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Collective Neutrino Oscillations: Beginning and End

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This is a summary of the plenary talk titled "Supernovae and Neutrinos" given by the author at *NOW 2022*. The talk highlighted two aspects of collective oscillations of neutrinos in supernovae and other similar environments: The first – *when and why does neutrino flavor change exponentially with time, even in dense matter where no flavor conversion is normally expected*; and the second – *what is the physical and astrophysical impact of such unstable flavor change*.

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1. Introduction

Neutrinos produced by supernovae and neutron star mergers can significantly impact the dynamics and nucleosynthesis of the stars they are produced in because they carry away a large amount of energy and interact differently with the background medium depending on their flavor states. Understanding the flavor evolution of these neutrinos in these environments is therefore important.

The flavor evolution of dense neutrino clouds can be complex due to a range of factors, including the initial trapping of neutrinos in the core, their subsequent leakage through diffusion, and the influence of background matter on flavor-mixing. While flavor-mixing is typically suppressed in environments with high background matter density, large collective flavor conversion can still occur due to an instability that arises when neutrinos forward-scatter off each other and influence each other's flavor evolution. This leads to an intricate, collective flavor evolution that can produce exponential growth in flavor conversion through nonlinear routes [1–5].

In the last three decades, a lot of insight has been obtained into the origin and impact of a variety of collective flavor transformations. However, it is only very recently that we have begun to consolidate our understanding of these perplexing dynamics under a single rubric. In this talk, I try to concisely answer two key questions:

- When and why does neutrino-flavor change exponentially with time?
- What is the eventual result of such unstable flavor change?

This talk is based on the papers [6] and [7–10], from which much of the material below is reproduced. Interested readers will also find more references to related literature cited therein. We use natural units throughout, with $\hbar = c = 1$.

2. The Set-up

We consider scenarios where the flavor-dependent occupation matrix for a neutrino evolves as

$$v^\alpha \partial_\alpha \rho_\mathbf{p} = -i\,[\mathsf{H}_\mathbf{p}, \rho_\mathbf{p}] + \mathsf{C}_\mathbf{p}\,, \tag{1}$$

where a summation over the spacetime indices $\alpha = 0, \ldots, 3$ is implied. $\rho_\mathbf{p}$ and $\bar{\rho}_\mathbf{p}$ are the 3×3 occupation number matrices for neutrinos and antineutrinos, and $\mathsf{H}_\mathbf{p}$ and $\mathsf{C}_\mathbf{p}$ are the Hamiltonian and collision matrices, respectively. The problem is nonlinear despite appearances because $\mathsf{H}_\mathbf{p}$ contains terms involving $\rho_\mathbf{p}$ and $\bar{\rho}_\mathbf{p}$, as does $\mathsf{C}_\mathbf{p}$. The equation of motion (EoM) for the antineutrino matrices $\bar{\rho}_\mathbf{p}$ is the same except for a sign-change in the mass-mixing term in $\mathsf{H}_\mathbf{p}$.

All flavor coherence effects depend only on the difference of the original neutrino phase space distributions. In particular, we may write the effective two-flavor neutrino matrices of occupation numbers in the form

$$\varrho_\mathbf{p}^{e\mu} = \frac{f_{\nu_e,\mathbf{p}} + f_{\nu_\mu,\mathbf{p}}}{2}\,\mathbb{I} + \frac{f_{\nu_e,\mathbf{p}} - f_{\nu_\mu,\mathbf{p}}}{2}\begin{pmatrix} s_\mathbf{p} & S_\mathbf{p} \\ S_\mathbf{p}^* & -s_\mathbf{p} \end{pmatrix}, \tag{2}$$

whose off-diagonal equals $\rho_\mathbf{p}^{e\mu}$, i.e., the off-diagonal of the three-flavor matrix, where $s_\mathbf{p}$ is a real number, and $S_\mathbf{p}$ complex with $s_\mathbf{p}^2 + |S_\mathbf{p}|^2 = 1$.

