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Geodesics on approximately analytic spacetimes

Justine Tarrant* **and Geoff Beck**

Centre for astrophysics and school of physics, University of the Witwatersrand
1 Jan Smuts Avenue, Johannesburg, South Africa

E-mail: justine.tarrant@wits.ac.za, geoffrey.beck@wits.ac.za

In order to derive models that realistically describe the dynamics of photons around black holes, the use of a geodesic integrator is required. Using such an integrator, we may track null geodesics in curved spacetimes surrounding black holes. Since the binary black hole solution to the Einstein equations is not analytic, one requires the use of computationally intensive $3 + 1$ numerical relativity. Instead, we consider a different route, by defining a novel metric created from analytic space-like hyper-surfaces. With this in hand, applying a ray-tracing algorithm requires minimal computational resources. Critically, we present a preliminary step to finding black hole shadows for general binaries.

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*Speaker

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1. Introduction

The era of multi-messenger astronomy is well on its way with the detections of merging binary black holes and neutron stars, made by the Laser Interferometer Gravitational wave Observatory (LIGO). The detection of a possible 1s duration gamma-ray burst was considered as a counterpart to the GW150914 event [1]. However, the weak transient was not detected by any other instrument. Soon after, it was clear that electromagnetic counterparts needed to be considered, therefore, forming a large community dedicated to modelling scenarios, where counterparts to binary mergers may be detected.

The initial data required for the evolution of the binary black hole system, is usually difficult to acquire [2], even for the simpler case of a binary made up of spherically symmetric, static, Schwarzschild black holes. Alvi's construction [3] provides a powerful tool for generating initial data. Presently, the evolution of systems such as binary black holes are studied using computationally intensive, $3 + 1$, numerical relativity, and these methods can be impractical [4].

Our novel approach for studying the evolution of binary black hole spacetimes uses a unique set of analytic space-like hyper-surfaces, built from the initial data in [3, 5], to build an approximately analytic metric. The evolution of the binary provides the evolutionary linkage between each hyper-surface. That is, each time-step is parametrised by the binary separation, b. We impose these evolutions via the relations from [6] according to the time assigned to that space-like hyper-surface.

For the current study, our primary goal is to predict binary black hole null geodesics. Therefore, a prudent initial step involves finding the null geodesics around a single Schwarzschild black hole. This is the simplest solution to the Einstein Equations and provides a test needed to determine whether our geodesic integrator works - via literature comparison [7]. This allows us to determine which errors are due to the integrator, versus those from the binary, which is far more complicated. We note that if the integrator is robust in the null case (i.e. the extremal case), it will be so for the time-like geodesics, as described in previous work [8]. Following the single black hole, we proceeded to binaries. Critically, we present a preliminary step for finding the shape of black hole shadows in binaries. Additionally, we isolate areas of concern for our metric composition technique.

We note that the most intensive part of our numerics is finding the geodesics, even so, the calculation requires very few resources. This is perhaps one of the greatest benefits of our technique.

This work forms part of a larger project where we aim to predict what a possible precursor electromagnetic signal, to binary inspiral, might look like. This is useful since it would allow us to alert multi-wavelength instruments, ahead of inspiral, allowing them to localise and view the event. The inverse of this has already been discussed in [9], where LIGO detection could provide a 10 to 15 minute pre-merger warning for follow-up by other instruments.

2. Binary spacetime structure

We begin by considering two equal mass Schwarzschild black holes in a clockwise circular orbit around each other, with Keplarian velocities. Note that any mass ratio is acceptable. Each black hole perturbs the other. They are widely separated ($\sim 20M$ in geometric units where $M = m_1 + m_2$), which allows for the subdivision of each hyper-surface occurring in [3, 5] and ensures that the binary is not near the inspiral phase. These results will apply to any choice of M provided $m_1 = m_2$.

