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Fractional calculus in modelling hereditariness and nonlocality in transmission lines

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Transmission lines are traditionally modelled by considering Heaviside's elementary circuit that contains a resistor and inductor in the series branch, accounting for the energy losses and magnetic effects, while the shunt branch contains a resistor and a capacitor, accounting for the energy losses and capacitive phenomena. Classical telegrapher's equations, modelling the signal propagation in a transmission line, are obtained by assuming the infinitesimal length of the elementary circuit and by passing to a continuum. The generalization of elementary circuit is two-fold: topological by adding the capacitor in the series branch in order to account for the charge accumulation effects along the line and constitutive in order to account for the memory effects that transmission line may display. The constitutive generalization is performed by changing the constitutive relation describing the capacitive and inductive material properties using the fractional calculus approach accounting for the short-tail memory. On the other hand, the inclusion of nonlocal material properties of a transmission line is performed by considering the magnetic coupling of inductors in the series branch of Heaviside's elementary circuit, so that the magnetic flux is obtained as a superposition of local and constitutively given nonlocal magnetic flux through the cross-inductivity kernel. Signal propagation is studied in the case of power, exponential, and Gauss type cross-inductivity kernels. The presented results are published in [1–3].

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1. Introduction

Small-scale structures, including transmission lines, may display non-locality effects and the aim of this theoretical study is to: model these effects, formulate the corresponding non-local telegrapher's equations, and analyze those equations analytically and numerically. The starting point is the k-th Heviside's elementary circuit, shown in Figure 1, modeling the transmission line physical properties, while non-local telegrapher's equations

$$
\begin{aligned}
-\frac{\partial}{\partial x} v(x,t) &= R(x,t)\, i(x,t) + \frac{\partial}{\partial t}\phi(x,t) - \mathcal{E}(x,t),\\
-\frac{\partial}{\partial x} i(x,t) &= G(x,t)\, v(x,t) + \frac{\partial}{\partial t}\big(C(x,t)\, v(x,t)\big),\\
\phi(x,t) &= L(x,t)\, i(x,t) + \int_a^b m(x,\zeta,t)\, i(\zeta,t)\,\mathrm{d}\zeta,
\end{aligned}
$$

mathematically describing the transmission line responses, are obtained from Kirchhoff's laws applied to the k-th elementary circuit in the limit when circuit length Δx_k tends to zero, while the number of circuits tends to infinity.

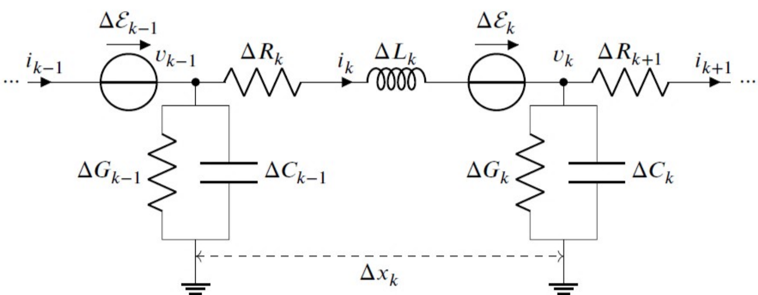

Figure 1: Elementary circuit of the nonlocal transmission line model.

The elementary circuit, as seen from Figure 1, consists of: series resistor and inductor, denoted by ΔR_k, ΔL_k, modeling dominantly conductive properties of transmission line; shunt capacitor and conductor, denoted by ΔC_k, ΔG_k, modeling its insulative properties; and electromotive force, denoted by $\Delta\mathcal{E}_k$, modeling the external influence on the line. The current at time-instant t, running through the series branch of the k-th elementary circuit, is denoted by $i_k = i_k(t)$, while $v_k = v_k(t)$ denotes the (time dependent) voltage on its shunt branch. The non-locality effects originate from the magnetic coupling of inductors in the series branch, namely by assuming that the magnetic flux within elementary circuit is the consequence of currents running through series branches of all elementary circuits rather than just one elementary circuit.

Another aim is to derive and analyze a generalization of the classical telegrapher's equation, where both capacitive and inductive phenomena are assumed to be of fractional-order type. In addition, the model under consideration takes into account the phenomena of charge accumulation along the line. Equivalent elementary circuit is shown in Figure 2. The generalized telegrapher's

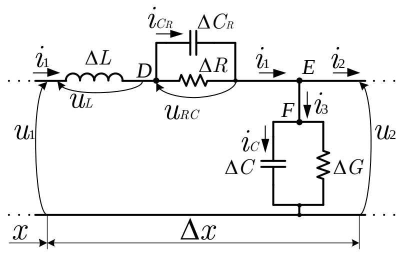

Figure 2: Elementary circuit of the hereditary transmission line model.

