE can be thought of as an eigenvalue equation

$$\lambda(P^2) \Gamma = K G \Gamma, \quad (1)$$

where Γ is the Bethe-Salpeter amplitude (BSA), K denotes the interaction kernel and G are the fully dressed quark propagators. We have also introduced an eigenvalue $\lambda(P^2)$ which depends on the hadron momentum P squared. Eq. (1) is solved if $\lambda(P^2 = -M^2) = 1$, i.e., the hadron goes on-shell.

In the following we will focus on the properties of four-quark states. Four-quark states are systems of two quarks and two anti-quarks, i.e., $qq\bar{q}\bar{q}$. Their exact four-body Bethe-Salpeter equation contains irreducible two-, three- and four-body interactions. For reasons of complexity, the latter two have been neglected so far. While on the surface this may be considered an uncontrolled and potentially severe oversimplification, arguments in favour of the dominance of two-body interactions have been discussed in [4]. We are then left with the system shown in Fig. 1, where we have three different two-body interaction topologies: two meson-meson and one diquark-antidiquark. For the two-body interaction kernels, we employ the *rainbow-ladder truncation*, i.e., the interaction reduces to exchanges of effective gluons, for details see, e.g., [10, 11]. In recent years there were some developments to systematically improve the Ansatz for the interaction in the light meson sector [12, 13].

The four-body BSA for the scalar four-quark state has the following form

$$\Gamma(k, q, p, P) = \sum_{i=1}^N f_i(\Omega) \tau_i(k, q, p, P) \otimes \Gamma_C \otimes \Gamma_F, \quad (2)$$

where k, q and p are relative momenta between (anti-)quark pairs, each associated with one specific interaction topology, and P is the total momentum of the four-quark state. The f_i are the dressing functions of the four-quark state, which depend on a set of nine Lorentz-invariants Ω and τ_i are the respective basis elements in Dirac space. For the scalar four-quark state we have $N = 256$ basis elements. Furthermore, there is also a colour and a flavour part, Γ_C and Γ_F respectively. We apply two different strategies to make this equation more manageable to solve. First, we consider the 16 s -wave tensors, which depend only on P but not on the relative momenta and form a Fierz-complete basis. This approximation has been done in the three-body equation for baryons and it was found to be reliable to $\sim 10\%$, which is sufficient for the current purposes. The second strategy is to recast the momenta k, q, p, P into multiplets of the permutation group S_4 [14]. One can then identify a singlet S_0 , a doublet D and two triplets T_0, T_1 . It has been shown in [4], that the dependence of the dressing functions f_i on the triplet variables is weak and can be neglected. The other two, however, are important: the S_0 -variable carries the scale, whereas D restricts the phase space. It is also in

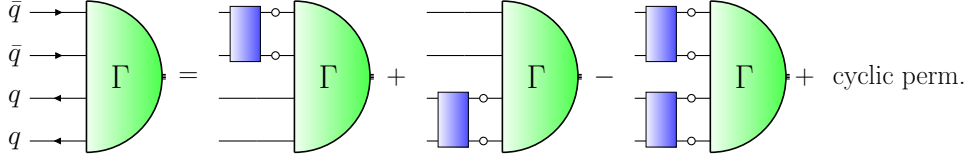


Figure 1: Four-quark BSE with two-body interactions. The green half-circles denote the BSA and the blue boxes indicate interactions between two quarks. There are three interaction topologies: two meson-meson ($q\bar{q})(q\bar{q})$ topologies and one a diquark-antidiquark ($qq)(\bar{q}\bar{q})$ topology (shown).

the doublet-variables D that intermediate 2-body pole structures arise dynamically. As a further approximation we use a physically motivated basis, i.e., we assume that the amplitude is dominated either by tensor structures corresponding to two physical channels identified by the decay products with lowest mass or by the diquark channels with lowest (unphysical) mass. Thus, we have three dressing functions $f^{\mathcal{M}_1}$, $f^{\mathcal{M}_2}$, $f^{\mathcal{D}}$ for the two meson and the diquark topology and can now put in the pole structures by hand, i.e., we make the replacement $f_i(S_0, D) \rightarrow f_i(S_0) \cdot P_{ab} \cdot P_{cd}$, where $P_{ab|cd}$ denote 2-body poles in certain topologies and a, b, c, d is the quark index. In this physical basis, the dressing functions f_i only depend on the singlet S_0 . A more detailed description can be found in [7]. Note, there is also a related BSE approach in which one assumes dominant two-body forces and can then further simplify the four-quark BSE. This approach will be termed *two-body approach* and deals with effective meson-meson and diquark-antidiquark degrees of freedom, which interact via quark exchange, see [7, 9] for details.