To study the initial growth of flavor-change, as in Sec. 3, we can linearize the EoM. To linear order in $|S_{\mathbf{p}}|$, one has $s_{\mathbf{p}} = 1$, so we can focus on the space-time evolution of $S_{\mathbf{p}}$ alone which holds all the information concerning flavor coherence. The two-flavor *spectrum* is

$$g_\Gamma = \sqrt{2} G_{\rm F} \begin{cases} f_{\nu_e,\mathbf{p}} - f_{\nu_\mu,\mathbf{p}} & \text{for } E > 0, \\ f_{\bar{\nu}_\mu,\mathbf{p}} - f_{\bar{\nu}_e,\mathbf{p}} & \text{for } E < 0, \end{cases} \tag{3}$$

with $\Gamma = \{E, \mathbf{v}\}$. The linearized EoM thus reads [11–13]

$$\Big(v^\alpha\,(i\,\partial_\alpha - \Lambda_\alpha) - \omega_E + i|\Delta_\Gamma|\Big) S_\Gamma = -v^\alpha \int d\Gamma'\, v'_\alpha\, g_{\Gamma'}\, S_{\Gamma'}\,, \tag{4}$$

where the phase-space integration is over

$$\int d\Gamma = \int_{-\infty}^{+\infty} \frac{E^2 dE}{2\pi^2} \int \frac{d\mathbf{v}}{4\pi}, \tag{5}$$

with $\int d\mathbf{v}$ an integral over the unit sphere, i.e., over the polar and azimuthal angles of $\mathbf{p}$. The ω_E encodes mass-square difference (note that the mixing has been set negligible to mimic dense matter), Λ_α contains refractive effects due to forward scattering, whereas $|\Delta_\Gamma|$encodes the effect of collisions [6].

On the other hand, for Sec. 4 relevant outside of the linear regime, one must solve the full nonlinear equation. Here, we specialize to fast oscillations without collisions. Axisymmetry restricts that the flavor evolution depends on a single spatial coordinate z, a single momentum coordinate v, and of course on time. This is a simple model for fast neutrino flavor evolution in a supernova, after it starts free streaming. Under these assumptions, the flavor content encoded in each ρ_v evolves as

$$(\partial_t + v\partial_z)\mathbf{S}_v = \mu_0 \int_{-1}^{+1} dv' G_{v'}\,(1 - vv')\,\mathbf{S}_{v'} \times \mathbf{S}_v\,. \tag{6}$$

Here $\mathbf{S}_v$ is the Bloch vector encoding the flavor state for neutrino modes with velocity v, with $|v| < 1$. The ELN distribution function G_v is the excess of the phase space distribution of ν_e over ν_μ (and $\bar{\nu}_\mu$ over $\bar{\nu}_e$), integrated over $E^2 dE$ and divided by a typical density, say n_ν. This is proportional to the spectrum g_Γ integrated over $E^2 dE$. Only the product of μ_0 and G_v appears; though, one defines a rate $\mu_0 \propto G_{\rm F} n_\nu$ as the collective potential. Hereafter, we set $\mu_0 = 1$, and express z and t in units of μ_0^{-1}. The ELN becomes dimensionless in these units.

3. Condition for Collective Instability

As usual, for a linear EoM we search for space-time dependent solutions of equation (4) in terms of its independent Fourier components

$$S_{\Gamma,r} = \sum_K Q_{\Gamma,K}\, e^{-i(K_0 t - \mathbf{K}\cdot\mathbf{r})}, \tag{7}$$

where $r^\mu = (t, \mathbf{r})$ and $K^\mu = (K^0, \mathbf{K})$. The quantity $Q_{\Gamma,K}$ is the eigenvector in Γ-space for the eigenvalue K. After some manipulations, the linearized EoM can be shown to imply

$$v_\alpha \Pi_k^{\alpha\beta} A_{k,\beta} = 0, \quad \text{where,} \tag{8}$$

$$\Pi_k^{\alpha\beta} = h^{\alpha\beta} + \int d\Gamma\, g_\Gamma \frac{v^\alpha v^\beta}{v_\gamma k^\gamma - \omega_E + i|\Delta_\Gamma|}, \tag{9}$$