equation corresponding to the transmission line model with elementary circuit as in Figure 2 takes the form

$$\begin{aligned}\big(\tau LC\,{}_0\mathrm{D}_t^{\alpha+\beta+\gamma}+\tau LG\,{}_0\mathrm{D}_t^{\alpha+\beta}+LC\,{}_0\mathrm{D}_t^{\alpha+\gamma}+LG\,{}_0\mathrm{D}_t^{\alpha}\\+RC\,{}_0\mathrm{D}_t^{\gamma}+RG\big)u\left(x,t\right)=\left(\tau\,{}_0\mathrm{D}_t^{\beta}+1\right)\frac{\partial^2}{\partial x^2}u\left(x,t\right),\end{aligned}\tag{1}$$

where u is voltage between line conductors at position x and at time instant t. The Riemann-Liouville fractional derivative of order ξ is defined by

$${}_0\mathrm{D}_t^{\xi}f\left(t\right)=\frac{\mathrm{d}^{\lceil\xi\rceil}}{\mathrm{d}t^{\lceil\xi\rceil}}{}_0\mathrm{I}_t^{\lceil\xi\rceil-\xi}f\left(t\right),\tag{2}$$

where $\lceil\xi\rceil$ is the smallest integer bigger or equal to ξ and where ${}_0\mathrm{I}_t^{\xi}$, $\xi>0$, is the operator of fractional integration, defined by

$${}_0\mathrm{I}_t^{\xi}f\left(t\right)=\frac{1}{\Gamma(\xi)}\int_0^t(t-\tau)^{\xi-1}f(\tau)\mathrm{d}\tau=\frac{t^{\xi-1}}{\Gamma\left(\xi\right)}*f\left(t\right),\tag{3}$$

where Γ is the Euler gamma function and $*$ denotes the convolution, which is for the causal function defined by $f\left(t\right)*g\left(t\right)=\int_0^t f\left(\tau\right)g\left(t-\tau\right)\mathrm{d}\tau$. The constant τ is related to charge accumulation effects. The equivalent form of the generalized telegrapher's equation (1) is

$$\begin{aligned}\big(\tau LC\,{}_0\mathrm{D}_t^{\alpha+\gamma}+\tau LG\,{}_0\mathrm{D}_t^{\alpha}+LC\,{}_0\mathrm{D}_t^{\alpha+\gamma}\,{}_0\mathrm{I}_t^{\beta}+LG\,{}_0\mathrm{D}_t^{\alpha}\,{}_0\mathrm{I}_t^{\beta}\\+RC\,{}_0\mathrm{D}_t^{\gamma}\,{}_0\mathrm{I}_t^{\beta}+RG\,{}_0\mathrm{I}_t^{\beta}\big)u\left(x,t\right)=\left(\tau+{}_0\mathrm{I}_t^{\beta}\right)\frac{\partial^2}{\partial x^2}u\left(x,t\right).\end{aligned}\tag{4}$$

The highest order of fractional differentiation in the generalized telegrapher's equation (1) is, in general, from the interval $(0,3)$, since $\alpha,\beta,\gamma\in(0,1)$. This would imply that (1), apart from sub-diffusion (the highest order is in the interval $(0,1)$) and diffusion-wave (the corresponding interval is $(0,2)$), would cover phenomena modelled by equation with the order higher than order of the wave equation. However, from the equivalent form of the generalized telegrapher's equation (4) is obvious that the highest order of fractional differentiation is $\alpha+\gamma \in (0,2)$, since the differentiation operator on the right-hand-side in (1), unlike the integral operator in (4), actually reduces the highest order of fractional differentiation. Thus, if $\alpha+\gamma \in (0,1)$ in (4), then one expects the response of the diffusive type, while if $\alpha+\gamma \in (1,2)$ in (4), then the wave type response is expected.

2. Models

2.1 Derivation of non-local telegrapher's equations

Application of Kirchhoff's laws to the k-th elementary circuit from Figure 1 imply the following two equations

$$-v_{k-1}(t)+\Delta R_k(t)\, i_k(t)+\Delta u_k(t)-\Delta\mathcal{E}_k(t)+v_k(t)=0, \tag{5}$$

$$i_k(t)-i_{k+1}(t)-\Delta G_k(t)\, v_k(t)-\frac{\mathrm{d}}{\mathrm{d}t}\big(\Delta C_k(t)\, v_k(t)\big)=0, \tag{6}$$

where the inductor voltage

$$\Delta u_k(t)=\frac{\mathrm{d}}{\mathrm{d}t}\Delta\phi_k(t) \tag{7}$$

is induced by the time-changes of magnetic flux within the k-th elementary circuit $\Delta\phi_k$. If the magnetic flux is due only to the current of the same circuit i_k, then the elementary circuit corresponds to the local (classical) model of transmission line and classical telegrapher's equation is obtained. Non-locality effects in transmission line modeling are introduced by assuming the magnetic flux within the k-th elementary circuit $\Delta\phi_k$ to be a consequence not only of the current i_k running through the k-th series branch, but also of currents i_j, $j=1,\dots,N$, $j\neq k$, running through all others series branches, so that the magnetic flux $\Delta\phi_k$ is written as

$$\Delta\phi_k(t)=\Delta L_k(t)\, i_k(t)+\sum_{\substack{j=1\\ j\neq k}}^{N}\Delta^2 m_{kj}(t)\, i_j(t), \tag{8}$$

where $\Delta^2 m_{kj}$ is the cross-inductivity coefficient, quantifying the influence of elementary current i_j to flux of the k-th elementary contour.