We build the approximately analytic metric by assigning times to analytic space-like hyper-surfaces in [3, 5], via the evolution of the binary spacing and rotation. Each hyper-surface is described by a 3+1 dimensional sub-divided metric, see Figure 1. The only assumption is that the black holes start out widely separated. In practice this requires that equal mass black holes have separation $b \gtrsim 5M$ to preserve the slow motion approximation for the black hole motion.

The sub-divided regions are described by the best suited approximate solution to the linearised Einstein equation in that region. To find the single, global metric we desire, a mathematical technique called *matched asymptotic expansions* [10] is used. In matching, one attempts to join two metrics along a boundary between them. In a buffer zone, around the boundary, both approximate metrics being matched must be valid. Matching is then achieved when both approximations are expanded to the same order in v^2/c^2 and set equal along the dividing surface in matching coordinate systems. Through matching, our four regions are thus "glued" together, creating a single global metric described by a single coordinate system.

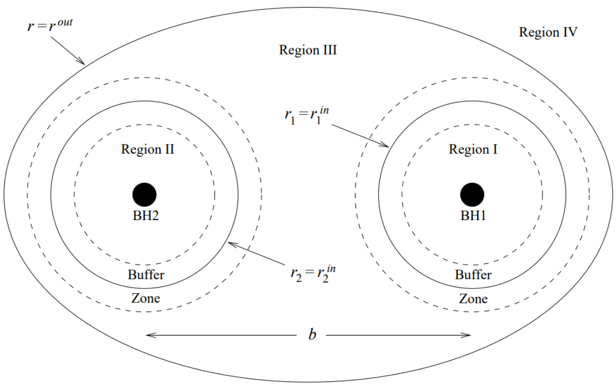

Figure 1: Here we illustrate the demarcated metric, divided into zones with different approximations to the linear Einstein Equations [3]. Regions I and II are perturbed Schwarzschild metrics. Region III is a post-Newtonian treatment of two rotating point masses. Finally, region IV is one where the two black holes are approximated as a single centre of mass (due to the large distance).

One may note that the photons we simulate will all be confined to regions I, II & III (see Fig. 1). Thus, they are contained within r^{out}, seen in Figure 1. A discontinuity is present at this boundary, where one moves from a Post-Newtonian (PN) expansion to Post-Minkowski expansion. These two expansions are no longer both valid in the buffer zone. This happens because the PN expansion breaks down since the PN expansions are valid for speeds $v^2 << c^2$. Therefore, as pointed out in [8], the discontinuity at the $r = r^{out}$ boundary proves significant for photons which are spuriously deflected across the boundary. An important question would address whether continuity in the matching, such as in [11], would mitigate this effect or if it is a consequence of the corotating coordinate system, as these are well-known to become pathological at large radii.

3. Calculating geodesics

Geodesics are the 'straight' lines in a curved spacetime and represent 'freely' moving particles. To find the null geodesics belonging to a photon, we solve the *geodesic equation*. Geodesics provide a simple probe for studying particle motion, and ultimately the fate of the photon in the system considered. Using ray-tracing, as in this work, allows us to compare future observations with predictions from current models. We create our own geodesic integrator using the geodesic equation and the Runge-Kutta-Fehlberg [12] method. The Christoffel symbols need to be found first, and are calculated using finite differencing techniques. The geodesic equation is:

$$\frac{d^2x^\mu}{dt^2} = -\Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt} + \Gamma^0_{\alpha\beta}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt}\frac{dx^\mu}{dt}, \tag{1}$$

where the Christoffel symbols are given by the $\Gamma^\mu_{\alpha\beta}$. Note that this equation depends on the underlying spacetime through the derivatives of the metric making up the Christoffel symbols.

4. Results and discussion

We consider two cases. First, we study a single Schwarzschild black hole and calculate the geodesics of photons around it. This step is a vital since it provides an independent test of our algorithm, and a means of comparison with the literature, see [7] for example. Secondly, we apply our ray-tracing algorithm to a binary black hole system. For the latter case, we try to predict what the null geodesics will look like.