with $h^{\alpha\beta} = \mathrm{diag}(+,-,-,-)$ being the metric tensor. This equation must hold for any v^α and thus consists of four independent equations $\Pi_k^{\alpha\beta} A_{k,\beta} = 0$. Nontrivial solutions require

$$\mathcal{D}(k) \equiv \det \Pi_k^{\alpha\beta} = 0, \tag{10}$$

establishing a connection between the components of $k = (k^0, \mathbf{k})$, i.e., the dispersion relation of the system. It depends only on the neutrino flavor spectrum g_Γ, which itself contains the neutrino density, the vacuum oscillation frequency ω_E, and the damping rate $|\Delta_\Gamma|$.

If the imaginary part of k^0 is positive, for any k that satisfies equation (10), equation (7) tells us that it leads to exponential growth of the off-diagonal flavor coherence between the two flavors under consideration, i.e., $S^{e\mu} \sim e^{t\,\mathrm{Im}\,k^0}$. In the limit of vanishing flavor-mixing, as relevant in dense matter, such flavor conversion is surprising and called a *collective instability*.

Now we prove that collective instabilities can arise only if there is a crossing of phase space distributions [6]. We prove the proposition by contradiction, following Morinaga [14]. We separate the real and imaginary parts of $k^0 = \kappa + i\sigma$, where $\kappa, \sigma \in \mathbb{R}$, and write the Π matrix as

$$\Pi^{\alpha\beta} = M^{\alpha\beta} - iN^{\alpha\beta}, \tag{11}$$

where M and N are real-symmetric matrices

$$\begin{aligned} M^{\alpha\beta} &= h^{\alpha\beta} + \int d\Gamma\, g_\Gamma \frac{(\kappa - \mathbf{v}\cdot\mathbf{k} - \omega_E)\, v^\alpha v^\beta}{(\kappa - \mathbf{v}\cdot\mathbf{k} - \omega_E)^2 + (\sigma + |\Delta_\Gamma|)^2}, \\ N^{\alpha\beta} &= \int d\Gamma\, g_\Gamma \frac{(\sigma + |\Delta_\Gamma|)\, v^\alpha v^\beta}{(\kappa - \mathbf{v}\cdot\mathbf{k} - \omega_E)^2 + (\sigma + |\Delta_\Gamma|)^2}. \end{aligned} \tag{12}$$

One can make an orthogonal transformation to diagonalize N to the diagonal matrix D, and then simple manipulations show that

$$\sum_\alpha D^{\alpha\alpha} |A_\alpha|^2 = 0. \tag{13}$$

In the equation above, $|A_\alpha|^2$ are non-negative and not all of them vanish. As proposed, we have $\sigma > 0$ and g_Γ has the same sign everywhere, so all $D^{\alpha\alpha}$ have the same signature as g_Γ.

There would appear to be two possibilities for equation (13). First, the singular case where $D^{\alpha\alpha} = 0$ for *all* α for which $|A_\alpha|^2 \neq 0$. However, in that case for some α, the integral of $(O^\alpha_\mu v^\mu)^2$ times an *everywhere-same-sign* function vanishes identically. This is possible only if $(O^\alpha_\mu v^\mu)^2 = 0$ for all points in Γ or if $g_\Gamma = 0$. That is, the same O makes the α-component of any v vanish or that there are no collective effects at all, respectively. These are either impossible or trivial, and therefore excluded. Second is the non-singular case, where $D^{\alpha\alpha} \neq 0$ for *some* $A_\alpha \neq 0$. In this case, in equation (13) at least one term is nonzero and all terms are non-negative. But then equation (13), which algebraically followed from our original assumptions, cannot be satisfied! The only resolution is that g_Γ must change sign if there exists a $\sigma > 0$. This completes the proof of the proposition. As a corollary, setting $\omega_E \to 0$ and $\Delta_\Gamma \to 0$, one recovers the necessary condition for collisionless fast instability. Other sub-classes of instabilities may be similarly obtained by taking appropriate limits, e.g., setting $\Delta_\Gamma \neq 0$ for collisional instabilities [15].