In order to obtain non-local transmission line model, the spatially discretized system of equations (5) - (7), with (8), corresponding the k-th elementary circuit, is rewritten as

$$\begin{aligned}
-\frac{v_k(t)-v_{k-1}(t)}{\Delta x_k} &= \frac{\Delta R_k(t)}{\Delta x_k}\, i_k(t)+\frac{\mathrm{d}}{\mathrm{d}t}\frac{\Delta\phi_k(t)}{\Delta x_k}-\frac{\Delta\mathcal{E}_k(t)}{\Delta x_k},\\
-\frac{i_{k+1}(t)-i_k(t)}{\Delta x_k} &= \frac{\Delta G_k(t)}{\Delta x_k}\, v_k(t)+\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\Delta C_k(t)}{\Delta x_k}\, v_k(t)\right),\\
\frac{\Delta\phi_k(t)}{\Delta x_k} &= \frac{\Delta L_k(t)}{\Delta x_k}\, i_k(t)+\sum_{\substack{j=1\\ j\neq k}}^{N}\frac{\Delta^2 m_{kj}(t)}{\Delta x_k\,\Delta x_j}\, i_j(t)\,\Delta x_j,
\end{aligned}$$

which in the limit when $\Delta x_k\to 0$ and $N\to\infty$ yields non-local telegrapher's equations

$$-\frac{\partial}{\partial x}v(x,t)=R(x,t)\, i(x,t)+\frac{\partial}{\partial t}\phi(x,t)-\mathcal{E}(x,t), \tag{9}$$

$$-\frac{\partial}{\partial x} i(x,t) = G(x,t)\, v(x,t) + \frac{\partial}{\partial t}\big(C(x,t)\, v(x,t)\big), \tag{10}$$

$$\phi(x,t) = L(x,t)\, i(x,t) + \int_a^b m(x,\zeta,t)\, i(\zeta,t)\, \mathrm{d}\zeta. \tag{11}$$

In the limit process, sequences of elementary quantities $\{v_k(t)\}_{k\in\{1,\dots,N\}}$ and $\{i_k(t)\}_{k\in\{1,\dots,N\}}$ become spatially distributed quantities $v = v(x,t)$ and $i = i(x,t)$, $x \in (a,b)$, $t > 0$, while $\left\{\frac{\Delta\mathcal{E}_k(t)}{\Delta x_k}\right\}_{k\in\{1,\dots,N\}}$ and $\left\{\frac{\Delta\phi_k(t)}{\Delta x_k}\right\}_{k\in\{1,\dots,N\}}$ become electromotive force and magnetic flux per-unit-length, defined by

$$\mathcal{E}(x,t) = \lim_{\Delta x_k\to 0} \frac{\Delta\mathcal{E}_k(t)}{\Delta x_k} \quad \text{and} \quad \phi(x,t) = \lim_{\Delta x_k\to 0} \frac{\Delta\phi_k(t)}{\Delta x_k}.$$

Similarly, model parameters per-unit-length: series resistance and inductance, shunt capacitance and conductance, as well the cross-inductivity kernel are defined by

$$R(x,t) = \lim_{\Delta x_k\to 0} \frac{\Delta R_k(t)}{\Delta x_k}, \quad L(x,t) = \lim_{\Delta x_k\to 0} \frac{\Delta L_k(t)}{\Delta x_k},$$

$$C(x,t) = \lim_{\Delta x_k\to 0} \frac{\Delta C_k(t)}{\Delta x_k}, \quad G(x,t) = \lim_{\Delta x_k\to 0} \frac{\Delta G_k(t)}{\Delta x_k}, \quad m(x,\zeta,t) = \lim_{\substack{\Delta x_k\to 0\\ \Delta x_j\to 0}} \frac{\Delta^2 m_{kj}(t)}{\Delta x_k\, \Delta x_j}.$$

Note that in the limit process finite differences become partial derivatives, while the sum becomes integral over the entire transmission line.