4.1 The single black hole

Figure 2 shows the passage of photons around a single Schwarzschild black hole with $M = 30$ $M_\odot$. All photons are emitted from the right. Those that have passed $x = 0$ are deflected. Photons that are at $y = 0$ and slightly above or below $y = 0$ are accreted. **As we move to larger $|y|$-values, the deflection of photons decreases and more photons are able to escape. Near the origin, however, photons make partial orbits and are captured by the black hole. We can start our paths from any direction for the single black hole, and get the same result, this is due to the spherical symmetry of a single Schwarzschild black hole.** This result pertains to an unperturbed black hole set-up. Importantly, these results match others in the literature [7], providing evidence our ray-tracing technique is working.

4.2 The binary system

The binary system is made up of two equal mass Schwarzschild black holes, i.e. $m_1 = m_2$. As already mentioned, this system has widely separated black holes, with $b = 20M$, ensuring that the binary is not undergoing the inspiral phase.

4.2.1 Constant y-values

In this case we have two streams of photons arriving at $y = -5M$ and $y = 5M$. These are then evolved via backward ray-tracing. The black holes making up the binary are rotating about their centre of mass in a clockwise direction around the z-axis with initial separation $20M$. The left

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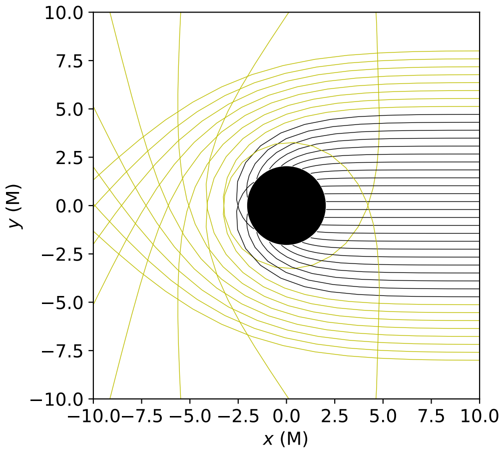

Figure 2: Plot illustrating the geodesics of photons around a single black hole. All photons are arrive at the right. Black trajectories correspond to parts of the shadow and yellow are for the escaping photons.

sub-plot in Figure 3 corresponds to the co-rotating frame. In this case an observer is moving with the binary rotation. Therefore, the left sub-plot does not illustrate what a distant observer would actually see. Instead, we consider the right sub-plot, which is in stationary coordinates. Here, we see what the spacetime is doing to the photons and what an observer, viewing the xy-plane, would see. The black circles are the black holes. In the right plot, corresponding to the stationary frame, the circles are transparent as they only represent the final position of the black holes. **Only a small fraction of photons are captured and evidence of an "eyebrow", the lensed shadow of the other black hole close to a main shadow, is apparent. These have been observed for very close binaries in full numerical simulations [13] but here we show they can occur at certain viewing angles even for widely spaced binaries. The stationary frame indicates that these eyebrows have far longer path lengths and are thus linked to a different orbital phase than the main shadows. Notably, the shadow regions seem to be far smaller than for stationary Schwarzschild black holes.**

4.2.2 Constant x-values

Next, we have two streams of photons arriving at $x = -17M$ and $x = 17M$ respectively. The left sub-plot in Figure 4 corresponds to the co-rotating frame. The physical geodesics of the photons can be viewed, from above the xy-plane, in the right sub-plot, which is in stationary coordinates. Once again, the black circles represent the final positions of the black holes and they are moving clockwise. Once again the shadow region is much smaller than the stationary Schwarzschild black hole. However, there are no "eyebrows" in evidence. The photons to the left of $x = 0$ and below the $y = 0$ line, are deflected by the left black hole, but are not captured. Additional curvature, observed in the stationary frame, of the paths far from the black holes is a result of the corotating coordinate system used in the computations. This effect occurs for long paths far from the centre of rotation.