2. Results

In previous works, the masses $M^2 = -P^2$ of the light and heavy-light four-quark states were extracted in the four-body and the two-body approaches, see [4–6, 9]. As most hadrons are resonances one strives to extract the real part of the mass plus the decay width of the hadrons from the theory framework to compare with the experimental values. In principle, it is straight forward (but numerically tedious) to solve the BSE, Eq. (1), for such complex P^2 . In the rest frame of the hadron this corresponds to $P^\mu = (iM + \Gamma/2) \hat{e}_4^\mu$ and the complex eigenvalue has to fulfil two conditions: $\text{Re}(\lambda(P^2)) = 1$ and $\text{Im}(\lambda(P^2)) = 0$.

There are however two caveats. With the techniques available so far, the computation is only possible in the first (unphysical) Riemann sheet. As we are interested in resonances, which are identified by poles lying in the (physical) second Riemann sheet, we need to perform an analytic continuation to extract the physical results. The second caveat is, that it is sometimes not straight forward to reach the P^2 such that $\lambda(P^2 = -M^2) = 1$ is fulfilled. This is because of the intermediate particle poles, which introduce branch cuts into the equation. Fortunately, in many cases path deformation techniques are available to circumvent the problem. These techniques have been explored for conventional mesons, see, e.g., [7] and references therein, and also in the context of four-quark states in the two-body approach [8]. In this paper, the authors used a combination of path deformation and analytic continuation in the two-body approach to compute the BSE in the complex plane above the threshold, see left plot in Fig. 2. It shows the real and imaginary part of the eigenvalue $\lambda(P^2)$ for the σ meson and one can clearly see a branch cut opening in the imaginary

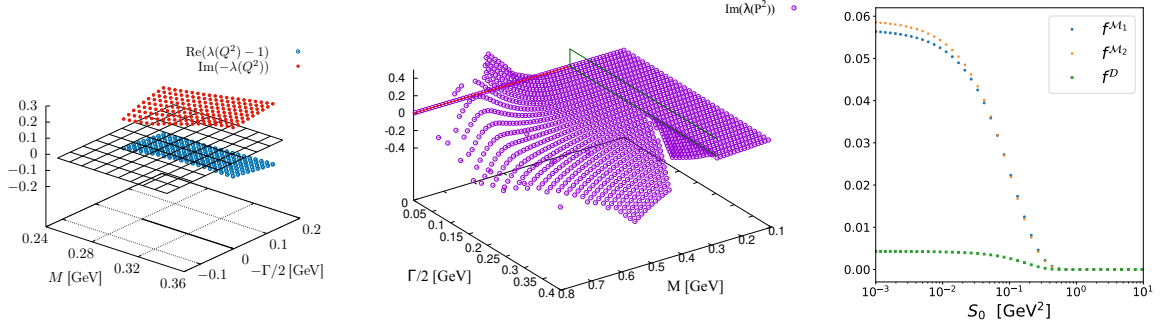


Figure 2: *Left:* Real and Imaginary part of the eigenvalue $\lambda(Q^2)$ for the σ in the two-body formalism. Here the total momentum is denoted by Q^2 . The black line on the bottom indicates the branch cut along the real axis. Figure taken from [8]. *Middle:* Preliminary result for the imaginary part of λ for the σ in the four-body formalism. The red line is the branch cut along the real axis and the green box is the particle threshold. *Right:* Preliminary result showing the dressing functions for the a_0 as a function of the singlet variable S_0 . From the magnitude, we find that the meson contributions are dominant and the diquark contribution is subleading.

part of $\lambda(Q^2)$ above a certain decay threshold, i.e., $\pi\pi$ for the σ . Extrapolating beyond the cut, they extracted the masses plus the decay widths of the light scalar nonet particles σ , a_0 , f_0 . The results match reasonably well with experimental values (see [8] for a full discussion).

The path deformation technique, however, has not yet been exploited in the four-body approach. This is work in progress. On the right in Fig. 2, we show a preliminary result for the imaginary part of the eigenvalue of the σ obtained in the four-body formalism. We have identified the branch cut opening above the two-pion threshold, similar to the two-body approach, and are now in the process of adapting the path deformation techniques to the problem at hand.

In the regions of P^2 which are directly assessable, we are now also in a position to address the internal structure of the four-quark states directly via a comparison of the size of different contributions to the Bethe-Salpeter amplitude, i.e. the shape and magnitude of the dressing functions f^{M_1} , f^{M_2} , f^D . On the right in Fig. 2 we plotted the dressing functions for the example case of the a_0 as functions of S_0 , i.e., the overall momentum scale. We find a clear dominance of the meson dressing functions over the diquark dressing function. This is completely in line with the findings from the two-body approach, [8], where this information was only indirectly available.

Closing remarks Since the DSE/BSE framework in principle makes no assumptions on the internal structure of four-quark states (i.e., e.g., diquark clustering vs. meson clustering), it is a very interesting tool to study the dynamical effects which generate these structures. For all states studied so far, the (heavy-light or light-light) meson-meson components dominate and diquark components are subleading. In recent time, the framework evolved to a level, which makes the extraction of decay widths possible [8], at least in the two-body approach. The generalisation of the corresponding techniques to the four-body approach is work in progress and may lead to more refined statements about the internal structure of four-quark states.

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