The magnetic flux accounting for non-local effects, assumed in the form (11), is a superposition of local and non-local magnetic fluxes, i.e.,

$$\phi = \phi_{\mathrm{L}} + \phi_{\mathrm{NL}}, \quad \text{with} \quad \phi_{\mathrm{L}}(x,t) = L\, i(x,t) \quad \text{and} \quad \phi_{\mathrm{NL}}(x,t) = \int_a^b m(x,\zeta)\, i(\zeta,t)\, \mathrm{d}\zeta, \tag{12}$$

where ϕ_{NL} is given constitutively through the cross-inductivity kernel m. In order to account for the non-locality assumption, i.e., assumption that the influence of physical quantity, measured at neighboring points ζ of point x, on the physical quantity measured at point x depends on their distance $|x-\zeta|$ and assuming that the influence should decrease as the distance between points increases, the cross-inductivity kernel is assumed as a decreasing function depending on $|x-\zeta|$, i.e., $m(x,\zeta) = m(|x-\zeta|)$. The non-locality assumption is inspired by the Neumann formula for calculating the cross-inductivity corresponding to two contours, where the infinitesimal cross-inductivity is inversely proportional to the distance between two elementary contours, see [18].

Several choices for the cross-inductivity kernel are given in Table 1. Each of the cross-inductivity kernels from Table 1 reduce to the Dirac distribution, transforming the non-local magnetic flux into the local one, so that flux, given by (12), becomes $\phi = (L+M)\, i$. On the other hand, the non-local flux is zero if cross-inductivity kernel m is either chosen to be zero, or in the limiting case when non-locality parameter ℓ tends to infinity, so that, by (12), the flux reduces to $\phi = L\, i$.

Constitutive equation determining the non-local magnetic flux, corresponding to the power type cross-inductivity kernel, is given through the symmetrized fractional integral, since

$$\phi_{\mathrm{NL}}(x,t) = \frac{M}{2\ell^{\alpha}} \left({}_aI_x^{\alpha} + {}_xI_b^{\alpha}\right) i(x,t),$$

Table 1: Several choices of cross-inductivity kernels.

Kernel type	$m(\lvert x-\zeta\rvert)$	$M\delta(\lvert x-\zeta\rvert)$
Power	$\frac{M}{2\Gamma(\alpha)}\frac{\lvert x-\zeta\rvert^{\alpha-1}}{\ell^{\alpha}}$	as $\alpha\to 0$
Exponential	$\frac{M}{2\ell}\mathrm{e}^{-\frac{\lvert x-\zeta\rvert}{\ell}}$	as $\ell\to 0$
Gaussian	$\frac{M}{\ell\sqrt{\pi}}\mathrm{e}^{-\frac{\lvert x-\zeta\rvert^{2}}{\ell^{2}}}$	as $\ell\to 0$

Model parameters are: cross-inductivity per-unit-length M, non-locality parameter (characteristic length) ℓ, and $\alpha\in(0,1)$.

where

$$ {}_aI_x^{\alpha} f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x \frac{f(\zeta)}{(x-\zeta)^{1-\alpha}}\mathrm{d}\zeta \quad\text{and}\quad {}_xI_b^{\alpha} f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b \frac{f(\zeta)}{(\zeta-x)^{1-\alpha}}\mathrm{d}\zeta $$

are, respectively, left and right fractional integrals, see [8, 16]. The non-local magnetic flux

$$ \phi_{\mathrm{NL}}(x,t)=\frac{M}{2\ell}\int_a^b \mathrm{e}^{-\frac{\lvert x-\zeta\rvert}{\ell}}\, i(\zeta,t)\,\mathrm{d}\zeta, $$

corresponding to the exponential type cross-inductivity kernel, is a solution of the differential equation

$$ \phi_{\mathrm{NL}}(x,t)-\ell^2\frac{\partial^2}{\partial x^2}\phi_{\mathrm{NL}}(x,t)=Mi(x,t), $$

known as Eringen's stress gradient constitutive equation [17], widely used in modeling non-local effects in continuum mechanics.

2.2 Derivation of hereditary telegrapher's equations

The generalization of transmission line model whose elementary circuit is shown in Figure 2 includes fractional-order inductive (ΔL) and capacitive (ΔC_R, ΔC) elements. We assume that the hereditariness of magnetization and polarization processes is quite prominent and therefore the connection between magnetic flux and current in coil, as well as connection between charge and voltage in capacitor, are modeled through the Riemann-Liouville fractional integral, i.e., we assume that the hereditariness kernel is the power type function. Such obtained flux is connected with coil voltage trough the Faraday law and charge is connected with capacitor current by definition, yielding that coil voltage and current, as well as capacitor current and voltage, are related trough the Riemann-Liouville fractional derivative.