4. Ultimate Fate of Collective Oscillations

Collective neutrino instabilities are typically much faster than ordinary neutrino oscillations, and occur deeper inside the star. They also have nontrivial dependence on spatial and time, because of their nontrivial dispersion relations. In detail, the full nonlinear evolution of these dense neutrino gases is very complex indeed. However, for the purposes of phenomenology we find that it may be useful to focus on the long-term final state of these oscillations. We find that if fast conversions occur, they typically lead to partial flavor equilibrium [7].

We performed some of the first nonlinear calculations of the fast collective oscillations depending on *both space and time* [8–10], as per Eq. (6). The initial evolution of the average flavor content is looks similar to the oscillatory motion of an inverted pendulum. However, this is better understood using the mapping to particle rolling in a quartic potential for a simple model [16]. As we show in refs. [8–10], including the advective terms in the EoM, this pendulum neither preserves its length nor retains its periodic motion. It settles down to a resting point, which is analytically known in terms of the ELN and its moments. The shrinking of the length of the pendulum and its settling down can be traced to a number of relaxation mechanisms. These fundamentally stem from the quenching of the transverse components of the flavor polarization vectors due to relative dephasing. Such dephasing begins already in the linear regime of flavor growth. However, the depolarization depends strongly on which velocity modes experience a large transverse Hamiltonian. In the nonlinear regime, n-multipole cascade and k-mode mixing lead to spreading of the flavor disturbance in momentum space and position space, respectively.

The flavor content eventually acquires an approximately time-independent character. This is called depolarization. The extent of depolarization is non-uniform over neutrino and antineutrino momentum. In general, it depends on the ELN. This is essentially because the net lepton asymmetry needs to remain conserved. The broad results on depolarization and its extent, as well as mixing of velocity multipoles and k-modes, are now confirmed by other groups [17–21].

The extent of depolarization, encoded in the depolarization factor, can be predicted – if the range of fully depolarized modes is assumed. For positive lepton asymmetry $A = \int_{-1}^{1} dv G_v > 0$, we find in the two-flavor approximation,

$$f_v^{\mathrm{D}} \approx \begin{cases} \frac{1}{2} - \frac{A}{4\gamma_0} - \frac{3A}{8\gamma_0}\, v & \text{if} \quad 1 \geq v \geq 0\,, \\ \frac{1}{2} & \text{if} \; -1 \leq v \leq 0\,, \end{cases} \tag{14}$$

where $\gamma_0 = \int_0^1 dv G_v$. A generalization to restricted-three-flavor scenario where the initial conditions and evolution of the μ and τ flavors are taken to be identical, is easy. One finds

$$f_v^{\mathrm{D,\,3\,flav}} = \begin{cases} \frac{4f_v^{\mathrm{D}}}{3} & \text{if } f_v^{\mathrm{D}} < \frac{1}{2}\,, \\ \frac{1+2f_v^{\mathrm{D}}}{3} & \text{if } f_v^{\mathrm{D}} > \frac{1}{2}\,. \end{cases} \tag{15}$$

The depolarized flavor distributions and the depolarized fluxes are given in terms of the original distributions and forward moments of the depolarization factor [10]. These are approximate but readily usable ingredients for implementation in supernova/nucleosynthesis simulations and for computations of neutrino signals. It is our expectation that results such as these can be easily used in prescription-based implementations of neutrino-mixing in simulations.

5. Conclusions

- We show that it is necessary for the phase space distributions of two flavors to cross each other in order to initiate a collective instability. This answers the first question, cementing the expectation articulated in ref. [22].
- We strongly indicate that neutrino flavors get almost democratically mixed-up, on coarse-grained scales, but for the conservation of lepton asymmetry. This asymmetry, prevents flavor-equalization of forward neutrinos for positive asymmetry (relevant for supernovae) and of backward neutrinos for negative asymmetry (relevant for neutron-star mergers). We expect that such effects will have astrophysical consequences.

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