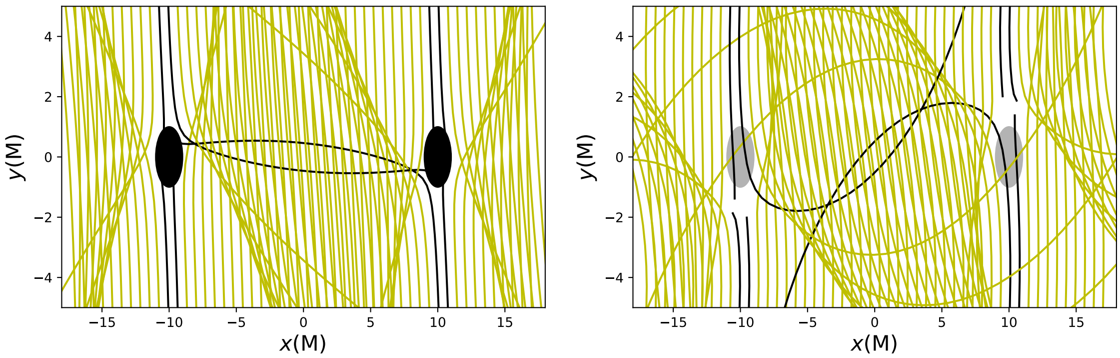

Figure 3: Illustration of geodesics around a Schwarzschild binary black hole with all the photons arriving at the same y value. The left sub-plot corresponds to the co-rotating reference frame, whilst the right sub-plot corresponds the stationary frame. The black holes shown by the light black circles represent the final position of the black holes. Black trajectories correspond to those in the shadow, while yellow are those which escape.

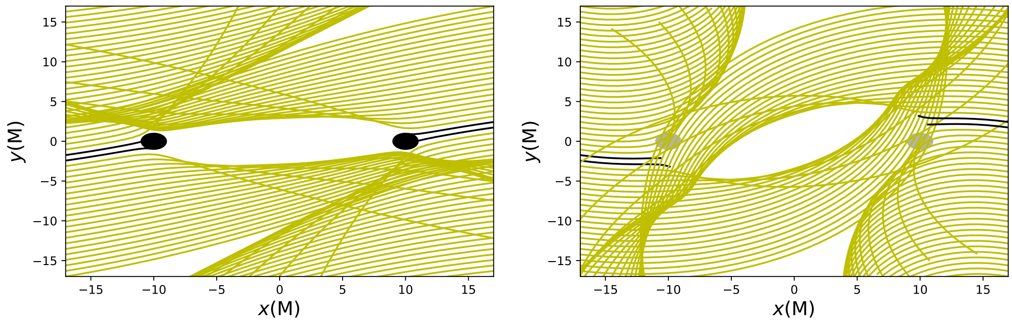

Figure 4: Illustration of geodesics around a Schwarzschild binary black hole with all the photons arriving at the same x values. The left sub-plot corresponds to the co-rotating reference frame, whilst the right sub-plot corresponds to the stationary frame. The black holes shown by the transparent black circles represent the final positions. Black trajectories correspond to those in the shadow, while yellow are those which escape.

5. Conclusion and future work

In this work we implemented a geodesic integrator to study isolated single and binary black holes. The latter are studied with an approximately analytic metric devised through the evolution of space-like hyper-surfaces found by Alvi [3]. First, we calculated the null geodesics around a single black hole to validate our approach. Then we examined a binary and found that the likely shadow regions for a rotating binary are far smaller than for stationary Schwarzschild black holes. Additionally, lensed shadows can be visible given certain viewing angles, despite the wide separation. The advantage of using Schwarzschild binaries is that we may discern the effect of the black hole motion on shadows independent of their own rotation. However, the co-rotating coordinates of our hyper-surfaces prove problematic for trajectories far from the centre of rotation, this motivates exploring hyper-surface implementations which do not use this coordinate choice. Finally, we note that these simulations do not require a vast amount of computing resources and are run in under a minute on a laptop.

Acknowledgements

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