Regarding the coil, since we want to include the memory effects, instead of the classical connection between magnetic flux ϕ and electric current i, established through inductance L and given by $\phi = Li_L$, we shall use

$$ \phi(t)=L\,{}_0\mathrm{I}_t^{1-\xi} i_L(t),\quad t>0,\ \xi\in(0,1), $$

where L is the (fractional) inductance and ${}_0\mathrm{I}_t^{1-\xi}$ denotes the operator of Riemann-Liouville fractional integration (3). Using the Faraday law of electromagnetic induction, we obtain

$$ u_L(t)=\frac{\mathrm{d}\phi(t)}{\mathrm{d}t}=L\frac{\mathrm{d}}{\mathrm{d}t}{}_0\mathrm{I}_t^{1-\xi} i_L(t)=L\,{}_0\mathrm{D}_t^{\xi} i_L(t),\quad t>0,\ \xi\in(0,1), \tag{13} $$

for the voltage of the coil, where ${}_0\mathrm{D}_t^\alpha$ stands for the Riemann-Liouville derivative of order $\alpha \in (0,1)$, defined by (2). Similar constitutive equation is used in [14] in modelling magnetic core coils. Note that fractional-order coil model (13) describes the element with performance between resistor and inductor. Indeed, such element can be seen as a series connection between a frequency-dependent resistor and a classical coil with frequency-dependent inductance.

The hereditariness will also be included by modelling capacitor in the framework of the fractional calculus. Namely, instead of the classical connection between the charge q and voltage u_C, given by $q = Cu_C$, where C is the capacitance, we use

$$q(t) = C\,{}_0\mathrm{I}_t^{1-\xi} u_C(t), \quad t>0, \ \xi \in (0,1),$$

so that

$$i_C(t) = \frac{\mathrm{d}}{\mathrm{d}t} q(t) = C\frac{\mathrm{d}}{\mathrm{d}t}{}_0\mathrm{I}_t^{1-\xi} u_C(t) = C\,{}_0\mathrm{D}_t^{\xi} u_C(t), \quad t>0, \ \xi \in (0,1), \tag{14}$$

where C is the (fractional) capacitance. In particular, fractional-order constitutive equation for capacitor is used in [4, 5, 7, 10, 11] describing electrochemical double-layer capacitors (EDLC), also known as super-capacitors or ultra-capacitors, with different assumptions on equivalent electrical scheme. Note that fractional-order capacitor model (14) describes the element with performance between conductor and capacitor, in the sense that it is equivalent to a parallel connection between frequency-dependent resistor and a classical capacitor with frequency-dependent capacitance.

Electrical circuits containing fractional order elements are considered in [6, 15], while in [9] skin effect is modelled using fractional-order coils and resistors. Fractional-order models for circuit elements are used to analyze RLC filters in [12, 13].

In order to formulate the mathematical model of the transmission line, modeled by elementary circuit shown in Figure 2, we use the first Kirchhoff law for the points D, E and F, as well as the second Kirchhoff law for the elementary circuit, to obtain

$$i_1 = \frac{u_{RC}}{\Delta R} + i_{C_R}, \quad i_1 = i_2 + i_3, \quad i_3 = i_C + \Delta G u_2, \tag{15}$$

$$u_L + u_{RC} + u_2 - u_1 = 0, \tag{16}$$

respectively. According to (13) and (14), we have

$$u_L = \Delta L\,{}_0\mathrm{D}_t^\alpha i_1, \quad i_{C_R} = \Delta C_R\,{}_0\mathrm{D}_t^\beta u_{RC}, \quad i_C = \Delta C\,{}_0\mathrm{D}_t^\gamma u_2.$$

Taking into account that the model shown in Figure 2 is the spatial discretization of the real material and that the unit cell ranges from x to $x+\Delta x$, we write

$$i_1 = i(x,t), \quad i_2 = i(x+\Delta x,t), \quad u_1 = u(x,t), \quad u_2 = u(x+\Delta x,t),$$

so that system (15), (16) becomes

$$\Delta R i(x,t) = u_{RC}(x,t) + \Delta R \Delta C_R\,{}_0\mathrm{D}_t^\beta u_{RC}(x,t), \tag{17}$$

$$i(x+\Delta x,t) - i(x,t) = -\Delta C\,{}_0\mathrm{D}_t^\gamma u(x+\Delta x,t) - \Delta G u(x+\Delta x,t), \tag{18}$$

$$u(x+\Delta x,t) - u(x,t) = -\Delta L\,{}_0\mathrm{D}_t^\alpha i(x,t) - u_{RC}(x,t). \tag{19}$$

In order to write the equations for the material itself, we have to pass to the continuum. This will be done in two steps. As the first step, we divide the equations (17) - (19) with Δx and obtain

$$\frac{\Delta R}{\Delta x} i(x,t) = \frac{u_{RC}(x,t)}{\Delta x} + \Delta R \Delta C_R \, {}_0\mathrm{D}_t^{\beta} \frac{u_{RC}(x,t)}{\Delta x}, \tag{20}$$

$$\frac{i(x+\Delta x,t) - i(x,t)}{\Delta x} = -\frac{\Delta C}{\Delta x} \, {}_0\mathrm{D}_t^{\gamma} u(x+\Delta x,t) - \frac{\Delta G}{\Delta x} u(x+\Delta x,t), \tag{21}$$

$$\frac{u(x+\Delta x,t) - u(x,t)}{\Delta x} = -\frac{\Delta L}{\Delta x} \, {}_0\mathrm{D}_t^{\alpha} i(x,t) - \frac{u_{RC}(x,t)}{\Delta x}. \tag{22}$$

In the second step, we introduce the model parameters: inductance, resistance, capacitance and conductance by-length respectively by

$$L = \lim_{\Delta x \to 0} \frac{\Delta L}{\Delta x}, \quad R = \lim_{\Delta x \to 0} \frac{\Delta R}{\Delta x}, \quad C = \lim_{\Delta x \to 0} \frac{\Delta C}{\Delta x}, \quad G = \lim_{\Delta x \to 0} \frac{\Delta G}{\Delta x}.$$

The generalized time-fractional model of transmission line is represented by system of equations

$$R i(x,t) = u'(x,t) + \tau \, {}_0\mathrm{D}_t^{\beta} u'(x,t), \tag{23}$$

$$\frac{\partial}{\partial x} i(x,t) = -C \, {}_0\mathrm{D}_t^{\gamma} u(x,t) - G u(x,t), \tag{24}$$

$$\frac{\partial}{\partial x} u(x,t) = -L \, {}_0\mathrm{D}_t^{\alpha} i(x,t) - u'(x,t), \tag{25}$$

that is obtained by letting $\Delta x \to 0$ in system (20) - (22), where

$$u'(x,t) = \lim_{\Delta x \to 0} \frac{u_{RC}(x,t)}{\Delta x}$$

is the voltage by-length and

$$\tau = \frac{R}{W_R} = \frac{\lim_{\Delta x \to 0} \frac{\Delta R}{\Delta x}}{\lim_{\Delta x \to 0} \frac{\Delta W_R}{\Delta x}} = \lim_{\Delta x \to 0} \frac{\frac{\Delta R}{\Delta x}}{\frac{\Delta W_R}{\Delta x}} = \lim_{\Delta x \to 0} \frac{\Delta R}{\Delta W_R} = \lim_{\Delta x \to 0} (\Delta R \Delta C_R),$$

where we used $\Delta W_R = \frac{1}{\Delta C_R}$.

3. Analytical solution of nonlocal and hereditary telegrapher's equations

3.1 Analytical solution of nonlocal telegrapher's equations

Non-local telegrapher's equations (9) - (11), modeling transmission line subject to external forcing, for infinite spatial domain and constant model parameters can be rewritten as

$$-\frac{\partial}{\partial x} v(x,t) = R\, i(x,t) + \frac{\partial}{\partial t} \phi(x,t) - \mathcal{E}(x,t), \tag{26}$$

$$-\frac{\partial}{\partial x} i(x,t) = G\, v(x,t) + C \frac{\partial}{\partial t} v(x,t), \tag{27}$$

$$\phi(x,t) = L\, i(x,t) + m(|x|) *_x i(x,t), \tag{28}$$

since in this case the integral in non-local magnetic flux (12) is represented by the convolution with respect to the spatial coordinate

$$\phi_{\mathrm{NL}}(x,t) = m(|x|) *_x i(x,t) = M \int_{-\infty}^{\infty} \bar{m}(|x-\zeta|)\, i(\zeta,t)\, \mathrm{d}\zeta, \tag{29}$$

with $\bar{m}$ denoting the normalized cross-inductivity kernel defined by $\bar{m} = \frac{m}{M}$.

Expressing the non-local flux in terms of convolution is in accordance with the assumption that the influence of the neighboring points ζ of point x depend on their distance $|x-\zeta|$, which implies the suitability of cross-inductivity kernels from Table 1, see (29). The Fourier transform of cross-inductivity kernels, along with their behavior in limiting cases is given in Table 2.

Table 2: Fourier transform of cross-inductivity kernels.

Kernel type	$\bar{m}(\lvert x\rvert)$	$\tilde{\bar{m}}(\lvert\xi\rvert)$	$\lim_{\xi\to 0}\tilde{\bar{m}}(\lvert\xi\rvert)$	$\lim_{\xi\to\infty}\tilde{\bar{m}}(\lvert\xi\rvert)$
Power	$\frac{\lvert x\rvert^{\alpha-1}}{2\Gamma(\alpha)\ell^{\alpha}}$	$\frac{\cos\frac{\alpha\pi}{2}}{\lvert\ell\xi\rvert^{\alpha}}$	∞	0
Exponential	$\frac{1}{2\ell}\mathrm{e}^{-\frac{\lvert x\rvert}{\ell}}$	$\frac{1}{1+(\ell\xi)^2}$	1	0
Gaussian	$\frac{1}{\ell\sqrt{\pi}}\mathrm{e}^{-\frac{\lvert x\rvert^2}{\ell^2}}$	$\mathrm{e}^{-\left(\frac{\ell\xi}{2}\right)^2}$	1	0

The analytical solution of non-local telegrapher's equations (26) - (28) is determined for zero initial voltage and magnetic flux (current) by the use of integral transform method. The voltage, as a solution of non-local telegrapher's equations (26) - (28), is obtained as a convolution of solution kernel Q and electromotive force $\mathcal{E}$ in both space and time, taking the form

$$v(x,t) = Q(x,t) *_{x,t} \mathcal{E}(x,t),$$

with convolution in time given by $f(t) *_t g(t) = \int_0^t f(\tau)\, g(t-\tau)\,\mathrm{d}\tau$. The solution kernel Q takes the following form

$$Q(x,t) = \frac{c^2}{\pi}\left(\int_0^{\xi_1} \frac{\sinh(\nu(\xi)\,t)}{\nu(\xi)}\,\mathrm{e}^{-\mu(\xi)\,t}\,\frac{\xi\sin(\xi x)}{1+\frac{\tau_M}{\tau_L}\tilde{\bar{m}}(|\xi|)}\,\mathrm{d}\xi \right.$$
$$\left. + \int_{\xi_1}^{\xi_2} f(\xi,t)\,\mathrm{e}^{-\mu(\xi)\,t}\,\frac{\xi\sin(\xi x)}{1+\frac{\tau_M}{\tau_L}\tilde{\bar{m}}(|\xi|)}\,\mathrm{d}\xi + \int_{\xi_2}^{\infty} \frac{\sin(\omega(\xi)\,t)}{\omega(\xi)}\,\mathrm{e}^{-\mu(\xi)\,t}\,\frac{\xi\sin(\xi x)}{1+\frac{\tau_M}{\tau_L}\tilde{\bar{m}}(|\xi|)}\,\mathrm{d}\xi\right),$$

since $\nu^2(\xi) \geq 0$ if $\xi \to 0$ and $\nu^2(\xi) < 0$ if $\xi \to \infty$, while $\nu^2(\xi)$ may change sign if $\xi \in (\xi_1, \xi_2)$, where the involved functions are defined as follows

$$\mu(\xi) = \frac{1}{2}\left(\frac{1}{\tau_L + \tau_M\tilde{\bar{m}}(|\xi|)} + \frac{1}{\tau_C}\right),$$
$$\nu^2(\xi) = \left(\frac{1}{2}\left(\frac{1}{\tau_L + \tau_M\tilde{\bar{m}}(|\xi|)} - \frac{1}{\tau_C}\right)\right)^2 - \frac{1}{\tau_C\left(\tau_L + \tau_M\tilde{\bar{m}}(|\xi|)\right)}\left(\frac{\xi}{K}\right)^2$$
$$= \left(\frac{1}{2}\left(\frac{1}{\tau_L + \tau_M\tilde{\bar{m}}(|\xi|)} - \frac{1}{\tau_C}\right)\right)^2 - \frac{(c\xi)^2}{1+\frac{\tau_M}{\tau_L}\tilde{\bar{m}}(|\xi|)}, \quad \text{with} \quad c = \frac{1}{K\sqrt{\tau_L\tau_C}},$$
$$f(\xi,t) = \frac{\mathrm{e}^{\nu(\xi)\,t} - \mathrm{e}^{-\nu(\xi)\,t}}{2\nu(\xi)} = \begin{cases} \frac{\sinh(\nu(\xi)\,t)}{\nu(\xi)}, & \text{if } \nu^2(\xi) \geq 0, \\ \frac{\sin(\omega(\xi)\,t)}{\omega(\xi)}, & \text{if } \nu^2(\xi) < 0, \text{ with } \omega(\xi) = -\mathrm{i}\nu(\xi), \end{cases}$$

containing the static attenuation coefficient together with time constants:

$$K = \sqrt{RG}, \quad \tau_L = \frac{L}{R}, \quad \tau_M = \frac{M}{R}, \quad \text{and} \quad \tau_C = \frac{C}{G}.$$

Note, the function $\nu^2(\xi)$, defined by (23), may be either positive or negative depending on the value of variable $\xi \in [0, \infty)$, implying that a real valued function function f has two forms: one for $\nu^2(\xi) \geq 0$ and the other for $\nu^2(\xi) < 0$, since then one switches to a real valued function $\omega^2 = -\nu^2$.

3.2 Analytical solution of hereditary telegrapher's equations

In the sequel, the initial-boundary value problem on the half axis, $x \in [0, \infty)$, for time $t > 0$, corresponding to system of equations (23) - (25) will be solved and therefore it is subject to initial

$$u'(x,0) = 0, \quad i(x,0) = 0, \quad \text{and} \quad u(x,0) = 0, \quad x \in [0,\infty), \tag{30}$$

and boundary conditions

$$u(0,t) = u_0(t) \quad \text{and} \quad \lim_{x\to\infty} u(x,t) = 0, \quad t > 0. \tag{31}$$

System of equations (23) - (25), subject to (30) and (31), after introduction of dimensionless quantities

$$T = \left(\frac{C}{G}\right)^{\frac{1}{\gamma}}, \quad \ell = \frac{1}{\sqrt{LG}}\left(\frac{C}{G}\right)^{\frac{\alpha}{2\gamma}}, \quad \bar{t} = \frac{t}{T}, \quad \bar{x} = \frac{x}{\ell}, \quad \bar{\tau} = \tau\left(\frac{G}{C}\right)^{\frac{\beta}{\gamma}},$$
$$\bar{u} = \frac{u}{U}, \quad \bar{u}' = \frac{\ell}{U}u', \quad I = U\sqrt{\frac{G}{L}}\left(\frac{C}{G}\right)^{\frac{\alpha}{2\gamma}}, \quad \bar{\imath} = \frac{i}{I},$$

where U is the nominal value of voltage u_0 at the boundary, see $(31)_1$, and after omitting the bars over dimensionless quantities becomes

$$\frac{R}{L}\left(\frac{C}{G}\right)^{\frac{\alpha}{\gamma}} i(x,t) = u'(x,t) + \tau\, {}_0\mathrm{D}_t^{\beta} u'(x,t), \tag{32}$$
$$\frac{\partial}{\partial x} i(x,t) = -{}_0\mathrm{D}_t^{\gamma} u(x,t) - u(x,t), \tag{33}$$
$$\frac{\partial}{\partial x} u(x,t) = -{}_0\mathrm{D}_t^{\alpha} i(x,t) - u'(x,t), \tag{34}$$

subject to (dimensionless) initial

$$u'(x,0) = 0, \quad i(x,0) = 0, \quad \text{and} \quad u(x,0) = 0, \quad x \in [0,\infty), \tag{35}$$

and (dimensionless) boundary conditions

$$u(0,t) = u_0(t) \quad \text{and} \quad \lim_{x\to\infty} u(x,t) = 0, \quad t > 0. \tag{36}$$

The solution to (32) - (34), with (35) and (36), is obtained by the method of Laplace transform as

$$u(x,t) = u_0(t) * u_\delta(x,t),$$

where

$$u_\delta = \mathcal{L}^{-1}\left[\mathrm{e}^{-k(s)x}\right]$$

is the impulse response (inversion of the transfer function), i.e., the response voltage to the boundary voltage assumed as a Dirac δ-distribution, involving the function

$$k(s) = \sqrt{\psi(s)} \tag{37}$$

$$\psi(s) = \frac{\left(s^{\alpha+\beta} + a s^{\alpha} + b\right)\left(s^{\gamma}+1\right)}{s^{\beta}+a},$$

with $a = \frac{1}{\tau}$ and $b = \frac{R}{\tau L}\left(\frac{C}{G}\right)^{\frac{\alpha}{\gamma}}$.

The form of the impulse response depends on the nature, number, and position of branching points of the function k, given by (37), so that in the case when k either has no branching points in addition to the point $s = 0$, or has a negative real branching point, the impulse response takes the form

$$u_\delta^{(\mathrm{I})}(x,t) = \frac{1}{2\pi\mathrm{i}}\int_0^\infty \left(\mathrm{e}^{-k\left(\rho\mathrm{e}^{-\mathrm{i}\pi}\right)x} - \mathrm{e}^{-k\left(\rho\mathrm{e}^{\mathrm{i}\pi}\right)x}\right)\mathrm{e}^{-\rho t}\mathrm{d}\rho.$$

On the other hand, if there are complex conjugated branching points of k having negative real part, denoted $s_0 = \rho_0\mathrm{e}^{\mathrm{i}\varphi_0}$ and $\bar{s}_0$, the impulse response takes the form

$$u_\delta^{(\mathrm{II})}(x,t) = \frac{1}{2\pi\mathrm{i}}\int_0^\infty \left(\mathrm{e}^{-k\left(\rho\mathrm{e}^{\mathrm{i}\varphi_0}\right)x}\mathrm{e}^{\mathrm{i}(\varphi_0+\rho t\sin\varphi_0)} - \mathrm{e}^{-k\left(\rho\mathrm{e}^{-\mathrm{i}\varphi_0}\right)x}\mathrm{e}^{-\mathrm{i}(\varphi_0+\rho t\sin\varphi_0)}\right)\mathrm{e}^{\rho t\cos\varphi_0}\mathrm{d}\rho.$$

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